Comment.Math.Univ.Carolin. 44,2 (2003)295–306 

# Notes on $cfp$-covers

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*Abstract.* The main purpose of this paper is to establish general conditions under which $T_2$-spaces are compact-covering images of metric spaces by using the concept of $cfp$-covers. We generalize a series of results on compact-covering open images and sequence-covering quotient images of metric spaces, and correct some mapping characterizations of $g$-metrizable spaces by compact-covering $\sigma$-maps and $mssc$-maps.

*Keywords:* $cfp$-covers, compact-covering maps, metrizable spaces, $g$-metrizable spaces, $\sigma$-maps, $mssc$-maps

*Classification:* 54E40, 54E18, 54C10

In 1964, E. Michael introduced the concept of compact-covering maps. Let $f : X \to Y$. $f$ is called a compact-covering map, if every compact subset of $Y$ is the image of some compact subset of $X$ under $f$. Because many important kinds of maps are compact-covering maps, such as closed maps on paracompact spaces, many topologists have aimed to seek the characterizations of certain compact-covering images of metric spaces since the seventies last century. E. Michael, K. Nagami, Y. Tanaka and some Chinese topologists have obtained the characterizations of images of metric spaces under the following maps: compact-covering and open maps, compact-covering and open $s$-maps, sequence-covering (quotient) $s$-maps, compact-covering (quotient) $s$-maps, compact-covering (quotient) compact maps. The key to prove these results is to construct compact-covering maps on metric spaces, but there is no method to unify these proofs. The purpose of this paper is to develop the concept of $cfp$-covers, and give some consistent methods to construct compact-covering maps.

We assume that all spaces are $T_2$, and maps are continuous and onto.

## 1. Compact-covering images of metric spaces

In 1960, V. Ponomarev proved that every first-countable space is an open image of some Baire zero-dimension metric space (Proposition 2.4.4 in [10]). Now, we generalize the Ponomarev's method. Let $\mathcal{P}$ be a network of $X$, $\mathcal{P} = \{P_\alpha\}_{\alpha\in\Lambda}$, let $\Lambda$ be endowed with discrete topology and $M = \{\alpha = (\alpha_i) \in \Lambda^\omega : \{P_{\alpha_i}\}_{i\in\mathbb{N}}$ forms a network at some point $x_\alpha$ in $X\}$, then $M$ is a metric space, and $x_\alpha$ is unique

Supported by the NNSF of China (10271026, 19971048, 10171043).

for each $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. Then $(f, M, X, \mathcal{P})$ is called a Ponomarev's system. The following lemma can be easily obtained by using the Ponomarev's method (Proposition 2.4.3 in [10]).

**Lemma 1** ([14]). *Let $(f, M, X, \mathcal{P})$ be a Ponomarev's system.*

(1) *$f$ is a map if there exists a countable subset of $\mathcal{P}$ which forms a network at $x$ for every $x \in X$.*
(2) *$f$ is an open map if $\mathcal{P}$ is a countable local base of $X$.*
(3) *For every non-empty subset $C$ of $X$, $f^{-1}(C)$ is a separable subspace of $M$ if $C$ meets only with countably many elements of $\mathcal{P}$.* $\square$

To ensure that $f$ in the Ponomarev's system is a compact-covering map, $\mathcal{P}$ must have some properties. Recall the concept of $cfp$-covers ([20]). Let $K$ be a subset of $X$. $\mathcal{F}$ is called a $cfp$-cover (i.e., compact-finite-partition-cover) of $K$, if $\mathcal{F}$ is a cover of $K$ in $X$ such that it can be precisely refined by some finite cover of $K$ consisting of compact subsets of $K$.

Let $\mathcal{P}$ be a collection of subsets of $X$, and $K$ be a subset of $X$. We say that $\mathcal{P}$ has the *cc*-property on $K$, if whenever $C$ is a non-empty compact subset of $K$, and $V$ a neighborhood of $C$ in $X$, then there exists a subset $\mathcal{F}$ of $\mathcal{P}$ such that $\mathcal{F}$ is a $cfp$-cover of $C$ and $\bigcup \mathcal{F} \subset V$.

The *cc*-property is related to the concept of strong $k$-networks posed by Chuan Liu and Mumin Dai ([11]).

**Theorem 2.** *Let $(f, M, X, \mathcal{P})$ be a Ponomarev's system. If $K$ is a non-empty compact subset of $X$ such that some countable subset $\mathcal{P}(K)$ of $\mathcal{P}$ has the cc-property on $K$, then there exists a compact subset $L$ of $M$ satisfying $f(L) = K$.*

Proof: Let $\mathcal{P} = \{P_\alpha\}_{\alpha \in \Lambda}$, and $K$ be a non-empty compact subset of $X$. $\mathcal{P}(K)$ is countable, hence there are only countably many $cfp$-covers of $K$ by elements of $\mathcal{P}(K)$. Let $\{\mathcal{P}_i\}$ enumerate these $cfp$-covers and $\mathcal{P}_i = \{P_\alpha\}_{\alpha \in \Gamma_i}$. Then $\mathcal{P}_i$ can be precisely refined by some finite cover $\mathcal{F}_i = \{F_\alpha\}_{\alpha \in \Gamma_i}$ of $K$ consisting of compact subset of $K$ with each $F_\alpha \subset P_\alpha$.

Let $L = \{(\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \bigcap_{i \in \mathbb{N}} F_{\alpha_i} \neq \emptyset\}$. Then

(2.1) $L$ is a closed subset of the compact set $\prod_{i \in \mathbb{N}} \Gamma_i$, so $L$ is a compact subset of $\Lambda^\omega$. Put $\alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i - L$. Then $\bigcap_{i \in \mathbb{N}} F_{\alpha_i} = \emptyset$. From the compactness of $K$, there exists $i_0 \in \mathbb{N}$ such that $\bigcap_{i \leq i_0} F_{\alpha_i} = \emptyset$. Let $W = \{(\beta_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \beta_i = \alpha_i$ for each $i \leq i_0\}$. Then $W$ is an open subset of $\prod_{i \in \mathbb{N}} \Gamma_i$ such that $\alpha \in W$ and $W \cap L = \emptyset$. Therefore, $L$ is a closed subset of $\prod_{i \in \mathbb{N}} \Gamma_i$.

(2.2) $L \subset M$ and $f(L) \subset K$. Let $\alpha = (\alpha_i) \in L$, then $\bigcap_{i \in \mathbb{N}} F_{\alpha_i} \neq \emptyset$. Pick $x \in \bigcap_{i \in \mathbb{N}} F_{\alpha_i}$. Then it will suffice to show that $\{P_{\alpha_i}\}_{i \in \mathbb{N}}$ is a network of $x$ in $X$. In this case, $\alpha \in M$ and $f(\alpha) = x \in K$, so $L \subset M$ and $f(L) \subset K$.

Let $V$ be a neighborhood of $x$ in $X$. Since $K$ is a regular subspace of $X$, there exists an open neighborhood $W$ of $x$ in $K$ such that $\overline{W} = \mathrm{cl}_K(W) \subset V$. Now $\overline{W}$

is a compact subset of $K$ and $\mathcal{P}(K)$ has the *cc*-property on $K$, so there exists a finite collection $\mathcal{P}'$ of $\mathcal{P}(K)$ such that $\mathcal{P}'$ is a $cfp$-cover of $\overline{W}$ and $\bigcup \mathcal{P}' \subset V$. On the other hand, $K - W$ is a compact subset of $K$ satisfying $K - W \subset X - \{x\}$, so there exists a finite collection $\mathcal{P}''$ of $\mathcal{P}(K)$ such that $\mathcal{P}''$ is a $cfp$-cover of $K - W$ and $\bigcup \mathcal{P}'' \subset X - \{x\}$. Put $\mathcal{P}^* = \mathcal{P}' \cup \mathcal{P}''$. Then $\mathcal{P}^*$ is a $cfp$-cover of $K$, so $\mathcal{P}_k = \mathcal{P}^*$ for some $k \in \mathbb{N}$. But $x \in F_{\alpha_k} \subset P_{\alpha_k} \in \mathcal{P}_k$, thus $P_{\alpha_k} \in \mathcal{P}'$ and $P_{\alpha_k} \subset V$. Hence $\{P_{\alpha_i}\}_{i \in \mathbb{N}}$ is a network of $x$ in $X$.

(2.3) $K \subset f(L)$. Let $x \in K$. For each $i \in \mathbb{N}$, pick $\alpha_i \in \Gamma_i$ such that $x \in F_{\alpha_i}$. Put $\alpha = (\alpha_i)$. Then $\alpha \in L$ and $f(\alpha) = x$ by the proof of (2.2). So $K \subset f(L)$.

In words, $L$ is a compact subset of $M$ such that $f(L) = K$. □

The *cc*-property provides that the compact subset $K$ of $X$ is the image of some compact subset of a metric space. "*cc*" means an abbreviation of "compact-covering". Next, we shall give some corollaries of Theorem 2.

The first corollary is an inner characterization of compact-covering and open images of metric spaces obtained by E. Michael and K. Nagami in 1973. Recall the concept of outer bases ([14]). A collection $\mathcal{B}$ of open subsets of a space $X$ is called an outer base of a subset $A$ in $X$, if there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$ for every $x \in A$ and an open neighborhood $U$ of $x$ in $X$. Michael and Nagami proved the following property of outer bases.

**Lemma 3** ([14]). *Let $K$ be a compact and metrizable subset of a space $X$. If $K$ has a countable neighborhood base in $X$, then there exists a countable outer base of $K$ in $X$.* □

**Lemma 4.** *Let $K$ be a subset of $X$. If $\mathcal{B}$ is an outer base of $K$, then $\mathcal{B}$ has the cc-property on $K$.*

PROOF: Let $C$ be a compact subset of $K$ and $V$ a neighborhood of $C$ in $X$. For each $x \in C$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset V$. From the regularity of $C$, we can choose a relatively open set $V_x$ of $C$ such that $x \in V_x \subset \overline{V}_x \subset B_x$. $\{V_x\}_{x \in C}$ is a relatively open cover of $C$, thus it has a finite subcover $\{V_{x_i}\}_{i \leq n}$. Hence, $C = \bigcup_{i \leq n} \overline{V}_{x_i} \subset \bigcup_{i \leq n} B_{x_i} \subset V$, and $\{\overline{V}_{x_i}\}_{i \leq n}$ is a precise refinement of $\{B_{x_i}\}_{i \leq n}$. This implies that $\mathcal{B}$ has the *cc*-property on $K$. □

**Corollary 5** ([14]). *A space $X$ is a compact-covering and open image of a metric space if and only if every compact subset of $X$ is a metrizable subspace and has a countable neighborhood base in $X$.*

PROOF: Let $f : M \to X$ be a compact-covering and open map, where $M$ is a metric space. Suppose $K$ is a compact subset of $X$. Then there exists a compact subset $L$ of $M$ such that $f(L) = K$. Since $L$ is a compact subset of $M$, $f|L : L \to M$ is a perfect map, whereas metrizability is persevered by perfect maps (Theorem 2.2.1 in [10]), so $K$ is a metrizable space. On the other hand, from the metrizability of $M$, $L$ has a countable neighborhood base $\{V_n\}_{n \in \mathbb{N}}$ in $M$, and $f$ is an open map, thus $\{f(V_n)\}_{n\in\mathbb{N}}$ is a countable neighborhood base of $K$ in $X$.

Conversely, suppose every compact subset of $X$ is metrizable and has a countable neighborhood base. For each compact subset $K$ of $X$, in view of Lemma 3, $K$ has a countable outer base $\mathcal{U}(K)$ in $X$. $\mathcal{U}(K)$ has the $cc$-property on $K$ by Lemma 4. $\mathcal{U} = \bigcup\{\mathcal{U}(K) : K$ is a compact subset of $X\}$ is a countable local base of $X$. Let $(f, M, X, \mathcal{U})$ be a Ponomarev's system. Then $f$ is a compact-covering and open map by Lemma 1 and Theorem 2. So $X$ is a compact-covering and open image of a metric space. □

The second corollary concludes inner characterizations of compact-covering $s$-images of metric spaces given by Pengfei Yan and Shou Lin in 1999 and compact-covering open $s$-images of metric spaces given by E. Michael and K. Nagami in 1973. Recall the concept of $cfp$-networks ([19]). Let $\mathcal{P}$ be a cover of a space $X$. $\mathcal{P}$ is called a $cfp$-network of $X$, if for every compact subset $K$ of $X$ and a neighborhood $V$ of $K$ in $X$ there exists $\mathcal{F} \subset P$ such that $\mathcal{F}$ is a $cfp$-cover of $K$ and $\bigcup\mathcal{F} \subset V$. Obviously, $cfp$-networks are related to $k$-networks introduced by P. O'Meara in 1971. Let $\mathcal{P}$ be a collection of subsets of $X$. $\mathcal{P}$ is called a $k$-network, if for every compact subset $K$ and a neighborhood $V$ of $K$ in $X$ there exists a finite subset $\mathcal{F}$ of $\mathcal{P}$ such that $K \subset \bigcup\mathcal{F} \subset V$. It is easy to see that every $k$-network consisting of closed sets is a $cfp$-network, and every $cfp$-network is a $k$-network.

**Lemma 6.** *Every base of a space $X$ is a $cfp$-network of $X$.*

Proof: Let $\mathcal{B}$ be a base of $X$. For each compact subset $K$ of $X$ and a neighborhood $V$ of $K$ in $X$, since $\mathcal{B}$ is an outer base of $K$, by Lemma 4, there exists a finite subset $\mathcal{F}$ of $\mathcal{B}$ such that $\mathcal{F}$ is a $cfp$-cover of $K$ and $\bigcup\mathcal{F} \subset V$. So $\mathcal{B}$ is a $cfp$-network of $X$. □

Certain point-countable covers can be used to characterize various $s$-images of metric spaces. The following $cfp$-property of point-countable collections is similar to the famous Miščenko's lemma. For every subset $A$ of $X$, $\mathcal{F}$ is called a minimal $cfp$-cover of $A$, if $\mathcal{F}$ is a $cfp$-cover of $A$, and $\mathcal{F} - \{F\}$ is not a $cfp$-cover of $A$ for every $F \in \mathcal{F}$.

**Lemma 7** ([19])**.** *Suppose $\mathcal{P}$ is a point-countable collection of subsets of a space $X$. Then every compact subset of $X$ has only countably many minimal $cfp$-covers by elements of $\mathcal{P}$.* □

**Corollary 8** ([19])**.** *A space $X$ is a compact-covering $s$-image of a metric space if and only if $X$ has a point-countable $cfp$-network.*

Proof: Necessity. Suppose $X$ is a compact-covering $s$-image of a metric space $M$. $M$ is a metrizable space, thus $M$ has a $\sigma$-locally finite base $\mathcal{P}$. By Lemma 6, $\mathcal{P}$ is a $cfp$-network of $M$. Since $cfp$-networks are preserved by compact-covering maps, $f(\mathcal{P})$ is a $cfp$-network of $X$. $f$ is an $s$-map, so $f(\mathcal{P})$ is a point-countable collection. Hence $f(\mathcal{P})$ is a point-countable $cfp$-network of $X$.

Sufficiency. Suppose $X$ has a point-countable $cfp$-network $\mathcal{P}$ and let $(f, M, X, \mathcal{P})$ be a Ponomarev's system. Then $f : M \to X$ is an $s$-map by Lemma 1. In view of Theorem 2, to prove that $f$ is a compact-covering map, it will suffice to show that there exists a countable subset $\mathcal{P}(K)$ of $\mathcal{P}$ with $cc$-property on $K$ for every compact subset $K$ of $X$.

By Lemma 7, if $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ is a collection of minimal $cfp$-covers of $K$ by elements of $\mathcal{P}$, then $\mathcal{P}(K) = \bigcup_{i\in\mathbb{N}} \mathcal{P}_i$ is countable and has the $cc$-property on $K$. In fact, for each non-empty compact subset $C$ of $K$ and a neighborhood $V$ of $C$ in $X$, since $K$ is a compact subset of $X$, $K$ is a normal subset of $X$, so there exists an open neighborhood $W$ of $C$ in $K$ such that $\overline{W} \subset V$. $\mathcal{P}$ is a $cfp$-network of $X$, thus there is a finite subset $\mathcal{P}'$ of $\mathcal{P}$ such that $\mathcal{P}'$ is a $cfp$-cover of $\overline{W}$ and $\bigcup \mathcal{P}' \subset V$. On the other hand, $K - W \subset X - C$, so we can pick a finite subset $\mathcal{P}''$ of $\mathcal{P}$ such that $\mathcal{P}''$ is a $cfp$-cover of $K - W$ and $\bigcup \mathcal{P}'' \subset X - C$. Let $\mathcal{P}^* = \mathcal{P}' \cup \mathcal{P}''$. Then $\mathcal{P}^*$ is a $cfp$-cover of $K$ and $\mathcal{P}_k \subset \mathcal{P}^*$ for some $k \in \mathbb{N}$. Suppose $\mathcal{P}_k = \{\mathcal{P}_\alpha\}_{\alpha\in\Gamma}$ is precisely refined by the finite cover $\{K_\alpha\}_{\alpha\in\Gamma}$ of $K$ consisting of compact subsets of $K$ and put $\mathcal{F} = \{P_\alpha \in \mathcal{P}_k : K_\alpha \cap C \neq \emptyset\}$. Then $\mathcal{F}$ is a $cfp$-cover of $C$ such that $\bigcup \mathcal{F} \subset V$. Hence $\mathcal{P}(K)$ has the $cc$-property on $K$.

Summarizing, $f$ is a compact-covering $s$-map. □

**Corollary 9** ([14]). *A space $X$ is a compact-covering and open $s$-image of a metric space if and only if $X$ has a point-countable base.*

Proof: It is easy to show that open $s$-images of metric spaces have point-countable bases. Let $\mathcal{B}$ be a point-countable base of $X$ and $(f, M, X, \mathcal{B})$ be a Ponomarev's system. In view of Lemma 1, $f$ is an open $s$-map. $f$ is a compact-covering map by Lemma 6 and Corollary 8. □

The third corollary is the inner characterization of sequence-covering and quotient $s$-images of metric spaces proved by Y. Tanaka in 1987. Recall the concepts of sequence-covering maps and $cs^*$-networks. A map $f : X \to Y$ is called a sequence-covering map ([6]), if each convergent sequence of $Y$ is the image of some compact subset of $X$ under $f$. A collection $\mathcal{P}$ of subsets of $X$ is called a $cs^*$-network ([5]), if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with $U$ open in $X$, then $\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

Obviously, every $cfp$-network of $X$ is a $cs^*$-network of $X$.

**Lemma 10.** *Let $\mathcal{P}$ be a point-countable $cs^*$-network, and $K$ be a convergent sequence (including its limit). Then there is a countable subset $\mathcal{P}(K)$ of $\mathcal{P}$ with the cc-property on $K$.*

Proof: Let $\mathcal{P}(K) = \{P \in \mathcal{P} : P \cap K \neq \emptyset\}$. Since $\mathcal{P}$ is point-countable, $\mathcal{P}(K)$ is a countable subset of $\mathcal{P}$. Suppose $C$ is a non-empty compact subset of $K$ and $V$ is a neighborhood of $C$ in $X$. If $C$ is a finite set, because $X$ is a $T_2$-space and $\mathcal{P}$ is a network of $X$, there is a finite subset $\mathcal{F}$ of $\mathcal{P}$ such that the intersection of $C$ with elements of $\mathcal{F}$ includes only one point and $C \subset \bigcup \mathcal{F} \subset V$. So $\mathcal{F}$ is a $cfp$-cover of $C$ and $\bigcup \mathcal{F} \subset V$. Suppose $C$ is an infinite set, put $C = \{x\} \cup \{x_n : n \in \mathbb{N}\}$, where $\{x_n\}$ converges to $x$, and $\mathcal{P}' = \{P \in \mathcal{P} : x \in P \subset V\} = \{P_i\}_{i \in \mathbb{N}}$. We shall show that there exists a $k_0 \in \mathbb{N}$ such that $x_n \in \bigcup_{i \leq k_0} P_i$ for all but finitely many $n \in \mathbb{N}$. If not, we can pick a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that each $x_{n_k} \in X - \bigcup_{i \leq k} P_i$. So each $P_i$ only includes finitely many elements of $\{x_{n_k}\}$. But $\mathcal{P}$ is a $cs^*$-network of $X$, hence there is a $P \in \mathcal{P}$ such that $P \subset V$ and some subsequence of $\{x_{n_k}\}$ is included in $P$, thus $P = P_m$ for some $m \in \mathbb{N}$. Hence $P_m$ includes infinitely many elements of $\{x_{n_k}\}$, a contradiction.

So $C - \bigcup_{i \leq k_0} P_i$ is a finite set and $C \cap P_i$ is a non-empty closed set. Without losing generality, we can assume that $C - \bigcup_{i \leq k_0} P_i$ is non-empty. Then there is a finite subset $\mathcal{P}'$ of $\mathcal{P}$ such that $\mathcal{P}'$ is a $cfp$-cover of $C - \bigcup_{i \leq k_0} P_i$, $\bigcup \mathcal{P}' \subset V$ and every element of $\mathcal{P}'$ meets with $C$. Let $\mathcal{F} = \{P_i\}_{i \leq k_0} \cup \mathcal{P}'$. Then $\mathcal{F}$ is a $cfp$-cover of $C$ and $\bigcup \mathcal{F} \subset V$, so $\mathcal{P}(K)$ has the $cc$-property on $K$. □

**Corollary 11** ([6], [17])**.** *The following are equivalent for a space $X$:*

(1) *$X$ is a quotient $s$-image of a metric space;*
(2) *$X$ is a sequence-covering and quotient $s$-image of a metric space;*
(3) *$X$ is a sequential space with a point-countable $cs^*$-network.*

Proof: (1) $\Rightarrow$ (3). Let $f : M \to X$ be a quotient $s$-map, where $M$ is a metric space. Suppose $\mathcal{B}$ is a $\sigma$-locally finite base of $M$, $\mathcal{P} = f(\mathcal{B})$. Since $f$ is a quotient $s$-map and sequentiality of spaces is preserved by quotient maps, $X$ is a sequential space and $\mathcal{P}$ is a point-countable $cs^*$-network of $X$.

(3) $\Rightarrow$ (2). Suppose $\mathcal{P}$ is a point-countable $cs^*$-network of $X$ and let $(f, M, X, \mathcal{P})$ be a Ponomarev's system. In view of Lemma 1, $f : M \to X$ is an $s$-map. $f$ is a sequence-covering map by Lemma 10 and Theorem 2. $X$ is a sequential space, so the sequence-covering map $f$ is a quotient map.

(2) $\Rightarrow$ (1). It is trivial. □

## 2. The characterizations of $g$-metrizable spaces

Recall the definition of $g$-metrizable spaces. A collection $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ of subsets of a space $X$ is called a weak base of $X$, if $\mathcal{P}$ satisfies that (1) $\mathcal{P}_x$ is a network of $x$ in $X$; (2) For each $U, V \in \mathcal{P}_x$, there exists $W \in \mathcal{P}_x$ such that $W \subset U \cap V$; (3) $G \subset X$ is open iff for each $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

$\mathcal{P}_x$ is called a weak neighborhood base of $x$ in $X$. $X$ is $gf$-countable if $X$ has a weak base $\bigcup_{x \in X} \mathcal{P}_x$, where each $\mathcal{P}_x$ is countable. Also, $X$ is $g$-metrizable if $X$ is a regular space with a $\sigma$-locally finite weak base.

Obviously, every metric space is $g$-metrizable. L. Foged obtained the following theorem for $g$-metrizable spaces.

**Lemma 12** ([3], [4]). *The following are equivalent for a regular space $X$:*

(1) *$X$ is a $g$-metrizable space;*
(2) *$X$ has a $\sigma$-discrete weak base;*
(3) *$X$ is a $gf$-countable space with a $\sigma$-locally finite $k$-network.* □

By Corollary 3.8.6 in [10], $g$-metrizable spaces are quotient and compact images of metric spaces, but the converse is not true (see Example 15). To find out a suitable map to characterize $g$-metrizable spaces, we need the concept of $\sigma$-maps ([8]). Let $f : X \to Y$. $f$ is called a $\sigma$-map, if there exists a base $\mathcal{B}$ of $X$ such that $f(\mathcal{B})$ is a $\sigma$-locally finite collection in $Y$. Every map defined on a separable metric space is a $\sigma$-map. In this section, we shall show that a regular space is $g$-metrizable iff it is a compact-covering, quotient, compact and $\sigma$-image of a metric space.

We extend the Ponomarev's system to cover sequences of spaces. Let $\{\mathcal{P}_i\}$ be a cover sequence of a space $X$. $\{\mathcal{P}_i\}$ is called a point-star network, if $\{\mathrm{st}(x, \mathcal{P}_i)\}_{i\in\mathbb{N}}$ is a network of $x$ for each $x \in X$. Suppose $\{\mathcal{P}_i\}$ is a point-star network of $X$, for each $i \in \mathbb{N}$, put $\mathcal{P}_i = \{P_\alpha\}_{\alpha\in\Lambda_i}$ and endow $\Lambda_i$ with the discrete topology. Then $M = \{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \{\mathcal{P}_{\alpha_i}\}_{i\in\mathbb{N}}$ forms a network at some point $x_\alpha$ in $X\}$ is a metric space and $x_\alpha$ is unique for each $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. Then $(f, M, X, \mathcal{P}_i)$ is also called a Ponomarev's system.

**Lemma 13.** *Let $(f, M, X, \mathcal{P}_i)$ be a Ponomarev's system.*

(1) *$f$ is a compact map if $\{\mathcal{P}_i\}$ is a point-star network consisting of point-finite covers.*
(2) *For a compact subset $K$ of $X$, if some finite subset of $\mathcal{P}_i$ is a $cfp$-cover of $K$ for each $i \in \mathbb{N}$, then there exists a compact subset $L$ of $M$ such that $f(L) = K$.*

Proof: Suppose $\{\mathcal{P}_i\}$ is a point-star network of $X$. For each $i \in \mathbb{N}$, put $\mathcal{P}_i = \{P_\alpha\}_{\alpha\in\Lambda_i}$. Then (1) holds in view of [7]. (A similar proof can be seen by Proposition 2.9.5(3) in [10].)

Next, we shall show that (2) is true. Let $K$ be a non-empty compact subset of $X$ such that for each $i \in \mathbb{N}$, there exists some finite subset $\mathcal{P}_i'$ of $\mathcal{P}_i$ which forms a $cfp$-cover of $K$. So there is a finite subset $\Gamma_i$ of $\Lambda_i$ such that $\mathcal{P}_i' = \{P_\alpha\}_{\alpha\in\Gamma_i}$ can be precisely refined by some finite cover $\{K_\alpha\}_{\alpha\in\Gamma_i}$ of $K$, where $K_\alpha$ is a non-empty compact subset of $K$ for each $\alpha \in \Gamma_i$. Put $L = \{(\alpha_i) \in \prod_{i\in\mathbb{N}} \Gamma_i : \bigcap_{i\in\mathbb{N}} K_{\alpha_i} \neq \emptyset\}$. Then

(13.1) $L$ is a closed subset of the compact subset $\prod_{i\in\mathbb{N}} \Gamma_i$, so $L$ is compact. Suppose $\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Gamma_i - L$. Then $\bigcap_{i\in\mathbb{N}} K_{\alpha_i} = \emptyset$. By the compactness of $K$, there exists $i_0 \in \mathbb{N}$ such that $\bigcap_{i\le i_0} K_{\alpha_i} = \emptyset$. Let $W = \{(\beta_i) \in \prod_{i\in\mathbb{N}} \Gamma_i : \beta_i = \alpha_i$ for each $i \le i_0\}$. Then $W$ is an open neighborhood of $\alpha$ in $\prod_{i\in\mathbb{N}} \Gamma_i$ and $W \cap L = \emptyset$. Hence $L$ is a closed subset of $\prod_{i\in\mathbb{N}} \Gamma_i$.

(13.2) $L \subset M$ and $f(L) \subset K$. Suppose $\alpha = (\alpha_i) \in L$. Then $\bigcap_{i\in\mathbb{N}} K_{\alpha_i} \neq \emptyset$. Pick $x \in \bigcap_{i\in\mathbb{N}} K_{\alpha_i}$, then $x \in \bigcap_{i\in\mathbb{N}} P_{\alpha_i}$, since $\{P_{\alpha_i}\}_{i\in\mathbb{N}}$ is a network of $x$ in $X$, so $\alpha \in M$ and $f(\alpha) = x \in K$. Hence $L \subset M$ and $f(L) \subset K$.

(13.3) $K \subset f(L)$. For every $x \in K$ and $i \in \mathbb{N}$, pick $\alpha_i \in \Gamma_i$ such that $x \in K_{\alpha_i}$. If $\alpha = (\alpha_i)$, then $\alpha \in L$ and $f(\alpha) = x$ by the proof of (13.2). So $K \subset f(L)$.

In words, $L$ is a compact subset of $M$ such that $f(L) = K$. □

**Theorem 14.** *A regular space is $g$-metrizable if and only if it is a compact-covering, quotient, compact and $\sigma$-image of a metric space.*

Proof: Let $X$ be a $g$-metrizable space. By Lemma 12, $X$ has a $\sigma$-discrete weak base $\mathcal{P}$. Since $X$ is a regular space, we can assume that each member of $\mathcal{P}$ is a closed set of $X$. Put $\mathcal{P} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i = \bigcup_{x\in X} \mathcal{P}_x$, where $\mathcal{B}_i$ is a discrete collection of closed sets of $X$, and $\mathcal{P}_x$ is a weak neighborhood base of $x$ in $X$. For each $i \in \mathbb{N}$, let $Q_i = \{x \in X : \mathcal{P}_x \cap \mathcal{B}_i = \emptyset\}$, $\mathcal{U}_i = \mathcal{B}_i \cup \{Q_i\}$. Then $\mathcal{U}_i$ is a locally finite cover of $X$. We shall show that for each non-empty compact subset $K$ of $X$, there exists a finite subset of $\mathcal{U}_i$ which forms a $cfp$-cover of $K$. In fact, since $\mathcal{B}_i$ is a discrete collection, $K$ meets only finitely many members of $\mathcal{U}_i$. Let $\Gamma_i = \{\alpha : B_\alpha \in \mathcal{B}_i, B_\alpha \cap K \neq \emptyset\}$. For each $\alpha \in \Gamma_i$, put $K_\alpha = B_\alpha \cap K$, $K_i = \overline{K - \bigcup_{\alpha\in\Gamma_i} K_\alpha}$. All $K_\alpha$ and $K_i$ are closed subset of $K$, and $K = K_i \cup (\bigcup_{\alpha\in\Gamma_i} K_\alpha)$. We only need to show $K_i \subset Q_i$. $K$ is metrizable as a compact subset of a $g$-metrizable space. Pick $x \in K_i$. There exists a sequence $\{x_n\}$ of $K - \bigcup_{\alpha\in\Gamma_i} K_\alpha$ converging to $x$. If $P \in \mathcal{P}_x \cap \mathcal{B}_i$, then $P$ is a weak neighborhood of $x$, thus $x_n \in P$ whenever $n \geqslant m$ for some $m \in \mathbb{N}$. Hence $x_n \in K_\alpha$ for some $\alpha \in \Gamma_i$, a contradiction. So $\mathcal{P}_x \cap \mathcal{B}_i = \emptyset$, and $x \in Q_i$. This implies that $K_i \subset Q_i$ and $\{Q_i\} \cup \{B_\alpha\}_{\alpha\in\Gamma_i}$ is a $cfp$-cover of $K$.

For each $x \in X$ and an open neighborhood $U$ of $x$, pick $P \in \mathcal{P}_x$ satisfying $P \subset U$. Then $P \in \mathcal{B}_i$ for some $i \in \mathbb{N}$. Thus $\mathrm{st}(x,\mathcal{U}_i) = P \subset U$ and $\{\mathrm{st}(x,\mathcal{U}_i)\}_{i\in\mathbb{N}}$ is a network of $x$ in $X$. So $\{\mathcal{U}_i\}$ is a point-star network of $X$. Let $(f, M, X, \mathcal{U}_i)$ be a Ponomarev's system. Then $f : M \to X$ is a compact-covering and compact map by Lemma 13.

Since $g$-metrizable spaces are sequential spaces, and $f$ is a compact-covering map, it is easily checked that $f$ is a quotient map. Next, we shall show that $f : M \to X$ is a $\sigma$-map. For each $i \in \mathbb{N}$, let $\mathcal{U}_i = \{U_\alpha\}_{\alpha\in\Lambda_i}$. For every $(\alpha_i) \in M$ and $n \in \mathbb{N}$, put $B(\alpha_1, \alpha_2, \dots, \alpha_n) = \{(\gamma_i) \in M : \gamma_i = \alpha_i \text{ for each } i \leq n\}$. Then $f(B(\alpha_1, \alpha_2, \dots, \alpha_n)) = \bigcap_{i\leq n} U_{\alpha_i}$. In fact, if $\gamma = (\gamma_i) \in B(\alpha_1, \alpha_2, \dots, \alpha_n)$, then $f(\gamma) \in \bigcap_{i\in\mathbb{N}} U_{\gamma_i} \subset \bigcap_{i\leq n} U_{\alpha_i}$, so $f(B(\alpha_1, \alpha_2, \dots, \alpha_n)) \subset \bigcap_{i\leq n} U_{\alpha_i}$. Let $z \in \bigcap_{i\leq n} U_{\alpha_i}$. Since $\mathcal{U}_i$ is a cover of $X$ for each $i \in \mathbb{N}$, pick $\beta_i \in \Lambda_i$ such that $z \in U_{\beta_i}$ and the following holds: (1) $\beta_i = \alpha_i$ for every $i \leq n$; (2) $U_{\beta_i} \in \mathcal{B}_i$ whenever $z \in \bigcup \mathcal{B}_i$ ($U_{\beta_i}$ is unique by the discreteness of $\mathcal{B}_i$). Then $\beta = (\beta_i) \in \prod_{i\in\mathbb{N}} \Lambda_i$. If $z$ is an isolated point in $X$, then there exists $m \in \mathbb{N}$ and $P \in \mathcal{B}_m$ such that $\{z\} = P$, thus $z \notin Q_m$ by the construction of $Q_m$, so $U_{\beta_m} = P$ and $\{U_{\beta_i}\}_{i\in\mathbb{N}}$ is a network of $z$ in $X$. Suppose that $z$ is an accumulation point in $X$. Since $\mathcal{P}_z$ is a weak neighborhood base of $z$ in $X$, $\mathcal{P}_z$ is an infinite set. Let $U$ be a neighborhood of $z$

in $X$. There exists $P \in \mathcal{P}_z \cap \mathcal{B}_m$ such that $P \subset U$ for some $m > n$, thus $U_{\beta_m} = P$, and $\{U_{\beta_i}\}_{i\in\mathbb{N}}$ is a network of $z$ in $X$. So $\beta \in B(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $z = f(\beta)$, $\bigcap_{i\le n} U_{\alpha_i} \subset f(B(\alpha_1, \alpha_2, \dots, \alpha_n))$. We have shown that $f(B(\alpha_1, \alpha_2, \dots, \alpha_n)) = \bigcap_{i\le n} U_{\alpha_i}$. Since $\{B(\alpha_1, \alpha_2, \dots, \alpha_n) : (\alpha_i) \in M, n \in \mathbb{N}\}$ is a base of $M$ and $\bigwedge_{i\le n} \mathcal{U}_i$ is a locally finite collection in $X$ for every $n \in \mathbb{N}$, $f$ is a $\sigma$-map.

Conversely, let $M$ be a metric space and $f : M \to X$ be a compact-covering, quotient, compact and $\sigma$-map. Since $f$ is a $\sigma$-map, $f(\mathcal{B})$ is a $\sigma$-locally finite collection in $X$ for some base $\mathcal{B}$ of $M$. $\mathcal{B}$ is a $k$-network of $M$ and $k$-networks are preserved by compact-covering maps, so $f(\mathcal{B})$ is a $\sigma$-locally finite $k$-network. Also, $X$ is a $gf$-countable space as a quotient compact image of a metric space (see Theorem 2.9.14 in [10]). This implies that $X$ is a $g$-metrizable space by Lemma 12. □

In 1977, E. Michael defined $\sigma$-locally finite maps to characterize $\sigma$-spaces ([13]). The definition of $\sigma$-locally finite maps is similar to $\sigma$-map's. A map $f : X \to Y$ is called $\sigma$-locally finite if for every $\sigma$-locally finite cover $\mathcal{P}$ of $X$, there exists a refinement $\mathcal{B}$ of $\mathcal{P}$ such that $f(\mathcal{B})$ is a $\sigma$-locally finite collection. E. Michael proved the following results in [13]: (1) A regular space $X$ is a $\sigma$-space if and only if $X$ is a $\sigma$-locally finite image of a metric space; (2) $f$ is a $\sigma$-locally finite map if there exists a network $\mathcal{B}$ of $X$ such that $f(\mathcal{B})$ is a $\sigma$-locally finite collection of $Y$. So every $\sigma$-map is a $\sigma$-locally finite map, but the converse is not true. See the following example.

**Example 15.** There exist a metric space $M$ and a compact-covering, open, compact and $\sigma$-locally finite map $f : M \to X$ such that $X$ is not $g$-metrizable.

Let $X$ be the non-normal space from Example 2.5 in [2] which can be represented as a union of two open metric subspaces. Since $X$ is first-countable but not metrizable, $X$ is not a $g$-metrizable space ([15]). Suppose that $M$ is the topological sum of a cover of $X$ consisting of two open metric subspaces. Then $M$ is metrizable. Let $f : M \to X$ be a natural map. Then $f$ is a finite-to-one open map, thus $f$ is a compact-covering map (see Corollary 1.2 in [12]). Also $X$ is the union of countable many closed metric subspaces. So there exists a network $\mathcal{B}$ of $M$ such that $f(\mathcal{B})$ is a $\sigma$-locally finite collection. Hence $f$ is a $\sigma$-locally finite map. By Theorem 14, $f$ is not a $\sigma$-map.

A completely regular, non-normal space which is the open and finite-to-one image of a metric space under $f$ is given directly in Example 3.2 in [16]. It can be showed that $f$ is a compact-covering and $\sigma$-locally finite map. □

The idea of Theorem 14 is inspired by a question posed by Y. Tanaka. For a metric space $(X, d)$, $f : X \to Y$ is called a $\pi$-map, if $d(f^{-1}(y), X - f^{-1}(V)) > 0$ for each $y \in Y$ and a neighborhood $V$ of $y$ in $Y$. Obviously, every compact map on metric spaces is a $\pi$-map. The following result is proved in [8]: A regular space $X$ is $g$-metrizable if and only if $X$ is a compact-covering, quotient, $\pi$ and $\sigma$-image of a metric space. In 2001, Y. Tanaka asked the first author of this paper the following question: Is the result above true? In the proof of the result above, the following lemma was used (see Lemma in [8]): Let $X$ be a metric space. If $f : X \to Y$ is a quotient map, then $Y$ is a symmetric space if and only if $f$ is a $\pi$-map. Next, we shall show that the lemma above is not true.

**Example 16.** There exist a metric space $M$ and a quotient map $f : M \to X$ such that $X$ is a symmetric space, but $f$ is not a $\pi$-map.

By Example 2.9.8 and Theorem 2.9.7 in [10], we can find out a symmetric space $X$ such that $X$ is not any quotient $\pi$-image of a metric space $M$.

Let $M$ be the topological sum of all convergent sequences in $X$, and $f : M \to X$ the natural map. Then $M$ is a metric space and $f$ is a quotient map, so $f$ is not a $\pi$-map. □

This wrong lemma is a modification of Proposition 1.3 in [18]. (Proposition 1.3: If $X$ is a symmetric space and $f : X \to Y$ a quotient map, then $Y$ is the symmetric space iff $f$ is a $\pi$-map for some equivalent symmetric on $X$.) By Example 16, this modification to replace symmetrics by metrics is not true. On the other hand, since the quotient $\pi$-images of metric spaces are $gf$-countable spaces (see Lemma 2.9.4 in [10]), in view of Theorem 14, a regular space $X$ is $g$-metrizable iff $X$ is the compact-covering, quotient, $\pi$ and $\sigma$-image of a metric space.

The first author of this paper proved the following mapping theorem on $g$-metrizable spaces in [9] by using the wrong lemma above: For a regular space $X$, $X$ is $g$-metrizable iff $X$ is the compact-covering, quotient, $\pi$ and *mssc*-image of a metric space. Recall the concept of *mssc*-maps ([9]). Let $f : X \to Y$, where $X$ is a subspace of a product space $\prod_{i\in\mathbb{N}} X_i$. $f$ is called a stratified strong compact map or *ssc*-map, if for each $y \in Y$ there exists a sequence $\{V_i\}$ of open neighborhoods of $y$ in $Y$ satisfying that $\overline{p_i(f^{-1}(V_i))}$ is a compact subspace of $X_i$ for each $i \in \mathbb{N}$, where $p_i : \prod_{i\in\mathbb{N}} X_i \to X_i$ is the projective map. $f$ is called a metrizable stratified strong compact map or *mssc*-map if $f$ is an *ssc*-map and $X_i$ is a metric space for each $i \in \mathbb{N}$.

**Lemma 17.** *mssc-maps are $\sigma$-maps.*

Proof: Let $f : X \to Y$ be an *mssc*-map. Then there exists a sequence $\{X_i\}$ of metric spaces satisfying the conditions of *mssc*-maps. For each $i \in \mathbb{N}$, $X_i$ has a $\sigma$-locally finite base $\mathcal{P}_i = \bigcup_{j\in\mathbb{N}} \mathcal{P}_{ij}$, where $\mathcal{P}_{ij}$ is a locally finite collection of $X_i$. Put $\mathcal{B}_{ij} = \{X \cap (\bigcap_{k\le i} p_k^{-1}(P_{kj})) : P_{kj} \in \mathcal{P}_{kj}, k \le i\}$, $\mathcal{B} = \bigcup_{i,j\in\mathbb{N}} \mathcal{B}_{ij}$. Then $\mathcal{B}$ is a base of $X$. For each $y \in Y$, there exists a sequence $\{V_i\}$ of open neighborhoods of $y$ in $Y$ such that $\overline{p_i(f^{-1}(V_i))}$ is a compact subspace of $X_i$ for each $i \in \mathbb{N}$. For each $i, j \in \mathbb{N}$, $p_i(f^{-1}(V_i))$ meets only elements of some finite subset $\mathcal{F}_{ij}$ of $\mathcal{P}_{ij}$. Since $p_i(f^{-1}(V_i)) \cap P \ne \emptyset$ if and only if $f^{-1}(V_i) \cap p_i^{-1}(P) \ne \emptyset$, $F_{ij} \in \mathcal{F}_{ij}$ for each $i \le n$ if $(\bigcap_{i\le n} f^{-1}(V_i)) \cap (\bigcap_{i\le n} p_i^{-1}(F_{ij})) \ne \emptyset$. For each $n \in \mathbb{N}$, let $V = \bigcap_{i\le n} V_i$. Then $\{Q \in f(\mathcal{B}_{nj}) : V \cap Q \ne \emptyset\}$ is a finite set, thus $f(\mathcal{B}_{nj})$ is a locally finite collection of $Y$, so $f(\mathcal{B})$ is a $\sigma$-locally finite collection of $Y$. Hence $f$ is a $\sigma$-map. $\square$

Generally, $\sigma$-maps need not be $mssc$-maps. For example, let $Y$ be a non-locally compact, separable metric space and put $X_i = Y$ for each $i \in \mathbb{N}$. Then $X_1$ is a subspace of $\prod_{i\in\mathbb{N}} X_i$. Let $f : X_1 \to Y$ be the identical map. Since $X_1$ is a separable metric space, $f$ is a $\sigma$-map. If $f$ is an $mssc$-map, then for each $y \in Y$, there exists a sequence $\{V_i\}$ of open neighborhoods of $y$ in $Y$ such that $\overline{p_i(f^{-1}(V_i))}$ is a compact subspace of $X_i$ for each $i \in \mathbb{N}$, thus $\overline{p_1(f^{-1}(V_1))} = \overline{f^{-1}(V_1)}$ is a compact subset of $X_1$, so $X_1$ is a locally compact space, a contradiction. Hence $f$ is not an $mssc$-map.

Next, we shall show that the $\sigma$-map $f$ in Theorem 14 is an $mssc$-map. For each $x \in X$ and $i \in \mathbb{N}$, since $\mathcal{U}_i$ is a locally finite cover of $X$, there exists an open neighborhood $V_i$ of $x$ in $X$ such that $V_i$ only meets with finitely many elements in $\mathcal{U}_i$. Let $\triangle_i = \{\alpha \in \Lambda_i : U_\alpha \cap V_i \neq \emptyset\}$. Then $\triangle_i$ is a finite subset and $\overline{p_i(f^{-1}(V_i))}(\subset \triangle_i)$ is a compact subset of $\Lambda_i$, so $f$ is an $mssc$-map. The following corollary holds by the above-mentioned discussions.

**Corollary 18.** *The following are equivalent for a regular space $X$:*

(1) *$X$ is a $g$-metrizable space;*
(2) *$X$ is a compact-covering, quotient, compact and $mssc$-image of a metric space;*
(3) *$X$ is a compact-covering, quotient, $\pi$ and $mssc$-image of a metric space;*
(4) *$X$ is a compact-covering, quotient, $\pi$ and $\sigma$-image of a metric space.* $\square$

**Question 19.** Let $\{X_i\}$ be a sequence of locally compact metric spaces and let $X$ be a subspace of $\prod_{i\in\mathbb{N}} X_i$. If $f : X \to Y$ is a $\sigma$-map, is $f$ an $mssc$-map?

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(*Received* May 21, 2002, *revised* January 28, 2003)