Biharmonic Green domains in a Riemannian manifold

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Abstract. Let R be a Riemannian manifold without a biharmonic Green function defined on it and Ω a domain in R. A necessary and sufficient condition is given for the existence of a biharmonic Green function on Ω.

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1. Introduction

In a Riemannian manifold R, we say that a domain Ω is a biharmonic Green domain if there exists a positive solution $Q_y(x)$ for the equation $\Delta^2 Q_y(x) = \delta_y(x)$ in Ω, where y is some point in Ω and Δ is the Laplace-Beltrami operator in R. Some necessary and sufficient conditions for R to be a biharmonic Green space are given in Sario et al. [8, Chapter VIII]. In this note we give a necessary and sufficient condition for a domain Ω in R to be a biharmonic Green domain when R itself is not a biharmonic Green space.

2. Preliminaries

Let R be an oriented Riemannian manifold of dimension $n \geq 2$ with local parameters $x = (x^1, \dots x^n)$ and a C^∞ metric tensor g_{ij} such that $g_{ij}x^i x^j$ is positive definite. If D is the determinant of g_{ij}, denote the volume element by $dx = D^{\frac{1}{2}} dx^1 \dots dx^n$; $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator acting on R in the sense of distributions; in the Euclidean case, Δ reduces to the form $\Delta u = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$. A continuous function h on an open set is harmonic, by definition, if $\Delta h = 0$. To every open set w in R, let $H(w)$ denote the class of harmonic functions on w. Then these harmonic functions have the *sheaf property*, solve locally the *Dirichlet problem* and possess the *Harnack property*; that is, they satisfy the axioms 1, 2, 3 of Brelot in the axiomatic potential theory ([5, pp. 13–14]). Consequently, we can use all the notions and the results of this axiomatic theory in the context of a Riemannian manifold; some of these are the following:

(1) Let w be a regular open set in R, that is w is relatively compact in R and each boundary point of w is regular for the Dirichlet problem. A compact set k in w is said to be *outerregular* if $w \setminus k$ is a regular open set. Given

a compact set k and a domain w, one can construct a regular domain w_0 and an outerregular compact set k_0 such that $k \subset \overset{\circ}{k_0} \subset k_0 \subset w_0 \subset w$ (see Loeb [6]).

(2) (See [5, pp. 37, 38 and 47]). If $s > 0$ is a superharmonic function on a domain $\Omega \subset R$ and if e is a subset of Ω, the *reduced function* by definition is

$$R_s^e(x) = \inf\{t(x) : t \geq 0 \ \text{ superharmonic on } \ \Omega \ \text{ and } \ t \geq s \ \text{ on } \ e\};$$

and its l.s.c. regularization is the *balayage* $\widehat{R}_s^e(x) = \liminf_{y \to x} R_s^e(y)$. In a domain Ω with a positive potential, a set e is *polar* if and only if $R_1^e(x) = 0$ at some point x, or, equivalently $\widehat{R}_1^e \equiv 0$.

(3) If there is a positive potential on Ω, we define on Ω the Green function $G(x, y) = G_y(x)$ with pole $y \in \Omega$, so that $\Delta G_y = \delta_y$. Then for any potential p on Ω, $\Delta p = \mu$ is a Radon measure and $p(x) = \int_\Omega G(x, y)\, d\mu(y)$. Also it is proved in [7] that given a Radon measure $\mu \geq 0$ on Ω, $\int_\Omega G(x, y)\, d\mu(y)$ is a potential if and only if

$$\int_\Omega \widehat{R}_1^w(x)\, d\mu(x) < \infty \ \text{ for some nonempty open set } w \text{ in } \Omega.$$

(4) More generally, we have the following result in [3]: Let Ω be a domain in R with or without positive potentials. Let $\mu \geq 0$ be a Radon measure on Ω. Then there exists a superharmonic function s on Ω such that $\Delta s = \mu$. This result is in fact a simple generalization of a classical result of Brelot [4] in $\mathbb{R}^n$.

Lemma 2.1. *Let Ω be a domain in R such that Ω has the Green function $G(x, y)$ defined on it. Then for a Radon measure $\mu \geq 0$ on Ω, $\int_\Omega G(x, y)\, d\mu(y)$ is a potential if and only if for one (and hence any) nonpolar compact set k in Ω, $\int_\Omega R_1^k\, d\mu < \infty$.*

Proof: This is a more useful reformulation of Theorem 3.1 [7]. First note that R_1^k is μ-measurable. For $R_1^k = \inf_n R_1^{w_n}$ where w_n is a decreasing sequence of relatively compact open sets such that $k = \bigcap w_n$. Since each $R_1^{w_n} = \widehat{R}_1^{w_n}$ is l.s.c., it is μ-measurable and hence R_1^k is μ-measurable.

(1) Suppose $\int_\Omega G(x, y)\, d\mu(y)$ is a potential on Ω and k is a nonpolar compact set in Ω. Then for some $x_0 \in k$, $\int_\Omega G(x_0, y)\, d\mu(y) < \infty$. If $G(x_0, y) \geq a$ on k, $G(x_0, y) \geq a R_1^k$ on Ω and hence $\int_\Omega R_1^k\, d\mu < \infty$.

(2) Conversely, suppose $\int_\Omega R_1^k\, d\mu < \infty$ for some nonpolar compact set k. Since $R_1^k = \inf_n R_1^{w_n}$, we can find an open set w and an outerregular compact set A such that $k \subset \overset{\circ}{A} \subset A \subset w$ and $\int_\Omega R_1^w\, d\mu < \infty$. Now $p(x) =$

$\int_A G(x,y)\,d\mu(y)$ is a potential on Ω; hence $p(x_0) < \infty$ for some $x_0 \in k$. If $a \le G(x_0,y) \le b$ on ∂A, then $aR_1^k(y) \le G(x_0,y) \le bR_1^A(y)$ on $\Omega \setminus A$ and hence $\int_{\Omega\setminus A} G(x_0,y)\,d\mu(y)$ is finite, which implies that $\int_\Omega G(x_0,y)\,d\mu(y)$ is finite and hence $\int_\Omega G(x,y)\,d\mu(y)$ is a potential on Ω.

□

The following form of Lemma 2.1, without an explicit reference to the reduced functions, is convenient for applications.

Lemma 2.2. *Let Ω be a domain in R with the Green function $G(x,y)$ defined on it; $\mu \ge 0$ is a Radon measure on Ω. Then the following are equivalent:*

1. *There exists a superharmonic function $s > 0$ on Ω such that $\int_\Omega s\,d\mu < \infty$.*
2. *$p(x) = \int_\Omega G(x,y)\,d\mu(y)$ is a potential on Ω.*
3. *For any locally bounded potential $q(x)$ with compact harmonic support on Ω, $\int_\Omega q\,d\mu < \infty$.*

Proof: (1)$\Rightarrow$(2): Let k be a nonpolar compact subset of Ω. If $s \ge \alpha > 0$ on k, then $\alpha R_1^k \le s$ on Ω and hence $\int_\Omega R_1^k\,d\mu \le \frac{1}{\alpha}\int_\Omega s\,d\mu < \infty$. Hence by Lemma 2.1, $p(x) = \int_\Omega G(x,y)\,d\mu(y)$ is a potential on Ω.

(2)$\Rightarrow$(3): Let q be a locally bounded potential on Ω, with compact harmonic support A. Let k be an outerregular compact set such that $A \subset \overset{\circ}{k}$. Then $R_q^k = q$ on $\Omega \setminus k$. For, $\widehat{R}_q^k \le q$ on Ω and hence $t = q - \widehat{R}_q^k$ on $\Omega \setminus k$ extended by 0 on k is a positive subharmonic function less or equal to q on Ω; hence $t \le 0$, so that $q = \widehat{R}_q^k = R_q^k$ on $\Omega \setminus k$. Consequently, if $q \le \alpha$ on k, then $q \le \alpha R_1^k$ on k; also on $\Omega \setminus k$, $q = R_q^k \le R_\alpha^k = \alpha R_1^k$. Thus $q \le \alpha R_1^k$ on Ω. Now assumption (2) along with Lemma 2.1 shows that $\int_\Omega R_1^k\,d\mu < \infty$. Hence $\int_\Omega q\,d\mu < \infty$.

(3)$\Rightarrow$(1): Let k be a nonpolar compact set. Let $s = \widehat{R}_1^k$ on Ω. Then $s > 0$ is a superharmonic function that is bounded on Ω and has compact harmonic support. Hence by (3), $\int_\Omega s\,d\mu < \infty$. □

3. Biharmonic Green domains

Let Ω be a domain in R. Given $y \in \Omega$, let w be a regular domain for the Dirichlet problem such that $y \in w \subset \overline{w} \subset \Omega$. Let $v_w(x,y)$ be the biharmonic Green function on w with biharmonic singularity y, that is $\Delta^2 v_w(x,y) = \delta_y(x)$, and with boundary conditions $v_w/\partial w = 0$ and $\Delta v_w/\partial w = 0$. Then v_w increases with w. Write $v_\Omega(x,y) = \lim_{w\to\Omega} v_w(x,y)$ if the limit exists for some regular exhaustion $\{w\}$. $v_\Omega(x,y)$ is called the *biharmonic Green function* on Ω and its existence is independent of the regular exhaustion $\{w\}$ and the choice of the singular point y (see Sairo et al. [8, pp. 300–307]). When $v_\Omega(x,y)$ exists on Ω, it can be written as $v_\Omega(x,y) = \int_\Omega G(x,z)G(z,y)\,dz$.

Definition 3.1. A domain Ω in R is said to be a *biharmonic Green domain* if and only if the biharmonic Green function $v_\Omega(x,y)$ exists on Ω.

The following theorem is a collection of known results about $v_\Omega(x,y)$.

Theorem 3.2. *Let Ω be a domain in R, carrying the harmonic Green function $G(x,y)$. Then the following are equivalent:*

(1) *Ω is a biharmonic Green domain.*
(2) *For one (and hence any) $y \in \Omega$, there exists a potential $q_y(x)$ on Ω such that $\Delta^2 q_y = \delta_y$.*
(3) *There exists a potential $Q(x) > 0$ on Ω such that $\Delta Q(x)$ is a superharmonic function.*
(4) *There exist potentials p and q on Ω such that $\Delta q = p$. (q is called a bipotential.)*

Proof: (1) ⇒(2): Let $v(x,y)$ be the biharmonic Green function on Ω. Since $v(x,y) = \int_\Omega G(x,z)G(z,y)\,dz$, for fixed y, $v_y(x) = v(x,y)$ is a potential on Ω and $\Delta v_y(x) = G_y(x)$ (see [8, p. 300]); hence $\Delta^2 v_y = \delta_y$.

(2) ⇒(3): For some potential q on Ω, let $\Delta^2 q = \delta_y$. Since $\Delta^2 q = \Delta G_y$, $\Delta q(x) = G_y(x)+$ (a harmonic function) on Ω. That is, $\Delta q = s$ is a superharmonic function on Ω; note that $s > 0$ since q is a potential > 0.

(3) ⇒(4): See Theorem 3.2 in [1].

(4) ⇒(1): This is a consequence of Theorem 4.2 in [1]. □

Theorem 3.3. *A domain Ω in R is a biharmonic Green domain if and only if there exists a superharmonic function $s > 0$ on Ω such that $\int_\Omega s^2\,dx < \infty$.*

Proof: (1) Let Ω be a biharmonic Green domain. Then there exist potentials $p > 0$ and $q > 0$ on Ω such that $\Delta q = p$. This means that if $G(x,y)$ is the Green function on Ω with $\Delta G_y = \delta_y$, $q(x) = \int_\Omega G(x,y)p(y)\,dy$ since q is a potential with the associated measure $d\mu(x) = (\Delta q)dx = pdx$ in the Riesz representation. This implies (by Lemma 2.1) that for any nonpolar compact set k in Ω, $\int_\Omega R_1^k(y)p(y)\,dy < \infty$. Moreover, since p is a potential on Ω, for some $\lambda > 0$, $R_1^k \le \lambda p$ on Ω. Consequently, with $s = \widehat{R}_1^k$ we have $\int_\Omega s^2\,dx < \infty$.

(2) Conversely, let $s > 0$ be superharmonic on Ω such that $\int_\Omega s^2\,dx < \infty$. Since for a nonpolar compact k in Ω, $R_1^k \le \lambda s$ for some $\lambda > 0$, $\int_\Omega R_1^k(y)\widehat{R}_1^k(y)\,dy < \infty$. This implies (Lemma 2.1) that $q(x) = \int_\Omega G(x,y)\widehat{R}_1^k(y)\,dy$ is a potential on Ω so that $\Delta q = \widehat{R}_1^k$. Since $\widehat{R}_1^k$ is a potential on Ω, we conclude that Ω is a biharmonic Green domain. □

Corollary 1. *Any domain in $\mathbb{R}^n$, $n \ge 5$, is a biharmonic Green domain; and $\mathbb{R}^n$ for $2 \le n \le 4$ is not a biharmonic Green space.* (Sario et al. [8, pp. 300–302] and [2, Theorem 5.5]).

PROOF: (a) Let Ω be a domain $\mathbb{R}^n$, $n \geq 5$. Note that $s(x) = |x|^{2-n}$ is a positive superharmonic function such that $\int_\Omega s^2\,dx \leq \infty$. Hence Ω is a biharmonic Green domain.

(b) Suppose $\mathbb{R}^n$, $2 \leq n \leq 4$, is a biharmonic Green space. Then there exists a superharmonic function $s > 0$ in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} s^2\,dx < \infty$. If B is the closed unit ball in $\mathbb{R}^n$, then for some $\lambda > 0$, $R_1^B \leq \lambda s$ and hence $\int_{\mathbb{R}^n}(R_1^B)^2\,dx < \infty$. But $R_1^B = |x|^{2-n}$ on $\mathbb{R}^n \setminus B$. Hence we should have $\int_1^\infty \int_{\partial B} r^{4-2n} r^{n-1}\,dr\,dw$ is finite, that is, $\int_1^\infty r^{3-n}\,dr$ is finite, a contradiction when $2 \leq n \leq 4$. $\square$

Corollary 2. *Suppose the Riemannian manifold R is not a biharmonic Green space. If Ω is a biharmonic Green domain in R, then $e = R \setminus \Omega$ is not compact.*

PROOF: Suppose e is compact. Let k be an outerregular compact set such that $e \subset \overset{\circ}{k} \subset k$. Since Ω is a biharmonic Green domain there exists $s > 0$ superharmonic on Ω such that $\int_\Omega s^2\,dx < \infty$. Suppose $\inf_{\partial k} s(x) = \lambda$. Then $\lambda \widehat{R}_1^k \leq s$ on $\Omega \backslash k = R \backslash k$ and hence $\int_{\Omega\backslash k}(\widehat{R}_1^k)^2\,dx < \infty$; also $\int_k(\widehat{R}_1^k)^2\,dx < \infty$, and hence $\int_R(\widehat{R}_1^k)^2\,dx < \infty$. This means that R is a biharmonic Green space, contradicting the hypothesis. $\square$

Corollary 3 ([2, Theorem 5.4])**.** *If Ω is a biharmonic Green domain in $\mathbb{R}^n$, $2 \leq n \leq 4$, then $e = \mathbb{R}^n \setminus \Omega$ is neither locally polar nor compact.*

PROOF: (a) Since $\mathbb{R}^n$, $2 \leq n \leq 4$, is not a biharmonic Green space, by the above corollary, e is not compact.

(b) Suppose e is locally polar. Since Ω is a biharmonic Green domain, there exists a superharmonic function $s > 0$ such that $\int_\Omega s^2\,dx < \infty$. Now $e = \mathbb{R}^n \setminus \Omega$ being locally polar by the assumption, $\int_e dx = 0$ and s extends as a superharmonic function $u > 0$ on $\mathbb{R}^n$. Hence $\int_{\mathbb{R}^n} u^2\,dx < \infty$ which means that $\mathbb{R}^n$, $2 \leq n \leq 4$, is a biharmonic Green space, a contradiction. $\square$

4. Biharmonic potentials and quasiharmonic potentials

If there exists a nonconstant positive harmonic function on Ω, then we can define the harmonic Green function $G(x, y)$ on Ω. However, we know that this sufficient condition for the existence of the harmonic Green function is not a necessary condition, as for example in $\mathbb{R}^n$, $n \geq 3$. A corresponding result for the biharmonic Green function is the following:

Proposition 4.1. *Suppose that there exists a biharmonic function which is a positive potential on Ω. Then Ω is a biharmonic Green domain.*

PROOF: Let b be a biharmonic function which is a positive potential on Ω. Since b is a potential such that Δb is harmonic, by Theorem 3.2(3), Ω is a biharmonic Green domain. $\square$

In view of the above proposition, we propose the following terminology.

Definition 4.2. In a domain Ω in R, let $u > 0$ be a potential.

(1) u is said to be a *biharmonic potential* if and only if $\Delta^2 u = 0$ on Ω.
(2) u is said to be a *quasiharmonic potential* if and only if $\Delta u = 1$ on Ω.

Remark. Let Ω be a harmonic Green domain in R. Then there exists a quasiharmonic potential on Ω if and only if $p(x) = \int_\Omega G(x,y)\,dy$ is a potential on Ω. For, suppose $p(x)$ is a potential. Then $\Delta p = 1$ so that $p(x)$ is a quasiharmonic potential on Ω. Conversely, suppose q is a quasiharmonic potential on Ω. Since q is a potential and $\Delta q = 1$, $q(x) = \int_\Omega G(x,y)\Delta q(y)\,dy = \int_\Omega G(x,y)\,dy$.

Theorem 4.3. *Let Ω be a harmonic Green domain in R. Then there exists a biharmonic (resp. quasiharmonic) potential on Ω if and only if there are a superharmonic function $s > 0$ and a harmonic function $h > 0$ on Ω such that $\int_\Omega s(x)h(x)\,dx < \infty$ (resp. $\int_\Omega s(x)\,dx < \infty$).*

Proof: Let $G(x,y)$ be the Green function on Ω. By Lemma 2.2, $\int_\Omega s(x)h(x)\,dx$ (resp. $\int_\Omega s(x)\,dx$) is finite if and only if $Q(x) = \int_\Omega G(x,y)h(y)\,dy$ (resp. $Q(x) = \int_\Omega G(x,y)\,dy$) is a potential on Ω which is equivalent to saying that Q is a biharmonic (resp. quasiharmonic) potential on Ω, since $\Delta Q = h$ (resp. $\Delta Q = 1$). □

Corollary 1. *Let Ω be a domain in R. If there exists a quasiharmonic potential on Ω, then for any potential p on Ω with compact harmonic support, $\int_\Omega p\,dx < \infty$. Consequently, there exists a unique bipotential q on Ω such that $\Delta q = p$.*

Proof: Since Ω has a quasiharmonic potential, there exists a superharmonic function $s > 0$ such that $\int_\Omega s\,dx < \infty$. Let p be a potential with compact harmonic support k. Let A be an outerregular compact set such that $k \subset \overset{\circ}{A} \subset A$. Then $p = B_A p$ on $\Omega \setminus A$ where $B_A p$ denotes the Dirichlet solution with boundary values p on ∂A and 0 at infinity. Hence $p \leq \lambda s$ on $\Omega \setminus A$ for some $\lambda > 0$ so that $\int_{\Omega\setminus A} p\,dx < \infty$. Since p is locally integrable on Ω, $\int_A p\,dx < \infty$. Hence $\int_\Omega p\,dx < \infty$. Consequently, for a nonpolar compact k, $\int_\Omega R_1^k(x)p(x)\,dx < \infty$. Hence $q(x) = \int_\Omega G(x,y)p(y)\,dy$ is a potential on Ω such that $\Delta q = p$ (Lemma 2.1). If q_1 is another bipotential on Ω such that $\Delta q_1 = p$, then $q_1 = q+$ (a harmonic function h) on Ω. Note $h \equiv 0$ by the uniqueness of the Riesz representation. □

Corollary 2. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, such that $\mathbb{R}^n \setminus \Omega$ is compact. If $u \geq 0$ is superharmonic on Ω and if Δu is constant, then u is harmonic and hence is of the form*

$$u(x) = \begin{cases} \alpha \log|x-a| + b(x) & \text{if } n = 2,\ a \notin \Omega \\ \alpha + b(x) & \text{if } n \geq 3, \end{cases}$$

where $\alpha \geq 0$ and $b(x)$ is harmonic on Ω such that $|b(x)| \leq \beta|x|^{2-n}$ near infinity.

Proof: First we note that there is no quasiharmonic potential on Ω. For, suppose Ω has a quasiharmonic potential. Then there exists a superharmonic function

$s > 0$ on Ω such that $\int_\Omega s\,dx < \infty$. Suppose $\mathbb{R}^n \setminus \Omega = e \subset \{x : |x| < a\}$. Let $\lambda = \inf_{|x|=a} s(x)$. Then $s(x) \geq \lambda \left|\frac{x}{a}\right|^{2-n}$ on $|x| > a$ by the minimum principle and hence $\int_a^\infty \int_{\partial B} \lambda \left(\frac{r}{a}\right)^{2-n} r^{n-1}\,dr\,dw \leq \int_\Omega s(x)\,dx < \infty$. This implies that $\int_a^\infty r\,dr < \infty$, a contradiction.

Now write $u = p + h$ on Ω where p is a potential and h is harmonic on Ω. Since $\Delta p = \Delta u$ is constant and since there is no quasiharmonic potential on Ω, $p \equiv 0$. Hence u is harmonic ≥ 0 outside a compact set. Then, applying an inversion in the unit ball to the classical representation of Bocher's, we get the stated expression for u. □

Remarks. (1) The above corollary implies that if a positive superharmonic function u on $\mathbb{R}^n$, $n \geq 3$, is biharmonic, then u is constant. Apparently, it generalizes the result that every positive harmonic function on $\mathbb{R}^n$ is constant.

(2) $\Omega = \{x : |x| \geq 1\}$ in $\mathbb{R}^n$, $n \geq 5$, is an example of a domain in which there exists a biharmonic potential but no quasiharmonic potential. For, if $s(x) = h(x) = |x|^{2-n}$, then $\int_\Omega sh\,dx < \infty$ and hence by Theorem 4.3, there exists a biharmonic potential on Ω. But there is no quasiharmonic potential on Ω. For, suppose $Q(x)$ is a potential > 0 on Ω such that $\Delta Q = 1$; then by the above Corollary 2, $Q(x)$ should be harmonic, a contradiction.

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