Strongly base-paracompact spaces

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Abstract. A space X is said to be *strongly base-paracompact* if there is a basis $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X has a star-finite open refinement by members of $\mathcal{B}$. Strongly paracompact spaces which are strongly base-paracompact are studied. Strongly base-paracompact spaces are shown have a family of functions $\mathcal{F}$ with cardinality equal to the weight such that every open cover has a locally finite partition of unity subordinated to it from $\mathcal{F}$.

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1. Introduction

The *weight*, $w(X)$, of a topological space X is the minimal cardinality of a basis for X. A space X is *base-paracompact* if there is a basis $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite refinement by members of $\mathcal{B}$. The author studied these spaces in [Po] as a weakening of totally paracompact spaces. Is strong paracompactness witnessed by a base in the same manner as base-paracompact spaces?

A family $\mathcal{A}$ of subsets of a topological space X is said to be *star-finite* if for every $A \in \mathcal{A}$ the set $\{B \in \mathcal{A} : B \cap A \neq \emptyset\}$ is finite. A topological space X is *strongly paracompact* if every open cover of X has a star-finite open refinement. A topological space X is *strongly base-paracompact* if there is a basis $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X has a star-finite open refinement by members of $\mathcal{B}$. In Section 2, topological spaces that are strongly base-paracompact are studied.

It is well known that a space X is paracompact if and only if every open cover has a locally finite partition of unity subordinated to it. A natural question arises: does there exist a family of functions $\mathcal{F}$ with cardinality equal to the weight of X such that every open cover has a locally finite partition of unity subordinated to it from $\mathcal{F}$? In Section 3, it is shown that strongly base-paracompact spaces have this property. The Hedgehog space of spininess $\kappa > \omega$ is shown to have a family of functions with cardinality equal to the weight such that every open cover has a locally finite partition of unity subordinated to it from this family but is not strongly paracompact.

All topological spaces are assumed to be Hausdorff, and the definitions of all terms not defined can be found in [E]. This paper constitutes a part of the Doctoral Thesis, completed under Professor Gary Gruenhage at Auburn University. The author would like to give thanks to Gary Gruenhage for his patience and useful suggestions.

2. Strongly base-paracompact spaces

It follows immediately that every strongly base-paracompact space is strongly paracompact. The converse is not known. A partial answer is given in the next theorem.

Lemma 2.1 ([Po]). *Let $\mathcal{B}$ be a basis for a topological space X with $|\mathcal{B}| = w(X)$. Then there is a basis $\mathcal{B}'$ for X with $|\mathcal{B}'| = w(X)$ and $\mathcal{B} \subset \mathcal{B}'$ which is closed under finite unions, finite intersections, and complements of closures.*

Theorem 2.2. *Every regular Lindelöf space is strongly base-paracompact.*

PROOF: Let $\mathcal{B}_0$ be a basis of X with cardinality equal to the weight which is closed under finite unions, finite intersections and complement of closures. For every $U, V \in \mathcal{B}_0$ with $\overline{V} \subset U$, there is an open set W_{UV} such that $\overline{V} \subset W_{UV} \subset \overline{W_{UV}} \subset U$. Let $\mathcal{B}_0' = \mathcal{B}_0 \cup \{W_{UV} : U, V \in \mathcal{B}_0 \text{ and } \overline{V} \subset U\}$. Let $\mathcal{B}_1$ be a basis with $|\mathcal{B}_1| = w(X)$ containing $\mathcal{B}_0'$ which is closed under finite unions, finite intersections, and complements of closures. Proceed by induction. Suppose $\mathcal{B}_n$ has been defined. For every $U, V \in \mathcal{B}_n$ with $\overline{V} \subset U$, there is an open set W_{UV} such that $\overline{V} \subset W_{UV} \subset \overline{W_{UV}} \subset U$. Let $\mathcal{B}_n' = \mathcal{B}_n \cup \{W_{UV} : U, V \in \mathcal{B}_n \text{ and } \overline{V} \subset U\}$. Let $\mathcal{B}_{n+1}$ be a basis with $|\mathcal{B}_{n+1}| = w(X)$ containing $\mathcal{B}_n'$ which is closed under finite unions, finite intersections, and complements of closures. Let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$.

<u>Claim</u>: $\mathcal{B}$ is the desired basis.

Clearly $|\mathcal{B}| = w(X)$. Let $\mathcal{U}$ be an open cover of X. For every $x \in X$, there exists $V_x, U_x \in \mathcal{B}$ such that $x \in V_x \subset \overline{V_x} \subset U_x \subset U$ for some $U \in \mathcal{U}$. Since X is Lindelöf, the V_x's have a countable subcover, say $V_0, V_1, \cdots, V_m, \cdots$. Then $\overline{V_m} \subset U_m$ for every $m < \omega$. Let $V_0^* = V_0$. Note that $\overline{V_0 \cup V_1} \subset U_0 \cup U_1$ and $V_0 \cup V_1, U_0 \cup U_1 \in B_n$ for some n. Then there is $V_1^* \in \mathcal{B}$ such that $\overline{V_0 \cup V_1} \subset V_1^* \subset \overline{V_1^*} \subset U_0 \cup U_1$. Proceed by induction. Suppose V_{k-1}^* has been defined. Note that $\overline{V_0 \cup \cdots \cup V_k} \subset U_0 \cup \cdots \cup U_k$ and $V_0 \cup \cdots \cup V_k, U_0 \cup \cdots \cup U_k \in B_n$ for some n. Then there is $V_k^* \in \mathcal{B}$ such that $\overline{V_0 \cup \cdots \cup V_k} \subset V_k^* \subset \overline{V_k^*} \subset U_0 \cup \cdots \cup U_k$. If we take $V_{-1}^* = \emptyset$, we will show $B_{k,j} = U_j \cap (V_{k+1}^* \backslash \overline{V_{k-1}^*})$ where $0 \leq j \leq k$ and $k < \omega$ form a star-finite refinement. Note that $\overline{V_k^*} \subset \bigcup_{j \leq k} U_k$. Let $x \in X$, and let k be the least integer such that $x \in \overline{V_k^*}$. Since $\overline{V_k^*} \subset \bigcup_{j \leq k} U_j$, there exists a $j \leq k$ such that $x \in U_j$. Then $x \in U_j \cap (\overline{V_k^*} \backslash \overline{V_{k-1}^*}) \subset B_{k,j}$, and the $B_{k,j}$'s where $0 \leq j \leq k$ and $k < \omega$ cover X. For every $j \leq k$, $B_{k,j} \subset V_{k+1}^* \subset \overline{V_{k+1}^*}$. Note that, for $m \geq k+2$ and $i \leq m$, $B_{k,j} \cap B_{m,i} = \emptyset$ which shows that the $B_{k,j}$'s where

$0 \le j \le k$ and $k < \omega$ are star-finite. Hence $\{B_{k,j}\}$ where $0 \le j \le k$ and $k < \omega$ is a star-finite refinement of $\mathcal{U}$ by members of $\mathcal{B}$. □

A space is called *non-archimedean* if it has a base which is a tree under reverse inclusion (see [Ny]). That is, the collection of all base members containing a given one is well-ordered by reverse inclusion.

Theorem 2.3. *Non-archimedean spaces are strongly base-paracompact.*

Proof: Let $\mathcal{B}$ be a base for X which is a tree under reverse inclusion and $|\mathcal{B}| = w(X)$. Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{B}' = \{B \in \mathcal{B} : B \subset U$ for some $U \in \mathcal{U}$ and if $B \subset B' \in \mathcal{B}'$, then $B = B'\}$. Note that $\mathcal{B}'$ is a disjoint cover, hence a star-finite cover of X, and X is strongly base-paracompact. □

Let $D(\kappa)$ be the discrete space of size $\kappa \ge \omega$. The *Baire space, $B(\kappa)$, of weight κ* is the space $[D(\kappa)]^\omega$ endowed with the product topology. Note that the Baire space of weight κ is a non-archimedean space and hence strongly base-paracompact. K. Morita [M] showed that every strongly paracompact metrizable space of weight κ is embedable in the product $B(\kappa) \times I^\omega$ where I is the unit interval. The next theorem gives us that the latter space is strongly base-paracompact.

Theorem 2.4. *Strong base-paracompactness is an inverse invariant of perfect mappings.*

Proof: Let $f : X \to Y$ be a perfect mapping onto a base-paracompact space Y. Let $\mathcal{B}_Y$ be a basis for Y which witnesses base-paracompactness. Note that $w(X) \ge w(Y)$ ([E, Theorem 3.7.19]). Let $\mathcal{B}_X$ be any basis for X with $|\mathcal{B}_X| = w(X)$. Let $\mathcal{B}'_X = \mathcal{B}_X \cup \{f^{-1}(B) : B \in \mathcal{B}_Y\} \cup \{B \cap f^{-1}(B') : B \in \mathcal{B}_X, B' \in \mathcal{B}_Y\}$.

<u>*Claim*</u>: $\mathcal{B}'_X$ witnesses strong base-paracompactness for X.

Clearly $|\mathcal{B}_X| = w(X)$. Let $\mathcal{U} = \{U_t : t \in T\} \subset \mathcal{B}'_X$ be an open cover of X. For every $y \in Y$, choose a finite subset $I(y) \subset T$ such that $f^{-1}(y) \subset \cup_{t \in I(y)} U_t$. Since f is a closed map, there exists a neighborhood V_y of y such that $f^{-1}(y) \subset f^{-1}(V_y) \subset \bigcup_{t \in I(y)} U_t$. The cover $\{V_y\}_{y \in Y}$ has a star-finite refinement $\mathcal{B}'_Y \subset \mathcal{B}_Y$. Then, $\{f^{-1}(B) : B \in \mathcal{B}'_Y\}$ is star-finite, and for each $B \in \mathcal{B}'_Y$, $f^{-1}(B) \subset f^{-1}(V_{y(B)}) \subset \bigcup_{t \in I(y(B))} U_t$ for some $y(B) \in Y$. Now, $\mathcal{B}''_X = \{f^{-1}(B) \cap U_t : B \in \mathcal{B}'_Y$ and $t \in I(y(B))\}$ is a star-finite refinement of $\mathcal{U}$ by members of $\mathcal{B}'_X$. Hence X is base-paracompact. □

Corollary 2.5. *Let X be strongly base-paracompact, and let Y be a compact space. Then $X \times Y$ is strongly base-paracompact.*

Proof: The projection map $p : X \times Y \to X$ is a perfect mapping. Hence $X \times Y$ is strongly base-paracompact. □

The space $B(\kappa) \times I^\omega$ is not hereditarily strongly paracompact. J. Nagata [N] showed that the product $B(\kappa)$, $\kappa > \omega$, with the interval $(0, 1)$ is not strongly paracompact. The next theorem shows that certain closed sets of strongly base-paracompact spaces are strongly base-paracompact.

Theorem 2.6. *Let X be a strongly base-paracompact space. If M is a closed subset of X with $w(X) = w(M)$, then M is base-paracompact.*

PROOF: Let X be strongly base-paracompact, and let M be closed. Let $\mathcal{B}$ be a basis which witnesses strong base-paracompactness for X. Let $\mathcal{B}(M) = \{B \cap M : B \in \mathcal{B}\}$. Let $\mathcal{U}$ be an open cover of M. For every $U \in \mathcal{U}$, there is an open set $O_U \subset X$ such that $U = O_U \cap M$. The open cover $\{O_U : U \in \mathcal{U}\} \cup \{X \backslash M\}$ has a star-finite refinement $\mathcal{B}'$ by members of $\mathcal{B}$. Then $\mathcal{W} = \{B \in \mathcal{B}' : B \cap M \neq \emptyset\}$ is a star-finite refinement of $\mathcal{U}$ by members of $\mathcal{B}(M)$, and M is strongly base-paracompact. □

It will be shown in Section 4 that removing the weight condition would result in proving that every strongly paracompact space is strongly base-paracompact, which is still an open question.

3. Partitions of unity

A family $\mathcal{F}$ of continuous functions from a space X to the closed interval $[0, 1]$ is called a *partition of unity* if $\sum_{f\in\mathcal{F}} f(x) = 1$ for every $x \in X$. We say that a partition of unity $\mathcal{F}$ is *locally finite on a space* X if the cover $\{f^{-1}((0, 1]) : f \in \mathcal{F}\}$ is locally finite. A partition of unity $\mathcal{F}$ is *subordinated to a cover* $\mathcal{U}$ of X if the cover $\{f^{-1}((0, 1]) : f \in \mathcal{F}\}$ of the space X is a refinement of $\mathcal{U}$.

Lemma 3.1. *Let X be a normal space X and $\mathcal{B}_0$ be a base for X with $|\mathcal{B}_0| = w(X)$. Then there is a base $\mathcal{B}$ of X containing $\mathcal{B}_0$ such that $|\mathcal{B}| = w(X)$ and every star-finite cover by members of $\mathcal{B}$ has a shrinking by members of $\mathcal{B}$.*

PROOF: Let $\mathcal{B}_0$ be a basis for X with cardinality equal to the weight of X. Define $\mathcal{F}_0 = \{F = B\backslash \bigcup_{i<k} B_i : B, B_i \in \mathcal{B}_0 \text{ and } F \text{ is closed}\}$. Note that $|\mathcal{F}_0| = w(X)$. By normality, for every $F \in \mathcal{F}_0$ there is an open set U_F such that $F \subset U_F \subset \overline{U_F} \subset B$. Let $\mathcal{B}_1 = \mathcal{B}_0 \cup \{U_F : F \in \mathcal{F}_0\}$. Proceed by induction. Suppose $\mathcal{B}_n$ has been defined. Define $\mathcal{F}_n = \{F = B\backslash \bigcup_{i<k} B_i : B, B_i \in \mathcal{B}_n \text{ and } F \text{ is closed}\}$. Note that $|\mathcal{F}_n| = w(X)$. By normality, for every $F \in \mathcal{F}_{n+1}$ there is an open set U_F such that $F \subset U_F \subset \overline{U_F} \subset B$. Let $\mathcal{B}_{n+1} = \mathcal{B}_n \cup \{U_F : F \in \mathcal{F}_n\}$. Let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$. Note that $|\mathcal{B}| = w(X)$.

Let $\mathcal{U} = \{U_\alpha : \alpha < \lambda\} \subset \mathcal{B}$ be a star-finite cover. Let $F_0 = U_0\backslash \bigcup_{0<\alpha} U_\alpha$. Note that F_0 is closed since $\mathcal{U}$ covers X. Since $\mathcal{U}$ is star-finite, U_0 meets only finitely many members of $\{U_\alpha : 0 < \alpha\}$. Hence $F_0 \in \mathcal{F}_i$ for some i. Then there is $B_0 \in \mathcal{B}_{i+1}$ such that $F_0 \subset B_0 \subset \overline{B_0} \subset U_0$. Proceed by induction. Suppose F_γ and B_γ have been defined for all $\gamma < \alpha$ such that $F_\gamma \subset B_\gamma \subset \overline{B_\gamma} \subset U_\gamma$, and $\bigcup_{\gamma<\alpha} B_\gamma$ covers $X\backslash \bigcup_{\gamma\geq\alpha} U_\gamma$. Define $F_\alpha = U_\alpha\backslash[(\bigcup_{\beta>\alpha} U_\beta) \cup (\bigcup_{\gamma<\alpha} B_\gamma)]$. Note that F_α is closed since $(\bigcup_{\beta\geq\alpha} U_\beta) \cup (\bigcup_{\gamma<\alpha} B_\gamma)$ covers X. Since $\mathcal{U}$ is star-finite,

U_α meets only finitely many members of $\{U_\alpha : 0 < \alpha\} \cup \{B_\gamma : \gamma < \alpha\}$. Hence $F_\alpha \in \mathcal{F}_i$ for some i. Then there is $B_\alpha \in \mathcal{B}_{i+1}$ such that $F_\alpha \subset B_\alpha \subset \overline{B_\alpha} \subset U_\alpha$. Note also that $\bigcup_{\gamma\leq\alpha} B_\gamma$ covers $X \setminus \bigcup_{\gamma>\alpha} U_\gamma$.

Claim: $\{B_\alpha : \alpha < \lambda\}$ is a shrinking of $\mathcal{U} = \{U_\alpha : \alpha < \lambda\}$ by members of $\mathcal{B}$.
Clearly $\{B_\alpha : \alpha < \lambda\} \subset \mathcal{B}$ and $\overline{B_\alpha} \subset U_\alpha$ for every α. We need to show $\{B_\alpha : \alpha < \lambda\}$ is a cover of X. Suppose there is $x \in X$ such that $x \notin \bigcup_{\alpha<\lambda} B_\alpha$. Since $\mathcal{U}$ is point finite, there is a maximum $\alpha < \lambda$ such that $x \in U_\alpha$. Then $x \notin \bigcup_{\beta>\alpha} U_\beta$, and hence $x \in \bigcup_{\gamma\leq\alpha} B_\gamma$ which is a contradiction. Therefore $\{B_\alpha : \alpha < \lambda\}$ covers X and $\{B_\alpha : \alpha < \lambda\}$ is a shrinking of $\{U_\alpha : \alpha < \lambda\}$ by members of $\mathcal{B}$. This completes the proof. □

Theorem 3.2. *Let X be a strongly base-paracompact space. Then there exists a family of functions $\mathcal{F}$ with $|\mathcal{F}| = w(X)$ such that every open cover of X has a locally finite partition of unity $\mathcal{G} \subset \mathcal{F}$ subordinated to $\mathcal{U}$.*

Proof: Let $\mathcal{B}'$ witness strong base-paracompactness for X, and let $\mathcal{B}$ be the basis containing $\mathcal{B}'$ described in Lemma 3.1. Then for every $U, V \in \mathcal{B}$ with $\overline{U} \subset V$ let $f_{(U,V)} : X \to [0,1]$ be a continuous function such that $f_{(U,V)}(\overline{U}) = 1$ and $f_{(U,V)}(V^c) = 0$. Let $\mathcal{F}$ be all functions of the form

$$g(x) = \begin{cases} \dfrac{f_{(U,V)}(x)}{\sum_{i=1}^n f_{(U_i,V_i)}(x)} & \text{if } \displaystyle\sum_{i=1}^n f_{(U_i,V_i)}(x) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which happen to be continuous. Note that $|\mathcal{F}| = w(X)$. Now, let $\mathcal{U}$ be an open cover of X, and let $\mathcal{V} \subset \mathcal{B}$ be a star-finite refinement of $\mathcal{U}$. Let $\{W_V : V \in \mathcal{V}\}$ be a shrinking of $\mathcal{V}$ by members of $\mathcal{B}$. Then $f_{(W_V,V)} : X \to [0,1]$ such that $f_{(W_V,V)}(\overline{W_V}) = 1$ and $f_{(W_V,V)}(V^c) = 0$. For every $V \in \mathcal{V}$, V meets only finitely many members of $\mathcal{V}$ say $V_1, \cdots, V_n$. Define

$$g_V(x) = \begin{cases} \dfrac{f_{(W_V,V)}(x)}{\sum_{i=1}^n f_{(W_{V_i},V_i)}(x)} = \dfrac{f_{(W_V,V)}(x)}{\sum_{V'\in\mathcal{V}} f_{(W_V,V)}(x)} & \text{if } x \in V, \\ 0 & \text{if } x \notin V. \end{cases}$$

Claim: $\mathcal{G} = \{g_V : V \in \mathcal{V}\}$ is the desired partition of unity.
Clearly, $\mathcal{G}$ is locally finite and subordinated to $\mathcal{U}$. So, it remains to be shown that $\mathcal{G}$ is a partition of unity. Let $x \in X$. Then

$$\sum_{g_V\in\mathcal{G}} g_V(x) = \sum_{V\in\mathcal{V}} \frac{f_{(W_V,V)}(x)}{\sum_{V'\in\mathcal{V}} f_{(W_{V'},V')}(x)} = \frac{\sum_{V\in\mathcal{V}} f_{(W_V,V)}(x)}{\sum_{V'\in\mathcal{V}} f_{(W_{V'},V')}(x)} = 1.$$

Therefore $\mathcal{G}$ is a locally finite partition of unity subordinated to $\mathcal{U}$.

Let I be the unit interval. Let κ be an infinite cardinal, and let $I_\alpha = I \times \{\alpha\}$ for every $\alpha \leq \kappa$. Define an equivalence relation E on $\bigcup_{\alpha \leq \kappa} I_\alpha$ by $(x, \alpha_1)E(y, \alpha_2)$ whenever $x = 0 = y$ or $x = y$ and $\alpha_1 = \alpha_2$. The formula

$$\rho([(x,\alpha_1)],[(y,\alpha_2)]) = \begin{cases} |x-y| & \text{if } \alpha_1 = \alpha_2, \\ x+y & \text{if } \alpha_1 \neq \alpha_2, \end{cases}$$

defines a metric on the equivalence classes of E. This metrizable space is called the *hedgehog space of spininess* κ and is denoted $J(\kappa)$. Note that $J(\kappa)$ is a connected space with $w(J(\kappa)) = \kappa$. One may also visualize a hedgehog space of spininess κ by taking κ-many copies of the unit interval and attaching each interval at the point 0. □

Theorem 3.3. *The Hedgehog space $J(\kappa)$, $\kappa > \omega$ is not strongly paracompact, but there exists a family of functions $\mathcal{F}$ with $|\mathcal{F}| = \kappa$ such that every open cover has a locally finite partition of unity $\mathcal{G} \subset \mathcal{F}$ subordinated to it.*

Proof: Suppose that $J(\kappa)$ is strongly paracompact. Let $\mathcal{U}$ be an open cover. Since $J(\kappa)$ is strongly paracompact, $\mathcal{U}$ has a star-finite refinement say $\mathcal{V}$. Choose $V \in \mathcal{V}$. Since $\mathcal{V}$ is star-finite, V meets only finitely many members of $\mathcal{V}$ say $\mathcal{V}_0 = \{V_i : i \in F_0\}$ where F_0 is finite. Suppose $\mathcal{V}_n = \{V_i : i \in F_n\}$, where F_n is finite, be all the elements of $\mathcal{V}$ which meet an element of $\mathcal{V}_{n-1}$. Since $\mathcal{V}_n$ is finite and $\mathcal{V}$ is star-finite, $\mathcal{V}_n$ only meets finitely many members of $\mathcal{U}$ say $\mathcal{V}_{n+1} = \{V_i : i \in F_{n+1}\}$. Let $F = \bigcup F_n$. Note that $\{V_i : i \in F\}$ is countable, and $\bigcup_{i \in F} V_i$ is an open and closed in X. Since $J(\kappa)$ is connected, $\bigcup_{i \in F} V_i = X$. This makes $J(\kappa)$ a Lindelöf space, but $J(\kappa)$ is not separable which is equivalent to the Lindelöf property in metrizable spaces. This contradicts $J(\kappa)$ being strongly paracompact.

Now we will show that $J(\kappa)$ has a family of functions with cardinality equal to the weight such that every open cover has a locally finite partition of unity subordinated to it. For $n \geq 1$, the interval $(\frac{1}{n}, 1]$ is a separable metric space and therefore is strongly base-paracompact by Theorem 2.2. Let $\mathcal{F}_n$ be a family of functions with $|\mathcal{F}_n| = \omega$ such that every open cover of $(\frac{1}{n}, 1]$ has a locally finite partition of unity $\mathcal{G} \subset \mathcal{F}$ subordinated to it. Let $\mathcal{H}_{n,\alpha}$ be the collection of all $h : J(\kappa) \to [0,1]$ with

$$h(x,\alpha) = \begin{cases} h(x,\alpha) = f(x) & \text{if } x \in (\dfrac{1}{n}, 1], \\ 0 & \text{otherwise}, \end{cases}$$

where $f \in \mathcal{F}_n$. Let $\mathcal{F} = \bigcup\{\mathcal{H}_{n,\alpha} : n < \omega \text{ and } \alpha < \kappa\}$. Consider the basis $B(0, \frac{1}{n})$, $n \in \omega$, at the point 0. Since $J(\kappa)$ is metrizable, there is a continuous $f_n : J(\kappa) \to [0,1]$ such that $f_n(B(0, \frac{1}{n-1})^c) = 0$ and $f_n(\overline{B(0, \frac{1}{n})}) = 1$. Let

$\mathcal{F}' = \mathcal{F} \cup \{f_n : n < \omega\} \cup \{\frac{f(x)}{1+f_n(x)} : f \in \mathcal{F} \text{ and } n < \omega\}$. We will show $\mathcal{F}'$ is the desired set of functions. Let $\mathcal{U}$ be an open cover of $J(\kappa)$. There is $U \in \mathcal{U}$ and $n \in \omega$ such that $0 \in B(0, \frac{1}{n}) \subset U$. Consider $J(\kappa) \backslash B(0, \frac{1}{n-1}) = \bigcup_{\alpha<\kappa} [\frac{1}{n-1}, 1]_\alpha$. Let $\mathcal{U}_\alpha = \{U \in \mathcal{U} : U \cap (\frac{1}{n}, 1]_\alpha \neq \emptyset\}$. Note that $\mathcal{U}_\alpha$ is a cover of $(\frac{1}{n}, 1]_\alpha$ so there exists a partition of unity $\mathcal{G}_\alpha \subset \mathcal{F}$ subordinated to $\mathcal{U}_\alpha$ such that $g^{-1}((0,1]) \subset (\frac{1}{n}, 1]$ for every $g \in \mathcal{G}_\alpha$. For every $g \in \mathcal{G}_\alpha$, define $g'(x) = \frac{g(x)}{1+f_n(x)}$. Let $\mathcal{G}'_\alpha = \{g'(x) : g \in \mathcal{G}_\alpha\}$.

<u>*Claim*</u>: $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}'_\alpha \cup \{f_n\}$ is the desired partition of unity.
Clearly $\mathcal{G} \subset \mathcal{F}'$ and $\mathcal{G}$ is locally finite and subordinates $\mathcal{U}$. Let $x \in X$.

<u>*Case* (1)</u>: $x \in \overline{B(0, \frac{1}{n})}$.
Then $\sum_{f \in \mathcal{G}} f(x) = f_n(x) = 1$ since $g(x) = 0$ for every $g \in \mathcal{G}_\alpha$ and every $\alpha < \kappa$.

<u>*Case* (2)</u>: $x \in (\frac{1}{n}, \frac{1}{n-1}]_\alpha$ for some $\alpha < \kappa$.
Then $\sum_{f \in \mathcal{G}} f(x) = \sum_{g' \in \mathcal{G}'_\alpha} g'(x) = \sum_{g \in \mathcal{G}_\alpha} \frac{g(x)}{1+f_n(x)} + \frac{f_n(x)}{1+f_n(x)} =$
$= \frac{(\sum_{g \in \mathcal{G}_\alpha} g(x)) + f_n(x)}{1+f_n(x)} = \frac{1+f_n(x)}{1+f_n(x)} = 1.$

<u>*Case* (3)</u>: $x \in [\frac{1}{n-1}, 1]_\alpha$ for some $\alpha < \kappa$.
Then $\sum_{f \in \mathcal{G}} f(x) = \sum_{g \in \mathcal{G}_\alpha} g(x) = 1$ since $f_n(x) = 0$, and this completes the proof. □

4. Questions

It is an open question whether all strongly paracompact spaces are strongly base-paracompact. The next theorem shows some of the difficulties with this problem. Suppose that there is a strongly paracompact space X that is not strongly base-paracompact. If we were to add an isolated point to X, then the resulting space would still be a strongly paracompact space that is not strongly base-paracompact. In conclusion, if there is a paracompact space that is not base-paracompact, then there is such a space that has an isolated point. This fact will be used in the proof of the next theorem.

Theorem 4.1. *The following are equivalent:*

(i) *Every strongly paracompact space is strongly base-paracompact.*
(ii) *Every closed subset of a strongly base-paracompact space is strongly base-paracompact.*
(iii) *Strongly base-paracompact spaces are preserved under open perfect mappings.*

Proof: Note that (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) (see [P]). It suffices to prove (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i). For these two cases, let X be a strongly paracompact space with an isolated point x_0 which is not base-paracompact. Let $\mathcal{O}$ be the family of all open sets of X and let $\kappa = |\mathcal{O}|$. Let $Y = X \oplus [0,\kappa]$ (the topological sum). Note that $w(Y) = \kappa$. Let $\mathcal{B}_\kappa$ be a basis for $[0,\kappa]$. Then $\mathcal{B} = \mathcal{B}_\kappa \cup \mathcal{O}$ is a basis for X that witnesses strong base-paracompactness for Y. Note that X is a closed subset of Y which is not strongly base-paracompact. This proves (ii)$\Rightarrow$(i).

Define $f : Y \to X$ by $f(x) = x$ if $x \in X$ and $f(x) = x_0$ if $x \in [0,\kappa]$. Note that f is an open perfect mapping from Y onto X. This proves (iii)$\Rightarrow$(i). $\square$

The author closes the paper by listing some open questions that the author was unable to answer.

Question 4.2. *Are strongly paracompact metric spaces strongly base-paracompact?*

Question 4.3. *Are paracompact linearly ordered spaces strongly base-paracompact?*

Question 4.4. *Do base-paracompact spaces have a family of functions $\mathcal{F}$ with cardinality equal to the weight such that every open cover has a locally finite partition of unity subordinated to it from $\mathcal{F}$?*

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