Perfect sets and collapsing continuum

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Abstract. Under Martin's axiom, collapsing of the continuum by Sacks forcing $\mathbb{S}$ is characterized by the additivity of Marczewski's ideal (see [4]). We show that the same characterization holds true if $\mathfrak{d} = \mathfrak{c}$ proving that under this hypothesis there are no small uncountable maximal antichains in $\mathbb{S}$. We also construct a partition of ${}^{\omega}2$ into $\mathfrak{c}$ perfect sets which is a maximal antichain in $\mathbb{S}$ and show that s^0-sets are exactly (subsets of) selectors of maximal antichains of perfect sets.

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1. General remarks

Let $(\mathbb{P}, \leq)$ be a partial order. We say that elements (conditions) $p, q \in \mathbb{P}$ are compatible and write $p \wedge q \neq 0$ if there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. Otherwise p and q are incompatible and we write $p \wedge q = 0$. A family of pairwise incompatible elements is called an antichain. For $p \in \mathbb{P}$, $\mathbb{P}{\restriction}p = \{q \in \mathbb{P} : q \leq p\}$. Let us recall some cardinal invariants for $\mathbb{P}$:

$$\begin{aligned}
\pi(\mathbb{P}) &= \min\{|X| : X \text{ is a dense subset of } \mathbb{P}\},\\
\mathrm{sat}(\mathbb{P}) &= \min\{\kappa : \text{every antichain has size} < \kappa\},\\
\mathfrak{a}(\kappa, \mathbb{P}) &= \min(\{\pi(\mathbb{P})\} \cup \{|A| : A \subseteq \mathbb{P} \text{ is a maximal antichain with } |A| \geq \kappa\}),\\
\mathrm{cf}_\pi(\mathbb{P}) &= \min\{\kappa : \Vdash_{\mathbb{P}} \mathrm{cf}(\pi^V(\mathbb{P})) \leq \kappa\}.
\end{aligned}$$

The hereditary version of a cardinal invariant $\kappa(\cdot)$ for partial orders is defined by $\mathrm{h}\kappa(\mathbb{P}) = \min\{\kappa(\mathbb{P}{\restriction}p) : p \in \mathbb{P}\}$. The symbols $\mathrm{h}\pi(\mathbb{P})$, $\mathrm{hsat}(\mathbb{P})$, $\mathrm{h}\mathfrak{a}(\kappa, \mathbb{P})$ denote the hereditary versions of the cardinals $\pi(\mathbb{P})$, $\mathrm{sat}(\mathbb{P})$, $\mathfrak{a}(\kappa, \mathbb{P})$, respectively.

A matrix on $\mathbb{P}$ is a sequence of antichains in $\mathbb{P}$ (the antichains may be maximal). Let $\mathcal{A}$ be a matrix on $\mathbb{P}$. A matrix $\mathcal{A}$ is shattering if for every $p \in \mathbb{P}$ there exists an antichain $A \in \mathcal{A}$ such that $|\{q \in A : p \wedge q \neq 0\}| \geq \pi(\mathbb{P})$. A matrix $\mathcal{A}$ is weakly shattering if $\sum_{A \in \mathcal{A}} |\{q \in A : p \wedge q \neq 0\}| \geq \pi(\mathbb{P})$ for every $p \in \mathbb{P}$. A matrix is a base matrix if $\bigcup \mathcal{A}$ is a dense subset of $\mathbb{P}$. The following two theorems contain some well known basic facts about all these notions.

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Theorem 1.1. (1) *A shattering matrix is weakly shattering.*

(2) *There exists a base matrix on* $\mathbb{P}$ *of size* $\pi(\mathbb{P})$.
(3) *If* $\mathrm{h}\pi(\mathbb{P}) = \pi(\mathbb{P})$, *then every base matrix on* $\mathbb{P}$ *is weakly shattering.*
(4) *There exists a shattering matrix on* $\mathbb{P}$ *if and only if* $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$.
(5) *If there is a weakly shattering matrix on* $\mathbb{P}$ *of size* $< \pi(\mathbb{P})$, *then* $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$.
(6) *For every weakly shattering matrix there exists a weakly shattering base matrix of the same size.*
(7) *If* $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$, *then for every base matrix on* $\mathbb{P}$ *there exists a shattering base matrix on* $\mathbb{P}$ *of the same size.*
(8) *If* $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$, *then there exists a shattering matrix on* $\mathbb{P}$ *of size* $\mathrm{cf}(\pi(\mathbb{P}))$.

Proof: The assertions (1)–(5) are easy to see. For the rest of the proof let us fix a dense set $D \subseteq \mathbb{P}$ with $|D| = \pi(\mathbb{P})$.

(6) Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ be a weakly shattering matrix on $\mathbb{P}$. There exists a one-to-one mapping $\varphi : D \to \bigcup_{\alpha<\kappa}\{\alpha\}\times A_\alpha$, $\varphi = (\varphi_1, \varphi_2)$, such that $p \wedge \varphi_2(p) \neq 0$ for every $p \in D$. For every $p \in D$ let us fix an element $r(p) \in P$ below p and $\varphi_2(p)$ and let $A'_\alpha = \{r(p) : \varphi_1(p) = \alpha\}$. The matrix $\mathcal{A} = \{A'_\alpha : \alpha < \kappa\}$ is a weakly shattering base matrix on $\mathbb{P}$.

(7) For $p \in \mathbb{P}$ let B_p be an antichain below p of size $\pi(\mathbb{P})$. If $\mathcal{A}$ is a base matrix on $\mathbb{P}$, then the matrix $\mathcal{A}' = \{\bigcup_{p\in A} B_p : A \in \mathcal{A}\}$ is a shattering base matrix on $\mathbb{P}$.

(8) Let $D = \bigcup\{D_\alpha : \alpha < \mathrm{cf}(\pi(\mathbb{P}))\}$ with $|D_\alpha| < \pi(\mathbb{P})$. By the Balcar-Vojtáš's Theorem (see [1] or [6]) for each α there is a disjoint refinement A_α of D_α. Therefore $\{A_\alpha : \alpha < \mathrm{cf}(\pi(\mathbb{P}))\}$ is a base matrix on $\mathbb{P}$ and by assertion (7) there exists a shattering matrix on $\mathbb{P}$ of the same size. □

From now on we assume that $\mathrm{h}\pi(\mathbb{P}) = \pi(\mathbb{P})$ and we define:

$$\mathrm{sh}(\mathbb{P}) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a weakly shattering matrix on } \mathbb{P}\},$$

$$\mathrm{sh}_\lambda(\mathbb{P}) = \min(\{\pi(\mathbb{P})\} \cup \{\kappa : \mathrm{r.o.}(\mathbb{P}) \text{ is } (\kappa, \pi(\mathbb{P}), \lambda)\text{-nowhere distributive}\}).$$

We use the definition of the three-parameter distributivity from [2]. Clearly, $\mathrm{sh}(\mathbb{P}) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a base matrix on } \mathbb{P}\} = \min(\{\pi(\mathbb{P})\} \cup \{|\mathcal{A}| : \mathcal{A} \text{ is a shattering matrix on } \mathbb{P}\}) = \mathrm{sh}_{\pi(\mathbb{P})}(\mathbb{P})$. Again, $\mathrm{hsh}(\mathbb{P})$ denotes the hereditary version of the cardinal $\mathrm{sh}(\mathbb{P})$.

Theorem 1.2. *Let us assume that* $\mathrm{h}\pi(\mathbb{P}) = \pi(\mathbb{P})$.

(1) *If* $\mathrm{r.o.}(\mathbb{P})$ *is* $(\kappa, \lambda, \lambda)$*-nowhere distributive*, *then* $\mathrm{r.o.}(\mathbb{P})$ *is* $(\kappa, \mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$*-nowhere distributive.*
(2) *If* $\mathrm{r.o.}(\mathbb{P})$ *is* $(\kappa, \mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$*-nowhere distributive*, *then* $\mathrm{r.o.}(\mathbb{P})$ *is* $(\kappa, \lambda, \mathrm{cf}\,\lambda)$*-nowhere distributive.*
(3) *If* $\kappa < \mathrm{cf}\,\lambda$, *then* $\mathrm{r.o.}(\mathbb{P})$ *is* $(\kappa, \mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$*-nowhere distributive if and only if* $\Vdash_{\mathbb{P}} \mathrm{cf}\,\lambda \leq \kappa$.

(4) *If* $\mathrm{hsh}(\mathbb{P}) = \mathrm{sh}(\mathbb{P})$, *then* $\Vdash_{\mathbb{P}} |\pi^V(\mathbb{P})| = \mathrm{sh}^V(\mathbb{P})$.
(5) $\Vdash_{\mathbb{P}} \pi(\mathbb{P}) = |\pi^V(\mathbb{P})|$.
(6) $\min\{\mathrm{sh}_{\mathrm{cf}\,\pi(\mathbb{P})}(\mathbb{P}), \mathrm{cf}(\pi(\mathbb{P}))\} \le \mathrm{cf}_\pi(\mathbb{P}) \le \min\{\mathrm{sh}(\mathbb{P}), \mathrm{cf}(\pi(\mathbb{P}))\}$ *and there are two possibilities*: *Either* $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$ *and* $\mathrm{sh}_{\mathrm{cf}\,\pi(\mathbb{P})}(\mathbb{P}) \le \mathrm{cf}_\pi(\mathbb{P}) \le \mathrm{sh}(\mathbb{P}) \le \mathrm{cf}(\pi(\mathbb{P}))$, *or* $\mathrm{hsat}(\mathbb{P}) \le \pi(\mathbb{P})$ *and* $\mathrm{sh}(\mathbb{P}) = \pi(\mathbb{P})$.
(7) *If* $\mathrm{sh}_{\mathrm{cf}\,\pi(\mathbb{P})}(\mathbb{P}) = \mathrm{sh}(\mathbb{P})$ (*e.g.*, *if* $\pi(\mathbb{P})$ *is regular*, *or if* $\mathfrak{a}(\mathrm{cf}(\pi(\mathbb{P})), \mathbb{P}) = \pi(\mathbb{P})$), *then* $\mathrm{cf}_\pi(\mathbb{P}) = \min\{\mathrm{sh}(\mathbb{P}), \mathrm{cf}(\pi(\mathbb{P}))\}$.
(8) *If* $\mathrm{hsat}(\mathbb{P}) \ge \lambda^+$, *then* $\mathrm{sh}_\lambda(\mathbb{P}) \le (\mathrm{cf}\,\lambda) \cdot \sup_{\kappa<\lambda} \mathrm{sh}_\kappa(\mathbb{P})$ *and* $\mathrm{sh}_{\mathrm{cf}\,\lambda}(\mathbb{P}) \le \mathrm{cf}\,\mathrm{sh}_\lambda(\mathbb{P})$.

Proof: The assertions (1) and (2) are easy.

(3) Let $\{\lambda_\xi : \xi < \mathrm{cf}\,\lambda\}$ be an increasing cofinal sequence in λ and let $\kappa < \mathrm{cf}\,\lambda$. Let $\dot f$ be a $\mathbb{P}$-name of an unbounded function from κ to λ. For $\alpha < \kappa$ let $A_\alpha = \{\|\dot f(\alpha) \in [\lambda_\xi, \lambda_{\xi+1})\| : \xi < \mathrm{cf}\,\lambda\} \setminus \{0\}$. The matrix $\{A_\alpha : \alpha < \kappa\}$ witnesses the $(\kappa, \mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$-nowhere distributivity of r. o.$(\mathbb{P})$. Conversely, if $\{A_\alpha : \alpha < \kappa\}$ is a matrix on r. o.$(\mathbb{P})$ with $A_\alpha = \{a_{\alpha,\xi} : \xi < \mathrm{cf}\,\lambda\}$ witnessing the $(\kappa, \mathrm{cf}\,\lambda, \mathrm{cf}\,\lambda)$-nowhere distributivity of r. o.$(\mathbb{P})$, then the formula $\|\dot f(\alpha) = \lambda_\xi\| = a_{\alpha,\xi}$ defines a $\mathbb{P}$-name of an unbounded function from κ to λ.

(4) Let us assume that p and μ are such that $p \Vdash_{\mathbb{P}} |\pi^V(\mathbb{P})| = \mu$. Let $\dot f$ be a $\mathbb{P}{\restriction}p$-name of a function from μ onto $\pi(\mathbb{P})$ and for $\alpha < \mu$ let A_α be a maximal antichain in $\mathbb{P}{\restriction}p$ consisting of $q \in \mathbb{P}{\restriction}p$ deciding $\dot f(\alpha)$. Since every $q \in \mathbb{P}{\restriction}p$ forces that $\dot f$ is onto $\pi(\mathbb{P}) = \pi(\mathbb{P}{\restriction}p)$, easily, it can be verified that $\{A_\alpha : \alpha < \mu\}$ is a weakly shattering matrix on $\mathbb{P}{\restriction}p$. Therefore $\mathrm{sh}(\mathbb{P}) = \mathrm{sh}(\mathbb{P}{\restriction}p) \le \mu$ and $p \Vdash_{\mathbb{P}} \mathrm{sh}^V(\mathbb{P}) \le |\pi^V(\mathbb{P})|$.

Let $\mathrm{sh}(\mathbb{P}) = \kappa$. If $\mathrm{sh}(\mathbb{P}) = \pi(\mathbb{P})$, then clearly, $\Vdash_{\mathbb{P}} |\pi^V(\mathbb{P})| \le \mathrm{sh}^V(\mathbb{P})$. Let us assume that $\mathrm{sh}(\mathbb{P}) < \pi(\mathbb{P})$. Then by Theorem 1.1(5), $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$. For every $q \in \mathbb{P}$ let us fix a maximal antichain $\{(q)_\xi : \xi < \pi(\mathbb{P})\}$ below q. As $\mathrm{sh}(\mathbb{P}) = \kappa$, there is a base matrix $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ (with all antichains maximal). We define a $\mathbb{P}$-name $\dot f$ of a function from κ onto $\pi^V(\mathbb{P})$ by $\|\dot f(\alpha) = \xi\| = \bigvee\{(q)_\xi : q \in A_\alpha\}$. Therefore $\Vdash_{\mathbb{P}} |\pi^V(\mathbb{P})| \le \mathrm{sh}^V(\mathbb{P})$.

(5) Clearly, $\Vdash_{\mathbb{P}} \pi(\mathbb{P}) \le |\pi^V(\mathbb{P})|$. Let p and κ be such that $p \Vdash_{\mathbb{P}} \pi(\mathbb{P}) = \kappa$ and $\mathrm{hsh}(\mathbb{P}{\restriction}p) = \mathrm{sh}(\mathbb{P}{\restriction}p)$. Let $\dot f$ be a $\mathbb{P}$-name of a function from κ into $\mathbb{P}$ such that $p \Vdash_{\mathbb{P}} (\forall q \in \mathbb{P})(\exists \alpha < \kappa)\, \dot f(\alpha) \le q$. Let A_α, $\alpha < \kappa$, be a maximal antichain of conditions below p deciding $\dot f(\alpha)$. For $q \le p$ let $B_{\alpha,q} = \{r \in A_\alpha : q \wedge r \ne 0\}$ and $B'_{\alpha,q} = \{s \in \mathbb{P} : (\exists r \in B_{\alpha,q})\, r \Vdash_{\mathbb{P}} \dot f(\alpha) = s\}$. The set $\bigcup_{\alpha<\kappa} B'_{\alpha,q}$ is a dense subset of $\mathbb{P}$ for every $q \le p$ and $|B_{\alpha,q}| \ge |B'_{\alpha,q}|$. Therefore $\sum_{\alpha<\kappa} |B_{\alpha,q}| \ge \pi(\mathbb{P}) = \pi(\mathbb{P}{\restriction}p)$ and hence the matrix $\{A_\alpha : \alpha < \kappa\}$ is weakly shattering on $\mathbb{P}{\restriction}p$. Hence $\mathrm{sh}(\mathbb{P}{\restriction}p) \le \kappa$ and by (4) we have $p \Vdash_{\mathbb{P}} |\pi^V(\mathbb{P})| \le \pi(\mathbb{P})$. A density argument proves that $\Vdash_{\mathbb{P}} |\pi^V(\mathbb{P})| \le \pi(\mathbb{P})$.

(6) By (1)–(3) we easily obtain the inequalities $\min\{\mathrm{sh}_{\mathrm{cf}\,\pi(\mathbb{P})}(\mathbb{P}), \mathrm{cf}(\pi(\mathbb{P}))\} \le \mathrm{cf}_\pi(\mathbb{P}) \le \min\{\mathrm{sh}(\mathbb{P}), \mathrm{cf}(\pi(\mathbb{P}))\}$. If $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$, then, by Theorem 1.1(8), $\mathrm{sh}(\mathbb{P}) \le \mathrm{cf}(\pi(\mathbb{P}))$. Since $\mathrm{sh}_{\mathrm{cf}\,\pi(\mathbb{P})}(\mathbb{P}) \le \mathrm{sh}(\mathbb{P})$, by (5), $\mathrm{sh}_{\mathrm{cf}\,\pi(\mathbb{P})}(\mathbb{P}) \le \mathrm{cf}_\pi(\mathbb{P})$. If

$\mathrm{hsat}(\mathbb{P}) \le \pi(\mathbb{P})$, then $\mathrm{sh}(\mathbb{P}) = \pi(\mathbb{P})$ by Theorem T1.1(5)

(7) immediately follows by (6), and (8) can be obtained by an easy computation. □

In the case $\mathrm{hsat}(\mathbb{P}) = \pi(\mathbb{P})^+$, in some special cases (e.g., if $\pi(\mathbb{P})$ is regular, or $\mathfrak{a}(\mathrm{cf}(\pi(\mathbb{P})), \mathbb{P}) = \pi(\mathbb{P})$, etc., see Theorem 1.2(7) or (8)), $\mathrm{sh}(\mathbb{P})$ is regular (even in $V^{\mathrm{r.o.}(\mathbb{P})}$). But in general it is not clear whether $\mathrm{sh}(\mathbb{P})$ is a regular cardinal.

We use the standard terminology. By $\mathcal{M}$ and $\mathcal{N}$ we denote the ideal of meager sets and the ideal of null sets, respectively, $\mathfrak{b}$ is the least cardinality of an unbounded family and $\mathfrak{d}$ is the least cardinality of a dominating family of functions in the ordering $\le^*$ on ${}^\omega\omega$ defined for $f, g \in {}^\omega\omega$ by $f \le^* g$ if and only if $f(n) \le g(n)$ for all but finitely many $n \in \omega$. $\mathrm{add}(I)$ is the additivity of an ideal I, $\mathrm{cov}(I)$ is the least size of a set $I_0 \subset I$ such that $\bigcup I_0 = \bigcup I$, $\mathrm{non}(I)$ is the least size of a subset of $\bigcup I$ not belonging to I, and $\mathrm{cof}(I)$ is the least size of a set $I_0 \subset I$ which is cofinal in $(I, \subseteq)$. Sacks forcing $\mathbb{S}$ is the set of perfect trees $p \subseteq {}^{<\omega}2$ where p is stronger than q, $p \le q$, if $p \subseteq q$. For $p \in \mathbb{S}$ and $s \in {}^{<\omega}2$ we denote $p_s = \{t \in p : s \subseteq t \text{ or } t \subseteq s\}$, $[p] = \{x \in {}^\omega 2 : \forall n\ x{\restriction}n \in p\}$, $[s] = \{x \in {}^\omega 2 : s \subseteq x\}$. Every perfect set in ${}^\omega 2$ is of the form $[p]$ for some $p \in \mathbb{S}$.

2. Maximal antichains in $\mathbb{S}$

Important is the question what the possible sizes of small maximal antichains in Sacks forcing are. By the next well-known theorem, $\mathfrak{a}(\omega_1, \mathbb{S}) \ge \mathrm{cov}(\mathcal{M})$ and we prove in Theorem 2.5 below that $\mathfrak{a}(\omega_1, \mathbb{S}) \ge \mathfrak{d}$.

Theorem 2.1. *For a cardinal κ the following conditions are equivalent*:

(1) $\kappa < \mathrm{cov}(\mathcal{M})$;
(2) *for every family B of perfect sets such that $|B| \le \kappa$ and ${}^\omega 2 \setminus \bigcup C$ is uncountable for every $C \in [B]^{\le\omega}$, ${}^\omega 2 \setminus \bigcup B \ne \emptyset$*;
(3) *for every family B of perfect sets such that $|B| \le \kappa$ and ${}^\omega 2 \setminus \bigcup C$ is uncountable for every $C \in [B]^{\le\omega}$, ${}^\omega 2 \setminus \bigcup B$ contains a perfect set.*

Proof: The implications (3) → (2) → (1) are obvious. We prove (1) → (3).

Let $\kappa < \mathrm{cov}(\mathcal{M})$ and let B be a family of perfect sets such that $|B| \le \kappa$ and ${}^\omega 2 \setminus \bigcup C$ is uncountable for every $C \in [B]^{\le\omega}$. Let q be the set of all $s \in {}^{<\omega}2$ such that $[s] \setminus \bigcup C$ is uncountable for every $C \in [B]^{\le\omega}$. By the assumption, $\emptyset \in q$ and it follows that q is a perfect tree and for every perfect set $[p] \in B$, $[p] \cap [q]$ is nowhere dense in $[q]$. As $\kappa < \mathrm{cov}(\mathcal{M})$, MA_κ(countable) implies the existence of a perfect tree $r \le q$ such that $[r] \cap [p] = \emptyset$ for all $[p] \in B$ (using a countable forcing for adding a perfect set of Cohen reals). □

We need the following special case of Exercise 7.13 in [5]:

Lemma 2.2. *If G is a dense G_δ subset of ${}^\omega 2$ such that ${}^\omega 2 \setminus G$ is dense in ${}^\omega 2$, then there exists a homeomorphism f from G onto ${}^\omega\omega$.*

Proof: By the assumptions no relatively clopen subset of G is compact. Let U_n, $n \in \omega$, be open sets in ${}^\omega 2$ such that $G = \bigcap_{n\in\omega} U_n$ and $U_{n+1} \subseteq U_n$ for all n. For $s \in {}^{<\omega}\omega$ let us define $t_s \in {}^{<\omega}2$ by induction on $|s|$ so that $s \subseteq s'$ if and only if $t_s \subseteq t_{s'}$, $t_\emptyset = \emptyset$, and $[t_s] \cap U_{n+1} = \bigcup_{i\in\omega}[t_{s^\frown\langle i\rangle}]$ for $|s| = n$. Then for $x \in G$ we let $f(x)$ be the unique element $y \in {}^\omega\omega$ such that $t_{y\restriction n} \subseteq x$ for all $n \in \omega$. □

Theorem 2.3. *If B is a family of perfect sets in ${}^\omega 2$ and $|B| < \mathfrak{d}$, then the set ${}^\omega 2 \setminus \bigcup B$ is either at most countable or contains a perfect set.*

Proof: Let us assume that $|B| < \mathfrak{d}$ and the set $X = {}^\omega 2 \setminus \bigcup B$ is uncountable. Let Y be a countable subset of X without isolated points. Let $q \in \mathbb{S}$ be such that $[q] = \overline{Y}$. By Lemma 2.2 there is a homeomorphism f from $G = [q] \setminus Y$ onto ${}^\omega\omega$. For $F \in B$, $F \cap Y = \emptyset$ and hence $F \cap G = F \cap [q]$. It follows that $f``(F \cap G)$ is compact and hence bounded in ${}^\omega\omega$. As $|B| < \mathfrak{d}$, there is an $y \in {}^\omega\omega$ not dominated by any member of the set $\bigcup_{F\in B} f``(F \cap G)$. Then the set $E = f^{-1}(\{x \in {}^\omega\omega : \forall n\; x(n) \geq y(n)\})$ is an uncountable relatively closed subset of G disjoint from $\bigcup B$. □

If $\mathfrak{d} = \mathfrak{c}$, then using Theorem 2.3 one can construct a partition of ${}^\omega 2$ into $\mathfrak{c}$ perfect sets. In the next theorem we prove that partitions of ${}^\omega 2$ into $\mathfrak{c}$ perfect sets exist in ZFC. We shall use the following notation:

Let $p \in \mathbb{S}$ and $x \in [p]$. Let $\{k_n : n \in \omega\}$ be the increasing enumeration of the set $\{k \in \omega : (x\restriction k)^\frown\langle 0\rangle \in p$ and $(x\restriction k)^\frown\langle 1\rangle \in p\}$ and let $\bar{x} \in {}^\omega 2$ be such that $\bar{x}(n) \neq x(n)$ for all $n \in \omega$. Let us define $\tau(p, x, n) = p_{(x\restriction k_n)^\frown\langle\bar{x}(k_n)\rangle} = \{s \in p : s \subseteq (x\restriction k_n)^\frown\langle\bar{x}(k_n)\rangle$ or $(x\restriction k_n)^\frown\langle\bar{x}(k_n)\rangle \subseteq s\}$. Then the system $[\tau(p, x, n)]$, $n \in \omega$, is a partition of $[p] \setminus \{x\}$. In particular, $[\tau({}^{<\omega}2, x, n)]$, $n \in \omega$, is a partition of ${}^\omega 2 \setminus \{x\}$ into clopen sets.

For $A \subseteq \mathbb{S}$ let $B_A = \{[p] : p \in A\}$ and let $\bigvee A$ denote the Boolean sum of A in r.o.($\mathbb{S}$). In the Boolean sums we will consider only those $A \subseteq \mathbb{S}$ for which $\bigvee A \in \mathbb{S}$. Notice that $\bigvee_n \tau(p, x, n) = \bigcup_n \tau(p, x, n) = p$.

Theorem 2.4. *Let D be a dense subset of $\mathbb{S}$.*

(1) *There exists a maximal antichain $A \subseteq D$ such that the family B_A is disjoint and for every $p \in \mathbb{S}$ with $[p] \subseteq \bigcup B_A$ there exists $C \in [B_A]^{<\mathfrak{c}}$ such that $[p] \subseteq \bigcup C$.*

(2) *There exist maximal antichains $A \subseteq D$ and $\bar{A} \subseteq \mathbb{S}$, both of size $\mathfrak{c}$, such that B_A is a disjoint family, $B_{\bar{A}}$ is a partition of ${}^\omega 2$, and the following conditions are satisfied:*

   (a) *for every $q \in \bar{A} \setminus A$ the set $A_q = \{p \in A : p \leq q\}$ is countable, $q = \bigvee A_q$, and $|[q] \setminus \bigcup B_{A_q}| = 1$;*

(b) *For every* $q \in \mathbb{S}$, *if* $|[q] \setminus \bigcup B_A| < \mathfrak{c}$, *then* $|\{p \in A : [q] \cap [p] \neq \emptyset\}| < \mathfrak{c}$;
(c) *for every* $q \in \mathbb{S}$, $|\{p \in A : q \wedge p \neq 0\}| < \mathfrak{c}$ *if and only if* $|\{p \in A : [q] \cap [p] \neq \emptyset\}| < \mathfrak{c}$.

In particular, by (b), $|{}^{\omega}2 \setminus \bigcup B_A| = \mathfrak{c}$.

Proof: The assertion (1) is Lemma 1.1 in [4] and it clearly follows from (2). The following proof of (2) is a modification of the proof in [4].

Let $\{q_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathbb{S}$ such that for each $q \in \mathbb{S}$, $q = q_\alpha$ for $\mathfrak{c}$ many α's, and let $\{y_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of ${}^{\omega}2$ without repetitions.

Let A' be a maximal antichain in $\mathbb{S}$ such that the set $\{[p] \cap [s] : p \in A'\}$ has size $\mathfrak{c}$ for every $s \in {}^{<\omega}2$ (for example, find pairwise disjoint perfect sets $[p_s] \subseteq [s]$, $s \in {}^{<\omega}2$ and split each $[p_s]$ into $\mathfrak{c}$ many disjoint perfect sets). Without loss of generality we can assume that $D \subseteq \{p : \exists q \in A'\ p \leq q\}$. By induction on $\alpha < \mathfrak{c}$ we construct $p_\alpha \in D$, countable $A'_\alpha \subseteq D$, and $x_\alpha \in {}^{\omega}2$. Let us assume that p_β, A'_β, x_β for $\beta < \alpha$ have been constructed and that the set $A''_\alpha = \bigcup_{\beta<\alpha} A'_\beta \cup \{p_\beta\}$ is an antichain.

If the set $[q_\alpha] \setminus (\{x_\beta : \beta < \alpha\} \cup \bigcup B_{A''_\alpha})$ is nonempty, then let x_α be its element; otherwise let $x_\alpha = x_0$.

If q_α is compatible with some $p \in A''_\alpha$, then we set $p_\alpha = p_0$. Otherwise the set

$$\begin{aligned} X_\alpha = \{x_\beta : \beta \leq \alpha\} \cup \{y_\beta : \beta < \alpha\} \cup ([q_\alpha] \cap \textstyle\bigcup B_{A''_\alpha}) \\ \cup \textstyle\bigcup \{[q_\beta] \cap [q_\alpha] : \beta < \alpha \text{ and } q_\beta \wedge q_\alpha = 0\} \end{aligned}$$

has size $< \mathfrak{c}$ and let $p_\alpha \in D$, $p_\alpha \leq q_\alpha$, be such that $[p_\alpha] \cap X_\alpha = \emptyset$. Notice that if $p_\alpha \neq p_0$, then $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

If $y_\alpha \in \bigcup B_{A''_\alpha \cup \{p_\alpha\}}$, then we set $A'_\alpha = \{p_0\}$. Assume that $y_\alpha \notin \bigcup B_{A''_\alpha \cup \{p_\alpha\}}$. By the assumption put on D the antichain $A''_\alpha \cup \{p_\alpha\}$ is nowhere locally maximal and for every $n \in \omega$ there is $r'_{\alpha,n}$ such that $p \wedge r'_{\alpha,n} = 0$ for $p \in A''_\alpha \cup \{p_\alpha\}$. The set

$$\begin{aligned} X_{\alpha,n} = \{x_\beta : \beta \leq \alpha\} \cup \{y_\beta : \beta \leq \alpha\} \cup ([r'_{\alpha,n}] \cap \textstyle\bigcup B_{A''_\alpha \cup \{p_\alpha\}}) \\ \cup \textstyle\bigcup \{[q_\beta] \cap [r'_{\alpha,n}] : \beta < \alpha \text{ and } q_\beta \wedge r'_{\alpha,n} = 0\} \end{aligned}$$

has size $< \mathfrak{c}$. Let $r_{\alpha,n} \in D$, $r_{\alpha,n} \leq r'_{\alpha,n}$ be such that $[r_{\alpha,n}] \cap X_{\alpha,n} = \emptyset$ and let $A'_\alpha = \{r_{\alpha,n} : n \in \omega\}$. Then $r_{\alpha,n} = \tau(\bigvee A'_\alpha, y_\alpha, n)$ and $[\bigvee A'_\alpha] = \{y_\alpha\} \cup \bigcup_{n\in\omega} [r_{\alpha,n}]$.

By the construction it is clear that $A = \bigcup \mathcal{A}$ is a maximal antichain in $\mathbb{S}$ refining the antichain A'. It follows that its size is $\mathfrak{c}$. Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the family $\mathcal{A}$ without repetitions and let $\bar{A} = \{\bigvee A_\alpha : \alpha < \mathfrak{c}\}$. Then $\bar{A}$ is a maximal antichain in $\mathbb{S}$. B_A is a disjoint family and as $A'_\alpha \neq \{p_0\}$ if and only if $y_\alpha \notin \bigcup B_A$, $[\bigvee A'_\alpha] = \{y_\alpha\} \cup \bigcup B_{A'_\alpha}$ whenever $A'_\alpha \neq \{p_0\}$. Therefore $B_{\bar{A}}$ is a partition of ${}^{\omega}2$ and condition (a) is satisfied. We prove conditions (b) and (c). Let $q \in \mathbb{S}$ be arbitrary.

(b) If the set $\{p \in A : [p] \cap [q] \neq \emptyset\}$ has size $\mathfrak{c}$, then, for every α such that $q_\alpha = q$, the set $[q_\alpha] \setminus \bigcup B_{A''_\alpha}$ has size $\mathfrak{c}$ and hence $x_\alpha \neq x_\beta$ for all $\beta < \alpha$. Therefore the set $\{x_\alpha : q_\alpha = q\}$ has size $\mathfrak{c}$ and is a subset of $[q] \setminus \bigcup B_A$.

(c) There is $\beta < \mathfrak{c}$ such that $q = q_\beta$. Let us assume that the set $B = \{p \in A : q \wedge p \neq 0\}$ has size $< \mathfrak{c}$. Let $\gamma > \beta$ be such that $B \subseteq A''_\gamma$. We prove that the set $\{p \in A : [q] \cap [p] \neq \emptyset\}$ is a subset of A''_γ and hence it has size $< \mathfrak{c}$.

For every $\alpha \geq \gamma$, if $p_\alpha \notin A''_\gamma$, then $p_\alpha \neq p_0$ and $q_\beta \wedge q_\alpha = 0$. Therefore $p_\alpha \leq q_\alpha$ is such that $[q_\beta] \cap [p_\alpha] = \emptyset$.

For every $\alpha \geq \gamma$, if $A'_\alpha \setminus A''_\gamma \neq \emptyset$, then $A'_\alpha \neq \{p_0\}$ and $A'_\alpha = \{r_{\alpha,n} : n \in \omega\}$ where $r_{\alpha,n} \leq r'_{\alpha,n}$ and $p \wedge r'_{\alpha,n} = 0$ for all $p \in A''_\alpha \supseteq A''_\gamma$, $n \in \omega$. It follows that $q_\beta \wedge r'_{\alpha,n} = 0$ and hence $r_{\alpha,n} \leq r'_{\alpha,n}$ is such that $[r_{\alpha,n}]$ is disjoint from $[q_\beta]$. So, if $A'_\alpha \neq \{p_0\}$, then $[q_\beta] \cap [p] = \emptyset$ for all $p \in A'_\alpha$. $\square$

Let us consider the following families:

$$\begin{aligned}
\mathcal{A}_1 &= \{A : A \text{ is a maximal antichain in } \mathbb{S} \text{ and } B_A \text{ is a disjoint family}\},\\
\mathcal{A}_2 &= \{B : B \text{ is a partition of } {}^\omega 2 \text{ into closed sets}\},\\
\mathcal{A}_3 &= \{A : A \text{ is a maximal antichain in } \mathbb{S},\ B_A \text{ is a disjoint family, and the}\\
&\qquad \text{set } {}^\omega 2 \setminus \textstyle\bigcup B_A \text{ has size } \mathfrak{c}\},\\
\mathcal{A}_4 &= \{A : A \text{ is a maximal antichain in } \mathbb{S},\ B_A \text{ is a disjoint family, and the}\\
&\qquad \text{set } {}^\omega 2 \setminus \textstyle\bigcup B_A \text{ is uncountable}\}.
\end{aligned}$$

By Theorem 2.4 all these families are nonempty and by Theorem 2.3 the families $\mathcal{A}_3$ and $\mathcal{A}_4$ do not contain countable antichains. Let us define the cardinals:

$$\begin{aligned}
\mathfrak{a}_i &= \min\{|A| : X \in \mathcal{A}_i \text{ and } |A| \geq \omega_1\}, && i = 1, 2, 3, 4,\\
\tilde{\mathfrak{a}}_i &= \sup\{|A|^+ : A \in \mathcal{A}_i \text{ and } |A| < \mathfrak{c}\} \cup \{\omega_1\}, && i = 1, 2, 3, 4.\\
\mathrm{cov}_1 &= \min\{|B| : B \text{ is a family of perfect sets such that the set } {}^\omega 2 \setminus \textstyle\bigcup B\\
&\qquad \text{is uncountable and does not contain a perfect set}\},\\
\mathrm{cov}_2 &= \min\{|B| : B \text{ is a family of perfect sets such that the set } {}^\omega 2 \setminus \textstyle\bigcup B\\
&\qquad \text{has size } \mathfrak{c} \text{ and does not contain a perfect set}\}.
\end{aligned}$$

Theorem 2.5. (1) $\mathfrak{d} = \mathrm{cov}_1 \leq \mathfrak{a}(\omega_1, \mathbb{S}) \leq \mathfrak{a}_1 = \mathfrak{a}_4 \leq \min\{\mathfrak{a}_2, \mathfrak{a}_3\}$; $\tilde{\mathfrak{a}}_1 = \tilde{\mathfrak{a}}_4$.

(2) $\mathrm{cov}_1 \leq \mathrm{cov}_2 \leq \mathfrak{a}_3$.

(3) *For every* i, $\tilde{\mathfrak{a}}_i \leq \mathfrak{a}_i$ *if and only if* $\tilde{\mathfrak{a}}_i = \omega_1$ *if and only if* $\mathfrak{a}_i = \mathfrak{c}$.

(4) *For every* i, $\tilde{\mathfrak{a}}_1 \leq \mathfrak{a}_i$ *if and only if* $\mathfrak{a}_i = \mathfrak{c}$.

(5) *If* $\mathfrak{a}_1 = \mathfrak{c}$, *then, for all* i, $\mathfrak{a}_i = \mathfrak{c}$ *and* $\tilde{\mathfrak{a}}_i = \omega_1$.

(6) *If* $\mathfrak{a}_2 = \mathfrak{c}$, *then* $\mathfrak{a}_1 = \mathfrak{a}_3$ *and* $\tilde{\mathfrak{a}}_1 = \tilde{\mathfrak{a}}_3$.

(7) *If* $\mathfrak{a}_3 = \mathfrak{c}$, *then* $\mathfrak{a}_1 = \mathfrak{c}$ *if and only if* $\mathfrak{a}_2 = \mathfrak{c}$.

(8) $\tilde{\mathfrak{a}}_1 = \max\{\tilde{\mathfrak{a}}_2, \tilde{\mathfrak{a}}_3\}$.

Proof: (1) The inequality $\mathfrak{d} \leq \mathrm{cov}_1$ is Theorem 2.3. To prove $\mathrm{cov}_1 \leq \mathfrak{d}$, without loss of generality let us assume that $\mathfrak{c} > \mathfrak{d}$. Let $X = \{x_\alpha, y_\alpha : \alpha < \omega_1\} \subseteq {}^\omega 2$ be a Hausdorff gap (see [3]), i.e., $x_\alpha \leq^* x_\beta \leq^* y_\beta \leq^* y_\alpha$ for $\alpha \leq \beta < \omega_1$, and for every $x \in {}^\omega 2$ there is $\alpha < \omega_1$ such that $x_\alpha \not\leq^* x$ or $x \not\leq^* y_\alpha$. Let $K_\alpha = \{x \in {}^\omega 2 : x_\alpha \not\leq^* x \text{ or } x \not\leq^* y_\alpha\}$ for $\alpha < \omega_1$. Then $K_\alpha \subseteq K_\beta$ for $\alpha \leq \beta$, $K_\alpha \cap X$ is countable, and consequently, the sets $K_\alpha \setminus X$, $\alpha < \omega_1$, are G_δ sets covering ${}^\omega 2 \setminus X$. The Baire space ${}^\omega\omega$ is a union of $\mathfrak{d}$ many compact sets and as every Polish space is a continuous image of ${}^\omega\omega$, every Polish space is a union of $\leq \mathfrak{d}$ compact sets. It follows that every set $K_\alpha \setminus X$ a union of $\leq \mathfrak{d}$ compact sets and hence ${}^\omega 2 \setminus X$ is a union of $\leq \mathfrak{d}$ compact sets. Considering the perfect kernels of these compacts (obtained by removing countable sets) we obtain a family of $\leq \mathfrak{d}$ perfect subsets of ${}^\omega 2$ whose union has uncountable complement of size $< \mathfrak{c}$ and hence $\mathrm{cov}_1 \leq \mathfrak{d}$.

Let us assume that $\mathfrak{a}(\omega_1, \mathbb{S}) < \mathrm{cov}_1$ and we prove a contradiction. Let $A \subseteq \mathbb{S}$ be a maximal antichain of size $\mathfrak{a}(\omega_1, \mathbb{S})$. The set $X = \bigcup\{[p] \cap [q] : p, q \in A, p \neq q\}$ has size $< \mathfrak{c}$. For every $p \in A$ let $x_p \in [p] \setminus X$ be arbitrary. The family $A' = \{\tau(p, x_p, n) : p \in A \text{ and } n \in \omega\}$ is a maximal antichain in $\mathbb{S}$ because if $[p] \cap [q]$ is uncountable for some $p \in A$, then $[\tau(p, x_p, n)] \cap [q]$ is uncountable for some n. The set $Y = {}^\omega 2 \setminus \bigcup B_{A'}$ is uncountable as it contains the set $\{x_p : p \in A\}$ and as $\mathfrak{a}(\omega_1, \mathbb{S}) < \mathrm{cov}_1$, there is a perfect set $[q] \subseteq Y$. But $[p] \cap [q] \subseteq \{x_p\}$ for all $p \in A$ which contradicts the assumption that A is maximal. Therefore $\mathrm{cov}_1 \leq \mathfrak{a}(\omega_1, \mathbb{S})$.

The inequality $\mathfrak{a}_4 \leq \mathfrak{a}_1$ can be easily proved by the same argument. Therefore $\mathfrak{a}_1 = \mathfrak{a}_4$ and by the same proof we obtain $\tilde{\mathfrak{a}}_1 = \tilde{\mathfrak{a}}_4$. The other inequalities are trivial.

(2) is an easy consequence of definitions.

(3–4) The implications from the right to the left are obvious. Let us assume that $\mathfrak{a}_i < \mathfrak{c}$ for some i. Then $\mathfrak{a}_i < \mathfrak{a}_i^+ \leq \tilde{\mathfrak{a}}_i$ and $\tilde{\mathfrak{a}}_i \leq \tilde{\mathfrak{a}}_1$.

(5) By (1), for all i, $\mathfrak{a}_i = \mathfrak{c}$ and by (3), $\tilde{\mathfrak{a}}_i = \omega_1$.

(6) If there is a maximal antichain $A \subseteq \mathbb{S}$ of size $< \mathfrak{c}$ such that the family B_A is disjoint and the set $X = {}^\omega 2 \setminus \bigcup B_A$ has size $< \mathfrak{c}$, then the partition $B = B_A \cup \{\{x\} : x \in X\}$ has size $< \mathfrak{c}$.

(7) Let $\mathfrak{a}_3 = \mathfrak{c}$. If $\mathfrak{a}_2 = \mathfrak{c}$, then, by (6), $\mathfrak{a}_1 = \mathfrak{a}_3 = \mathfrak{c}$.

(8) $\tilde{\mathfrak{a}}_1 \geq \tilde{\mathfrak{a}}_2$ and $\tilde{\mathfrak{a}}_1 \geq \tilde{\mathfrak{a}}_3$. Let us assume that $\tilde{\mathfrak{a}}_3 < \tilde{\mathfrak{a}}_1$. For any κ with $\tilde{\mathfrak{a}}_3 \leq \kappa < \tilde{\mathfrak{a}}_1$ there is an antichain $A \in \mathcal{A}_1 \setminus \mathcal{A}_3$ of size $< \mathfrak{c}$ and $\geq \kappa$. Then the partition $B_A \cup \{\{x\} : x \in {}^\omega 2 \setminus \bigcup B_A\}$ has size $< \mathfrak{c}$ and $\geq \kappa$. Therefore $\tilde{\mathfrak{a}}_2 > \kappa$ and so $\tilde{\mathfrak{a}}_2 = \tilde{\mathfrak{a}}_1$. □

Clearly, $\mathfrak{a}(\omega, \mathbb{S}) = \omega$. There are known several constructions of small uncountable antichains in $\mathbb{S}$. J. Stern and independently K. Kunen (for the proof see [8]) under CH constructed a partition of ${}^\omega 2$ into ω_1 compact sets. L. Newelski [9] pointed out that under MA the same construction produces a partition into $\mathfrak{c}$ compact sets which is preserved by forcing with measure algebras and he proved that after adding ω_1 dominating reals, the Baire space ${}^\omega\omega$ (and hence, by Lemma 2.2, also the Cantor space ${}^{\omega}2$) can be partitioned into ω_1 disjoint compact perfect sets. A. Rosłanowski and S. Shelah [10], by a finite support iteration of c.c.c. forcing notions of length ω_1, constructed a maximal antichain A such that the family B_A is disjoint and every tree $p \in A$ has on each level at most one branching node. Moreover, the set $\bigcup B_A$ does not contain any ground model reals and therefore $\mathfrak{a}_3 = \omega_1$ holds in the extension.

We say that a set $a \subseteq {}^{<\omega}2$ is saturated if for every $s, t \in {}^{<\omega}2$ whenever $s \subseteq t$ and $t \in a$, then $s \in a$. Easily, it can be observed that $\mathfrak{a}_2$ is the minimal size of a family A, maximal with respect to the inclusion, such that A is an uncountable almost disjoint family of infinite saturated sets. Notice that such a family A cannot be a maximal almost disjoint family of infinite subsets of ${}^{<\omega}2$. To see this, let $a \in A$ be such that the set of all infinite branches in a is nowhere dense in ${}^{\omega}2$ and let $x \in a$ be arbitrary. For every n choose $s_n \in {}^{<\omega}2$ such that $x{\restriction}n \subseteq s_n$ and $s_n \notin a$. Then the set $\{s_n : n \in \omega\}$ has a finite intersection with every $b \in A$. The similarity of this characterization of $\mathfrak{a}_2$ with maximal almost disjoint families suggests the question whether there is some relation between $\mathfrak{a}_2$ and $\mathfrak{a}$ (the minimal size of a maximal almost disjoint family of subsets of ω).

3. Marczewski's ideal and the collapse by Sacks forcing

A subset X of ${}^{\omega}2$ is an s^0-set if for every $p \in \mathbb{S}$ there is $q \leq p$ such that $[q] \cap X = \emptyset$. This notion is due to E. Marczewski [7]. It is known that $\omega_1 \leq \mathrm{add}(s^0) \leq \mathrm{cov}(s^0) \leq \mathrm{cf}(\mathfrak{c}) \leq \mathrm{non}(s^0) = \mathfrak{c} < \mathrm{cf}(\mathrm{cof}(s^0))$ (see [4]) and $\mathrm{add}(s^0) \leq \mathfrak{b}$ (in fact $\mathrm{sh}(\mathbb{S}) \leq \mathfrak{b}$ see [11]; this is not true for $\mathrm{cov}(s^0)$ because in the iterated Sacks forcing model $\mathrm{cov}(s^0) = \omega_2$ see [4] but $\mathfrak{b} = \mathrm{cof}(\mathcal{N}) = \omega_1$). Notice that $\mathrm{add}(I) \leq \mathrm{cf}(\mathrm{non}(I))$ for each ideal I. If $y \in {}^{\omega}2$ is a new real, then the perfect set $A_y = \{x \in {}^{\omega}2 : (\forall n)\, x(2n) = y(n)\}$ does not contain old reals. This explains why in iterations of length ω_1 the set of old reals is an s^0-set and $\mathrm{cov}(s^0) = \omega_1$. To see that there are s^0-sets of size $\mathfrak{c}$ (see also [4]), take any maximal antichain $\{p_\alpha : \alpha < \mathfrak{c}\}$ of size $\mathfrak{c}$ in $\mathbb{S}$ so that the system of perfect sets $B_A = \{[p_\alpha] : \alpha < \mathfrak{c}\}$ is disjoint and clearly, every selector of this system is an s^0-set. By Theorem 2.4(2) every s^0-set has this form. If B_A is not disjoint, then its selectors need not be s^0-sets (observe that the system $\{A_y : y \in {}^{\omega}2\}$ has a perfect selector).

The next theorem refines Theorem 1.1 in [4].

Theorem 3.1. (1) $\mathrm{sh}_{\mathfrak{a}_3}(\mathbb{S}) \leq \mathrm{add}(s^0) \leq \mathrm{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S}) \leq \mathrm{sh}(\mathbb{S}) \leq \min\{\mathrm{cf}\,\mathfrak{c}, \mathfrak{b}\}$.

(2) $\mathrm{sh}_{\omega_1}(\mathbb{S}) = \mathrm{sh}_{\mathfrak{a}_1}(\mathbb{S}) = \min\{\mathrm{sh}_{\mathfrak{a}_2}(\mathbb{S}), \mathrm{add}(s^0)\} \leq \mathrm{sh}_{\mathfrak{a}_3}(\mathbb{S})$.

(3) $\mathrm{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S}) \leq \max\{\mathrm{sh}_{\tilde{\mathfrak{a}}_3}(\mathbb{S}), \mathrm{add}(s^0)\} = \mathrm{sh}_{\tilde{\mathfrak{a}}_1}(\mathbb{S}) = \mathrm{sh}(\mathbb{S})$.

(4) $\mathrm{sh}_{\omega_1}(\mathbb{S}) \leq \mathrm{sh}_{\mathrm{cf}\,\mathfrak{c}}(\mathbb{S}) \leq \mathrm{cf}_\pi(\mathbb{S}) \leq \mathrm{sh}(\mathbb{S})$.

(5) $\mathrm{sh}_{\mathrm{cf}\,\mathfrak{c}}(\mathbb{S}) \leq \mathrm{cf}\,\mathrm{sh}(\mathbb{S})$, *and if* $\mathrm{sh}(\mathbb{S})$ *is singular*, *then* $\mathrm{sh}_\kappa(\mathbb{S}) < \mathrm{sh}(\mathbb{S})$ *for* $\kappa < \mathfrak{c}$, $\tilde{\mathfrak{a}}_1 = \tilde{\mathfrak{a}}_3 = \mathfrak{c}$, *and* $\mathfrak{c}$ *is singular*.

(6) *If* $\max\{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3\} = \mathfrak{c}$, *then* $\mathrm{add}(s^0) = \mathrm{sh}_{\mathfrak{a}_3}(\mathbb{S}) = \mathrm{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S})$.

(7) *If* $\mathfrak{a}_1 = \mathfrak{c}$, *then*, *for every* κ *with* $\omega_1 \leq \kappa \leq \mathfrak{c}$, $\mathrm{add}(s^0) = \mathrm{sh}_\kappa(\mathbb{S}) = \mathrm{cf}_\pi(\mathbb{S})$.

(8) *If* $\mathfrak{a}_2 = \mathfrak{c}$, *then* $\mathrm{add}(s^0) = \mathrm{sh}_{\omega_1}(\mathbb{S})$.
(9) *If* $\mathfrak{a}_3 = \mathfrak{c}$, *then* $\mathrm{add}(s^0) = \mathrm{sh}(\mathbb{S})$.
(10) *If* $\mathfrak{a}(\mathrm{cf}\,\mathfrak{c}, \mathbb{S}) = \mathfrak{c}$, *then* $\mathrm{sh}(\mathbb{S}) = \mathrm{cf}_\pi(\mathbb{S}) = \mathrm{sh}_{\mathrm{cf}\,\mathfrak{c}}(\mathbb{S})$.

In particular, if $\mathfrak{d} = \mathfrak{c}$, *then the assumptions of* (6)–(10) *are satisfied, and if* $\mathfrak{c}$ *is regular, then the assumption of* (10) *is satisfied.*

Here is the picture of the inequalities between the cardinals:

$$\begin{tikzcd}
 & \mathrm{sh}_{\mathfrak{a}_2}(\mathbb{S}) \arrow[drrr] & & \mathrm{sh}_{\tilde{\mathfrak{a}}_3}(\mathbb{S}) \arrow[dr] & \\
\mathrm{sh}_{\omega_1}(\mathbb{S}) = \mathrm{sh}_{\mathfrak{a}_1}(\mathbb{S}) \arrow[ur] \arrow[urrr] \arrow[r] \arrow[dr] & \mathrm{sh}_{\mathfrak{a}_3}(\mathbb{S}) \arrow[r] & \mathrm{add}(s^0) \arrow[r] & \mathrm{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S}) \arrow[r] & \mathrm{sh}_{\tilde{\mathfrak{a}}_1}(\mathbb{S}) = \mathrm{sh}(\mathbb{S}) \\
 & \mathrm{sh}_{\mathrm{cf}\,\mathfrak{c}}(\mathbb{S}) \arrow[rr] & & \mathrm{cf}_\pi(\mathbb{S}) \arrow[ur] &
\end{tikzcd}$$

Proof: (1) $\mathrm{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S}) \le \mathrm{sh}(\mathbb{S})$ because $\tilde{\mathfrak{a}}_2 \le \mathfrak{c}$, $\mathrm{sh}(\mathbb{S}) \le \mathrm{cf}\,\mathfrak{c}$ by Theorem 1.1(8). We shall sketch a proof of the inequality $\mathrm{sh}(\mathbb{S}) \le \mathfrak{b}$ which a little simplifies the proof presented in [11]. Let us recall some notation.

For $p \in \mathbb{S}$ let $f_p \in {}^\omega\omega$ be such that for every n and every $s \in {}^{f_p(n)}2$ there is a splitting node $t \in {}^{<f_p(n+1)}2$ above s in p. For $p \in \mathbb{S}$ and $a \subseteq \omega$, $p[a]$ is a subtree of p defined by induction: (i) $\emptyset \in p[a]$; (ii) Let $s \in p[a]$ and $\mathrm{dom}\, s = n$. If $n \in a$, then, for $i = 0,\ 1$, $s^\frown i \in p[a]$ if and only if $s^\frown i \in p$. If $n \notin a$, then, for $i = 0$, 1, $s^\frown i \in p[a]$ if and only if $i = 0$ and $s^\frown 0 \in p$ or $i = 1$ and $s^\frown 0 \notin p$.

If $p, q \in \mathbb{S}$ and $a, b \subseteq \omega$, then $p[a] \cap q[b] = (p \cap q)[a \cap b]$, and if $[f_p(n), f_p(n+1)) \subseteq a$ for infinitely many n, then $p[a] \in \mathbb{S}$.

We shall construct a base matrix on $\mathbb{S}$ of size $\mathfrak{b}$ using the fact that $\mathfrak{h} \le \mathfrak{b}$ where $\mathfrak{h}$ is the minimal size of a base matrix on $\mathcal{P}(\omega)/\mathit{fin}$ (see [2]). Let $\mathcal{F} \subseteq {}^\omega\omega$ be an unbounded family of increasing functions and let $\{B_\alpha : \alpha < \mathfrak{h}\}$ be a base matrix on $\mathcal{P}(\omega)/\mathit{fin}$. If $p \in \mathbb{S}$, then there is an $f \in \mathcal{F}$ such that the set $x_p = \{n : |[f(n), f(n+1)) \cap \mathrm{rng}\, f_p| \ge 2\}$ is infinite and so there is $\alpha < \mathfrak{h}$ and $a \in B_\alpha$ such that $a \subseteq^* x_p$. Now for $f \in \mathcal{F}$ and $a \in \bigcup_{\alpha<\mathfrak{h}} B_\alpha$ let $\mathbb{S}_{f,a}$ be the set of all $p \in \mathbb{S}$ such that $|[f(n), f(n+1)) \cap \mathrm{rng}\, f_p| \ge 2$ for all but finitely many $n \in a$. As $\mathbb{S}_{f,a}$ has size $\le \mathfrak{c}$, we can assign, in a one-to-one way, for each $p \in \mathbb{S}_{f,a}$ an infinite set $b_{f,a,p} \subseteq a$ so that the system $\{g_{f,a,p} : p \in \mathbb{S}_{f,a}\}$ is almost disjoint. Let $c_{f,a,p} = \bigcup\{[f(n), f(n+1)) : n \in b_{f,a,p}\}$. Then $\{c_{f,a,p} : a \in B_\alpha \text{ and } p \in \mathbb{S}_{f,a}\}$ is an almost disjoint family and hence the system $A_{f,\alpha} = \{p[c_{f,a,p}] : a \in B_\alpha \text{ and } p \in \mathbb{S}_{f,a}\}$ is an antichain in $\mathbb{S}$ refining $\bigcup_{a\in B_\alpha} \mathbb{S}_{f,a}$. Therefore $\{A_{f,\alpha} : f \in \mathcal{F} \text{ and } \alpha < \mathfrak{h}\}$ is a base matrix on $\mathbb{S}$.

$\mathrm{sh}_{\mathfrak{a}_3}(\mathbb{S}) \le \mathrm{add}(s^0)$: Let $\kappa < \mathrm{sh}_{\mathfrak{a}_3}(\mathbb{S})$ and let X_α, $\alpha < \kappa$, be s^0-sets. We prove that the set $X = \bigcup_{\alpha<\kappa} X_\alpha$ is an s^0-set and hence $\kappa < \mathrm{add}(s^0)$. Let A_α, $\alpha < \kappa$, be maximal antichains in $\mathbb{S}$ such that $X_\alpha \cap B_{A_\alpha} = \emptyset$. By Theorem 2.4(1) we can assume that for every $\alpha < \kappa$, B_{A_α} is a disjoint family. Let $q \in \mathbb{S}$ be arbitrary. By $(\kappa, \mathfrak{c}, \mathfrak{a}_3)$-distributivity of r. o.($\mathbb{S}$) there is $q' \le q$ such that for every α the set $A'_\alpha = \{p \in A_\alpha : q' \wedge p \ne 0\}$ has size $< \mathfrak{a}_3$. By the definition of $\mathfrak{a}_3$ it follows that every set $Y_\alpha = [q'] \setminus \bigcup B_{A'_\alpha}$ has size $< \mathfrak{c}$ and as $\kappa < \operatorname{cf} \mathfrak{c}$, the set $X \cap [q'] \subseteq \bigcup_{\alpha<\kappa} Y_\alpha$ has size $< \mathfrak{c}$. Therefore there is $r \le q'$ such that $X \cap [r] = \emptyset$.

$\operatorname{add}(s^0) \le \operatorname{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S})$: Let $\kappa < \operatorname{add}(s^0)$ and let $\{A_\alpha : \alpha < \kappa\}$ be a system of maximal antichains in $\mathbb{S}$. We prove that for every $q \in \mathbb{S}$ there is $r \le q$ such that for every $\alpha < \kappa$ the set $\{p \in A_\alpha : r \wedge p \ne 0\}$ has size $< \mathfrak{a}_2$ and hence $\kappa < \operatorname{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S})$. By refining the antichains, if necessary, we can assume without loss of generality that they all satisfy the conditions in Theorem 2.4(1). By the additivity assumption, the set $X = \bigcup_{\alpha<\kappa}({}^\omega 2 \setminus \bigcup B_{A_\alpha})$ is an s^0-set. Let $q \in S$. There is $r \le q$ such that $X \cap [r] = \emptyset$ and hence for every α, $[r] \subseteq \bigcup B_{A_\alpha}$. By Theorem 2.4(1) then, for every α, $C_\alpha = \{p \in A_\alpha : [r] \cap [p] \ne \emptyset\}$ has size $< \mathfrak{c}$ and by the definition of $\tilde{\mathfrak{a}}_2$ we have $|C_\alpha| < \tilde{\mathfrak{a}}_2$.

(2) We prove only $\min\{\operatorname{sh}_{\mathfrak{a}_2}(\mathbb{S}), \operatorname{add}(s^0)\} \le \operatorname{sh}_{\omega_1}(\mathbb{S})$; all the remaining inequalities of this part of the theorem hold due to the monotonicity of the invariants $\operatorname{sh}_\kappa(\mathbb{S})$ and part (1).

Let $\kappa < \min\{\operatorname{sh}_{\mathfrak{a}_2}(\mathbb{S}), \operatorname{add}(s^0)\}$ and let A_α, $\alpha < \kappa$, be maximal antichains in $\mathbb{S}$. We show that for every $q \in \mathbb{S}$ there is $r \le q$ such that for every $\alpha < \kappa$ the set $\{p \in A_\alpha : r \wedge p \ne 0\}$ is countable. Without loss of generality we can assume that all the antichains A_α satisfy conditions in Theorem 2.4(2). Given $q \in \mathbb{S}$ by the κ-additivity of s^0 and $(\kappa, \mathfrak{c}, \mathfrak{a}_2)$-distributivity of r. o.($\mathbb{S}$) there is $q' \le q$ such that for each $\alpha < \kappa$, $[q'] \subseteq \bigcup B_{A_\alpha}$ and the set $\{p \in A_\alpha : q' \wedge p \ne 0\}$ has size $< \mathfrak{a}_2$. By condition (c) in Theorem 2.4(2), as $\kappa < \operatorname{cf} \mathfrak{c}$, the set $X = \bigcup_{\alpha<\kappa} \bigcup\{[q'] \cap [p] : p \in A_\alpha$ and $q' \wedge p = 0\}$ has size $< \mathfrak{c}$. Let $r \le q'$ be such that $X \cap [r] = \emptyset$. Then for each $\alpha < \kappa$ the set $\{p \in A_\alpha : [r] \cap [p] \ne \emptyset\}$ has size $< \mathfrak{a}_2$ and therefore it is countable.

(3) It is clear that $\operatorname{sh}_{\tilde{\mathfrak{a}}_2}(\mathbb{S}) \le \operatorname{sh}(\mathbb{S}) = \operatorname{sh}_{\tilde{\mathfrak{a}}_1}(\mathbb{S})$. Let $\kappa_1 = \operatorname{sh}_{\tilde{\mathfrak{a}}_3}(\mathbb{S})$ and $\kappa_2 = \operatorname{add}(s^0)$. We prove that $\max\{\kappa_1, \kappa_2\} = \operatorname{sh}(\mathbb{S})$. We know that the inequality $\le$ holds true. Let us assume that $\kappa_1, \kappa_2 < \operatorname{sh}(\mathbb{S})$ and we prove a contradiction. Let $\{A'_\alpha : \alpha < \kappa_1\}$ be a system of maximal antichains in $\mathbb{S}$ witnessing the $(\kappa, \mathfrak{c}, \tilde{\mathfrak{a}}_3)$-nowhere distributivity of r. o.($\mathbb{S}$) and let $\{X_\beta : \beta < \kappa_2\}$ be a system of s^0-sets such that for every $q \in \mathbb{S}$, $[q] \cap \bigcup_{\beta<\kappa} X_\beta$ has size $\mathfrak{c}$. For each pair $(\alpha, \beta) \in \kappa_1 \times \kappa_2$ let $A_{\alpha,\beta}$ be a maximal antichain in $\mathbb{S}$ such that $A_{\alpha,\beta}$ refines A'_α and $X_\beta \cap \bigcup B_{A_{\alpha,\beta}} = \emptyset$. We can find $A_{\alpha,\beta}$'s so that the conditions in Theorem 2.4(2) are satisfied. We claim that the system $\{A_{\alpha,\beta} : (\alpha, \beta) \in \kappa_1 \times \kappa_2\}$ is a witness for the $(\kappa_1 \cdot \kappa_2, \mathfrak{c}, \mathfrak{c})$-nowhere distributivity of r. o.($\mathbb{S}$) which contradicts the inequality $\kappa_1 \cdot \kappa_2 < \operatorname{sh}(\mathbb{S})$. To see this let $q \in \mathbb{S}$ be arbitrary. As $\kappa_1 \cdot \kappa_2 < \operatorname{sh}(\mathbb{S})$ there is $r \le q$ such that for every $(\alpha, \beta) \in \kappa_1 \times \kappa_2$ the set $A'_{\alpha,\beta} = \{p \in A_{\alpha,\beta} : r \wedge p = 0\}$ has size $< \mathfrak{c}$. As $[r] \cap \bigcup_{\beta<\kappa_2} X_\beta$ has size $\mathfrak{c}$ and $\kappa_2 < \operatorname{cf} \mathfrak{c}$ there is $\beta < \kappa_2$ such that $[r] \cap X_\beta$ has size $\mathfrak{c}$. As for every α the antichain $A_{\alpha,\beta}$ refines the antichain A'_α, there is $\alpha < \kappa_1$

such that $|A'_{\alpha,\beta}| \geq \tilde{\mathfrak{a}}_3$. Now $[r] \cap X_\beta$ is disjoint from $\bigcup B_{A'_{\alpha,\beta}}$ and $|A'_{\alpha,\beta}| < \mathfrak{c}$. It follows that $\tilde{\mathfrak{a}}_3 \geq |A'_{\alpha,\beta}|^+$ while $|A'_{\alpha,\beta}| \geq \tilde{\mathfrak{a}}_3$. A contradiction.

(4) The inequalities hold true by Theorem 1.2(6) because $\mathrm{sh}_{\omega_1}(\mathbb{S}) \leq \mathrm{sh}_{\mathrm{cf}\,\mathfrak{c}}(\mathbb{S}) \leq \mathrm{sh}(\mathbb{S}) \leq \mathrm{cf}\,\mathfrak{c}$.

(5) The inequalities hold true by Theorem 1.2(8) by which $\mathrm{sh}_\kappa(\mathbb{S})$ is regular for κ regular. Hence if $\mathrm{sh}(\mathbb{S})$ is singular, then $\mathfrak{c}$ is singular, and as $\mathrm{add}(s^0)$ is regular, by (3), $\mathrm{sh}_{\tilde{\mathfrak{a}}_3}(\mathbb{S}) = \mathrm{sh}_{\tilde{\mathfrak{a}}_1}(\mathbb{S}) = \mathrm{sh}(\mathbb{S})$. Therefore, $\tilde{\mathfrak{a}}_1 = \tilde{\mathfrak{a}}_3 = \mathfrak{c}$.

(6)–(9) are easy consequences of the above proved inequalities using the fact that $\mathfrak{a}_i = \mathfrak{c}$ if and only if $\tilde{\mathfrak{a}}_i = \omega_1$.

(10) follows by (4) since under the assumption $\mathrm{sh}(\mathbb{S}) = \mathrm{sh}_{\mathrm{cf}\,\mathfrak{c}}(\mathbb{S})$. □

By Theorem 3.1(10), if the continuum is regular, then it is collapsed to a regular cardinal of the extension. MA(countable) does not imply the continuum is regular. Anyway, by Theorem 3.1(7), under MA(countable) (even under $\mathfrak{d} = \mathfrak{c}$) Sacks forcing collapses the continuum to a regular cardinal in $V^{\mathrm{r.o.}(\mathbb{S})}$. We think that it is an open question whether Sacks forcing can collapse the continuum to a singular cardinal.

Under some hypotheses (see Theorem 3.1), there is $\kappa \leq \mathfrak{c}$ such that $\mathrm{add}(s^0) = \mathrm{sh}_\kappa(\mathbb{S})$. We do not know whether the same is true in ZFC.

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