Closed subsets of absolutely star-Lindelöf spaces II

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Abstract. In this paper, we prove the following two statements: (1) There exists a discretely absolutely star-Lindelöf Tychonoff space having a regular-closed subspace which is not CCC-Lindelöf. (2) Every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented in a Hausdorff (regular, Tychonoff) absolutely star-Lindelöf space as a closed G_δ subspace.

Keywords: star-Lindelöf space, absolutely star-Lindelöf space

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1. Introduction

By a space, we mean a topological space. A space X is *absolutely star-Lindelöf* (see [1]) (*discretely absolutely star-Lindelöf* (see [8])) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a countable subset $F \subseteq D$ such that $\mathrm{St}(F,\mathcal{U}) = X$ (respectively, F is discrete and closed in X and $\mathrm{St}(F,\mathcal{U}) = X$), where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$.

A space X is *star-Lindelöf* (see [4], [8] – under different name) if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X such that $\mathrm{St}(F,\mathcal{U}) = X$. It is clear that very separable space is star-Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

A space X is *centered-Lindelöf* (*linked-Lindelöf*, *CCC-Lindelöf*) (see [2], [3]) if every open cover has a σ-centered (σ-linked, CCC, respectively) subcover. A family of sets is *centered* (*linked*) if every finite subfamily (every two elements, respectively) has non-empty intersection and a family is σ-centered (σ-linked) if it can be represented as the union of countably many centered-subfamilies (linked-subfamilies, respectively). A family of nonempty sets is a *CCC-family* if there is no uncountable pairwise disjoint subfamily.

From the above definitions, it is not difficult to see that every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf, every absolutely star-Lindelöf space is star-Lindelöf, every star-Lindelöf space is centered-Lindelöf, every centered-Lindelöf space is linked-Lindelöf and every linked-Lindelöf space is CCC-Lindelöf.

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Bonanzinga and Matveev [2] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed subspace in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. They asked if every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed G_δ subspace in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. Song [8] gave a positive answer to their question. Moreover, Song and Shi [9] showed that every centered-Lindelöf Tychonoff space can be represented as a closed G_δ subspace in a Tychonoff absolutely star-Lindelöf space. But their construction does not work in the classes of Hausdorff and regular spaces. Thus, it is natural for us to consider the following more general question:

Question. Is it true that every Tychonoff (Hausdorff, regular) star-Lindelöf (centered-Lindelöf, linkered-Lindelöf) space can be embedded into some Tychonoff (Hausdorff, regular, respectively) absolutely star-Lindelöf space as a closed subspace? And can it be embedded as a closed G_δ subspace?

Throughout the paper, the cardinality of a set A is denoted by $|A|$. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. Let $\mathfrak{c}$ denote the cardinality of the continuum and ω the first infinite cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols will be used as in [5].

2. Closed subspaces of absolutely star-Lindelöf spaces

In [1], Bonanzinga showed that a regular-closed subspace of a star-Lindelöf space need not be star-Lindelöf, and in [9], Song and Shi showed that a regular-closed subspace of an absolutely star-Lindelöf space need not be absolutely star-Lindelöf. In the following, we give a stronger example to show that a regular-closed subspace of a discretely absolutely star-Lindelöf space need not be CCC-Lindelöf.

Recall that the Alexandroff duplicate of a space X, denoted by $A(X)$, is constructed in the following way: the underlying set of $A(X)$ is $X \times \{0,1\}$ and each point of $X \times \{1\}$ is isolated; a basic neighborhood of a point $\langle x,0\rangle \in X \times \{0\}$ is a set of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x,1\rangle\})$, where U is a neighborhood of x in X. It is well-known that $A(X)$ is Hausdorff (regular, Tychonoff, normal) iff so is X and $A(X)$ is compact iff so is X.

Recall from [6] that a space X is *absolutely countably compact* if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a finite subset $F \subseteq D$ such that $\mathrm{St}(F,\mathcal{U}) = X$. Vaughan [10] proved that every countably compact GO-space is absolutely countably compact. Thus, every cardinal with uncountable cofinality is absolutely countably compact. In the next example we use the following lemma from [11].

Lemma 2.1. *If X is countably compact, then $A(X)$ is acc.*

Given a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.2. *There exists a discretely absolutely star-Lindelöf Tychonoff space X having a regular-closed subspace which is not CCC-Lindelöf.*

PROOF: Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Define $S_1 = (\mathfrak{c}^+ \times \omega) \cup \mathcal{R}$. We topologize X as follows: $\mathfrak{c}^+ \times \omega$ has the usual product topology and is an open subspace of S_1, and a basic neighborhood of $r \in \mathcal{R}$ takes the form

$$G_{\beta,K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}^+\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}^+$ and a finite subset K of ω. Then, the space S_1 is Tychonoff and $e(S_1) = \mathfrak{c}$, because $\mathcal{R}$ is discrete and closed in X. Now, we show that S_1 is discretely absolutely star-Lindelöf. For this end, let $\mathcal{U}$ be an open cover of S_1. Let S be the set of all isolated points of $\mathfrak{c}^+$ and let $T = S \times \omega$. Then, T is dense in X and every dense subspace of X includes T. Thus, it suffices to show that there exists a countable subset $F \subseteq T$ such that F is discrete closed in X and $\mathrm{St}(F,\mathcal{U}) = S_1$. For each $n < \omega$, since $\mathfrak{c}^+ \times \{n\}$ is absolutely countably compact, there exists a finite subset $F_n \subseteq S \times \{n\}$ such that $\mathfrak{c}^+ \times \{n\} \subseteq \mathrm{St}(F_n,\mathcal{U})$. Let $F' = \bigcup\{F_n : n \in \omega\}$. Then, $\mathfrak{c}^+ \times \omega \subseteq \mathrm{St}(F',\mathcal{U})$. For each $x \in \mathcal{R}$, take $U_x \in \mathcal{U}$ with $x \in U_x$, and fix $\alpha_x < \mathfrak{c}^+$ and $n_x \in \omega$ such that $\{\langle n_x, \alpha\rangle : \alpha_x < \alpha < \mathfrak{c}^+\} \subseteq U_x$. For each $n \in \omega$, let $X_n = \{x \in \mathcal{R} : n_x = n\}$ and choose $\beta_n \in S$ with $\beta_n > \sup\{\alpha_x : x \in X_n\}$. Then, $X_n \subseteq \mathrm{St}(\langle \beta_n, n\rangle,\mathcal{U})$. It is quicker to choose $\beta \in S$ such that $\beta_n < \beta$ for all $n \in \omega$. Thus, if we put $F'' = \{\langle \beta, n\rangle : n \in \omega\}$, then $\mathcal{R} \subseteq \mathrm{St}(F'',\mathcal{U})$. Let $F = F' \cup F''$. Then, F is a countable subset of D such that $S_1 = \mathrm{St}(F,\mathcal{U})$. Since $F \cap (\mathfrak{c}^+ \times \{n\})$ is finite for each $n < \omega$, F is discrete and closed in S_1, which shows that S_1 is discretely absolutely star-Lindelöf.

Let D be a discrete space of cardinality $\mathfrak{c}$ and let

$$S_2 = A((\beta D \times (\mathfrak{c}^+ + 1)) \setminus ((\beta D \setminus D) \times \{\mathfrak{c}^+\})) \setminus ((D \times \{\mathfrak{c}^+\}) \times \{1\}).$$

We show that S_2 is not CCC-Lindelöf. Since $|D| = \mathfrak{c}$, we can enumerate D as $\{d_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = (\{d_\alpha\} \times [0, \mathfrak{c}^+]) \times \{0, 1\} \setminus \{\langle\langle d_\alpha, \mathfrak{c}^+\rangle, 0\rangle\}.$$

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{[0, \mathfrak{c}) \times \beta D\} \times \{0, 1\}$$

of S_2. This contains an uncountable pairwise disjoint subfamily $\{U_\alpha : \alpha < \mathfrak{c}\}$ and since U_α is the only element of $\mathcal{U}$ containing $\langle\langle \mathfrak{c}^+, d_\alpha\rangle, 0\rangle$, S_2 is not CCC-Lindelöf.

We assume that $S_1 \cap S_2 = \emptyset$. Let $\varphi : \mathcal{R} \to \{\mathfrak{c}^+\} \times D \times \{0\}$ be a bijection. Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying r with $\varphi(r)$ for each $r \in \mathcal{R}$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map. It is easy to check that $\pi(S_2)$ is a regular-closed subset of X, however, it is not CCC-Lindelöf, since it is homeomorphic to S_2.

Now, we show that X is discretely absolutely star-Lindelöf. For this end, let $\mathcal{U}$ be an open cover of X. Let

$$S' = \pi((D \times S \times \{0\}) \cup (\beta D \times [0, \mathfrak{c}^+) \times \{1\})) \cup \pi(T),$$

where S is the set of all isolated point of $\mathfrak{c}^+$. Then, S' is dense in X and every dense subspace of X includes S'. Thus, it is suffices to show that there exists a countable subset $F \subseteq S'$ such that F is discrete and closed in X and $\mathrm{St}(F, \mathcal{U}) = X$. Since $\pi(S_1)$ is homeomorphic to S_1, $\pi(S_1)$ is absolutely discretely star-Lindelöf, and there is a countable subset F_1 of S' such that F_1 is discrete and closed in $\pi(S_1)$ and $\pi(S_1) \subseteq \mathrm{St}(F_1, \mathcal{U})$. Since $\pi(S_1)$ is a closed subset of X, F_1 is discrete and closed in X. On the other hand, since $\mathfrak{c}^+$ is locally compact and countably compact, it follows from [4, Theorem 3.10.13] that $\beta D \times \mathfrak{c}^+$ is countably compact. Thus $\pi(A(\beta D \times \mathfrak{c}^+))$ is acc by Lemma 2.1. Hence, there exists a finite subset F_2 of S' such that

$$\pi(A(\beta D \times \mathfrak{c}^+)) \subseteq \mathrm{St}(F_2, \mathcal{U}).$$

If we put $F = F_1 \cup F_2$, then F is a countable subset of S' such that $X = \mathrm{St}(F, \mathcal{U})$. Since F_1 is discrete and closed in X and F_2 is finite, F is discrete and closed in X, which completes the proof. □

Remark 1. Bonanzinga and Matveev [2] proved that a regular-closed subspace of a star-Lindelöf space need not be CCC-Lindelöf. In fact, they used Example 2.32 from [4] under Continuum Hypothesis. Example 2.1 is stronger than theirs.

Remark 2. Example 2.2 also shows that regular-closed subspaces of discretely absolutely star-Lindelöf (absolutely star-Lindelöf, star-Lindelöf, centered-Lindelöf, linked-Lindelöf and CCC-Lindelöf) spaces need not be discretely absolutely star-Lindelöf (absolutely star-Lindelöf, star-Lindelöf, centered-Lindelöf and linked-Lindelöf and CCC-Lindelöf, respectively).

Song and Shi [9] proved that every centered-Lindelöf Tychonoff space is representable as a closed G_δ subspace of a star-Lindelöf Tychonoff space. The following theorem is a generalization of this fact.

Theorem 2.3. *Every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed G_δ subspace in a Hausdorff (regular, Tychonoff, respectively) absolutely star-Lindelöf space.*

Proof: First, we state a construction from [8, Theorem 2.1] which is a minor improvement of [2, Theorem 1]. Let X be a linked-Lindelöf space, $\mathcal{T}(X)$ the topology of X and $\mathcal{L}$ the collection of all linked subfamilies of $\mathcal{T}(X)$, and consider $\mathcal{L}$ as a discrete space. Let $\mathcal{A} = \beta\mathcal{L} \times \omega$ and define $S(X) = X \cup \mathcal{A}$. We topologize $S(X)$ as follows: the subspace $\mathcal{A}$ has the usual product topology and is an open subspace of $S(X)$; and a basic neighborhood of $x \in X$ in $S(X)$ is a set of the form

$$G_{U,n} = U \cup (\mathrm{cl}_{\beta\mathcal{L}} L(U) \times \{m : n < m < \omega\})$$

for an open neighborhood U of x in X and $n < \omega$, where

$$L(U) = \{\mathcal{U} \in \mathcal{L} : (\exists V \in \mathcal{U})(V \subseteq U)\}.$$

Then, it was proved in [2, Theorem 1] and [8, Theorem 2.1] that $S(X)$ is Hausdorff (regular, Tychonoff) iff so is X and $S(X)$ is star-Lindelöf if X is linked-Lindelöf.

Now, we modify the above construction. Let

$$\mathcal{R}(S(X)) = A(S(X)) \setminus (X \times \{1\}).$$

Let $X_\omega = X \times \{0\}$ and $X_n = (\beta\mathcal{L} \times \{n\}) \times \{0, 1\}$ for each $n < \omega$. Then

$$\mathcal{R}(S(X)) = X_\omega \cup \bigcup_{n<\omega} X_n.$$

Then, X can be represented as $\mathcal{R}(S(X))$ as a closed-G_δ subspace, since X is homeomorphic to X_ω and $A(S(X))$ is Hausdorff (regular, Tychonoff) if X is Hausdorff (regular, Tychonoff). Thus it suffices to show that $A(S(X))$ is absolutely star-Lindelöf. Let $\mathcal{U}$ be an open cover of $\mathcal{R}(S(X))$. Without loss of generality, we can assume that $\mathcal{U}$ consists of basic open sets. Let

$$D_n = ((\mathcal{L} \times \{n\}) \times \{0\}) \cup ((\beta\mathcal{L} \times \{n\}) \times \{1\})$$

for each $n < \omega$ and let

$$D = \bigcup_{n<\omega} D_n.$$

Then, every dense subspace of $\mathcal{R}(S(X))$ includes D. Thus, it suffices to show that there exists a countable subset $F \subseteq D$ such that $\mathrm{St}(F, \mathcal{U}) = \mathcal{R}(S(X))$. Let $\mathcal{U}_X = \{U \cap X_\omega : U \in \mathcal{U}\}$. Then, $\mathcal{U}_X$ is an open cover of X_ω. Since X_ω is homeomorphic to X and X is linked-Lindelöf, $\mathcal{U}_X$ has a σ-linked open refinement $\bigcup\{\mathcal{U}_m : m < \omega\}$. Let

$$F' = \{\langle\langle \mathcal{U}_m, n\rangle, i\rangle : m < \omega, n < \omega, i = 0, 1\} \subseteq D.$$

To show that $X_\omega \subseteq \mathrm{St}(F', \mathcal{U})$, let $x \in X$ be fixed. Then there exist $m, n < \omega$, $V \in \mathcal{U}_m$ and $U \in \mathcal{T}(X)$ such that

$$\langle x, 0\rangle \in V \subseteq G'_{U,n} = (U \times \{0\}) \cup (\mathrm{cl}_{\beta\mathcal{L}} L(U) \times \{m : n < m < \omega\} \times \{0, 1\}) \in \mathcal{U}.$$

Since $\langle\langle \mathcal{U}_m, n+1\rangle, 1\rangle \in F' \cap G'_{U,n}$, $\langle x, 0\rangle \in \mathrm{St}(F', \mathcal{U})$. Hence, $X_\omega \subseteq \mathrm{St}(F', \mathcal{U})$. On the other hand, since X_n is compact for each $n < \omega$, it is not difficult to find a finite subset $F_n \subseteq D_n$ such that $X_n \subseteq \mathrm{St}(F_n, \mathcal{U})$. If we put $F = F' \cup \bigcup_{n<\omega} F_n$, then F is a countable subset of D and $\mathcal{R}(S(X)) = \mathrm{St}(F, \mathcal{U})$, which completes the proof. □

Since every star-Lindelöf space is centered-Lindelöf and every centered-Lindelöf space is linked-Lindelöf, the next corollary follows from Theorem 2.3.

Corollary 2.4 (Song and Shi [9]). *Every Hausdorff (regular, Tychonoff) centered-Lindelöf (star-Lindelöf) space can be embedded in a Hausdorff (regular, Tychonoff, respectively) absolutely star-Lindelöf space as a closed-G_δ subspace.*

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