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Linear extensions of relations between vector spaces

Árpád Száz

Abstract. Let X and Y be vector spaces over the same field K. Following the terminology of Richard Arens [Pacific J. Math. 11 (1961), 9–23], a relation F of X into Y is called linear if $\lambda F(x) \subset F(\lambda x)$ and $F(x) + F(y) \subset F(x + y)$ for all $\lambda \in K \setminus \{0\}$ and $x, y \in X$.

After improving and supplementing some former results on linear relations, we show that a relation Φ of a linearly independent subset E of X into Y can be extended to a linear relation F of X into Y if and only if there exists a linear subspace Z of Y such that $\Phi(e) \in Y|Z$ for all $e \in E$. Moreover, if E generates X, then this extension is unique.

Furthermore, we also prove that if F is a linear relation of X into Y and Z is a linear subspace of X, then each linear selection relation Ψ of $F|Z$ can be extended to a linear selection relation Φ of F. A particular case of this Hahn-Banach type theorem yields an easy proof of the existence of a linear selection function f of F such that $f \circ F^{-1}$ is also a function.

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0. Introduction

Let X and Y be vector spaces over the same field K. A relation F on X to Y (i.e., a subset F of the product set $X \times Y$) is called linear if $\lambda F(x) \subset F(\lambda x)$ and $F(x) + F(y) \subset F(x + y)$ for all $\lambda \in K \setminus \{0\}$ and $x, y \in X$.

Here, for the sake of simplicity, we shall assume that $F \neq \emptyset$. Namely, in this case, $0 \in F(0)$, and hence $0F(x) \subset F(0x)$ for all $x \in X$. Therefore, linear relations on X to Y are actually linear subspaces of the product space $X \times Y$.

Clearly, a linear function is in particular a linear relation. Moreover, the inverse of a linear function or relation is also a linear relation. This is the main reason why linear relations are frequently more convenient means than linear functions.

In [20], by using a Hamel basis of X, we proved that a relation F of X into Y is linear if and only if there exist a linear function f of X into Y and a linear subspace Z of Y such that $F(x) = f(x) + Z$ for all $x \in X$.

Therefore, by using quotient spaces, the study of linear relations can, in principle, be reduced to that of linear functions. However, the study of quotient spaces actually rests on the theory linear equivalence relations.

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Besides inversion, several other important operations lead out from the class of linear functions. For instance, if f is a linear function or relation on one topological vector space X to another Y, then the closure of f is also a linear relation.

To motivate the study of linear relations, it is also worth mentioning that Banach's closed graph and open mapping theorems can, most naturally, be generalized in terms of linear relations of one vector relator space to another.

It is a curious fact that the corresponding extensions of the Banach-Steinhaus and Hahn-Banach theorems need sublinear relations [16]. Sublinear relations can, most easily, be obtained by taking unions of linear relations.

Linear relations were first introduced by Arens [2]. They have later been investigated by several authors under various names. However, the first systematic account and a most complete bibliography can only be found in Gross [4].

Our main purpose here is to investigate the possibility of extending arbitrary and linear relations of a part of X to linear relations of the whole of X. For this, we shall first improve and supplement some former results on linear relations.

The necessary prerequisites concerning relations, which seem not to be universally agreed upon, and some basic facts on linear operations in the power set $\mathcal{P}(X)$ of the vector space X will be briefly laid out in the next preparatory section.

1. Relations and vector spaces

A subset F of a product set $X \times Y$ is called a relation on X to Y. In particular, the relations $\Delta_X = \{(x,x) : x \in X\}$ and $X^2 = X \times X$ are called the identity and the universal relations on X, respectively.

Namely, if in particular $X = Y$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in Y : (x,y) \in F\}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively. Whenever $A \in X$ seems unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} of F can be defined so that $F^{-1}(y) = \big\{x \in X : y \in F(x)\big\}$ for all $y \in Y$.

Moreover, if F is a relation on X to Y and G is a relation on Y to Z, then the composition $G \circ F$ of G and F can be defined so that $(G \circ F)(x) = G\big(F(x)\big)$ for all $x \in X$. Thus, we have $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

If F is a relation on X to Y, then the sets $D_F = F^{-1}(Y)$ and $R_F = F(X)$ are called the domain and the range of F, respectively. If in particular $X = D_F$ (and $Y = R_F$), then we say that F is a relation of X into (onto) Y.

A relation F of X into Y is called a function if for each $x \in X$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If F is a relation of X into Y and $E \subset X$, then the relation $F|E = F \cap (E \times Y)$ is called the restriction of F to E. Moreover, the relation F is called an extension to X of a relation Φ of E into Y if $\Phi = F|E$.

On the other hand, if F is a relation of X into Y, then a relation Φ of X into Y is called a selection of F if $\Phi \subset F$. In terms of selections, the axiom of choice can be briefly reformulated by saying that every relation has a selection function.

Throughout in the sequel, X and Y will denote vector spaces over the same field K. For any $\lambda \in K$ and $A, B \subset X$, we write $\lambda A = \{\lambda x : x \in A\}$ and $A + B = \{x + y : x \in A,\ y \in B\}$.

Note that thus only two axioms of a vector space may fail to hold for the family $\mathcal{P}(X)$ of all subsets of X. Namely, only the one-point subsets of X can have additive inverses. Moreover, in general, we only have $(\lambda + \mu)A \subset \lambda A + \mu A$.

Whenever F and G are relations on X to Y, then in contrast to the above notations the relations λF and $F + G$ are still to be defined so that $(\lambda F)(x) = \lambda F(x)$ and $(F + G)(x) = F(x) + G(x)$ for all $x \in X$.

Finally, we note that, for any $E \subset X$, we denote by $\operatorname{lin}(E)$ the intersection of all linear subspaces of X containing E. Moreover, for a linear subspace Z of Y, we define the quotient set $Y|Z$ by $Y|Z = \{y + Z : y \in Y\}$.

2. Linear relations

Definition 2.1. A nonvoid relation F on X to Y is called linear if

$$\lambda F(x) \subset F(\lambda x) \quad \text{and} \quad F(x) + F(y) \subset F(x + y)$$

for all $\lambda \in K \setminus \{0\}$ and $x, y \in X$.

Remark 2.2. Since $F \neq \emptyset$, there exist $x \in X$ and $y \in Y$ such that $y \in F(x)$. Hence, by the linearity of F, it follows that

$$0 = y - y \in F(x) - F(x) \subset F(x) + F(-x) \subset F(x - x) = F(0).$$

Therefore, we also have $0 \in F(0)$, and thus $0F(x) \subset F(0x)$ for all $x \in X$.

This remark and the next simple theorem show that our present definition of a linear relation is equivalent to those of Berge [3, p. 133], Arens [2] and Kelley and Namioka [11, p. 101].

Theorem 2.3. *If F is a relation on X to Y, then the following assertions are equivalent:*

(1) *F is linear;*
(2) *F is a linear subspace of $X \times Y$.*

The importance of linear relations lies mainly in the following obvious consequence of the above theorem.

Corollary 2.4. *If F is a linear relation on X to Y, then F^{-1} is a linear relation on Y to X.*

The particular case $D_F = X$ of the following theorem was already proved in [21], while the more general case has only been treated by Cross [4, p. 9]. However, the proofs given there are less natural than the one given here.

Theorem 2.5. *If F is a linear relation on X to Y, then*

$$F(\lambda A) = \lambda F(A) \quad \text{and} \quad F(A+B) = F(A) + F(B)$$

for all $\lambda \in K \setminus \{0\}$, $A \subset D_F$ and $B \subset X$.

PROOF: By using Definition 2.1 and Remark 2.2, we can easily see that

$$\lambda F(A) \subset F(\lambda A) \quad \text{and} \quad F(A) + F(B) \subset F(A+B)$$

for all $\lambda \in K$ and $A, B \subset X$. Now, if in particular $\lambda \neq 0$, then it is clear we also have

$$F(\lambda A) = \lambda\lambda^{-1} F(\lambda A) \subset \lambda F(\lambda^{-1}\lambda A) = \lambda F(A).$$

Moreover, if in particular $A \subset D_F$, then for each $x \in A$ there exists $y \in F(x)$. Hence, it is clear that

$$\begin{aligned} F(x+B) = y - y + F(x+B) \subset F(x) - F(x) + F(x+B) \subset \\ \subset F(A) + F(-x+x+B) = F(A) + F(B). \end{aligned}$$

Therefore, we also have

$$F(A+B) = F\Big(\bigcup_{x \in A} (x+B)\Big) = \bigcup_{x \in A} F(x+B) \subset F(A) + F(B).$$

□

Remark 2.6. By defining $F = \Delta_{\Delta_{\mathbb{R}}}$, $A = \mathbb{R} \times \{0\}$ and $B = \{0\} \times \mathbb{R}$, we can at once see that F is a linear relation on $\mathbb{R}^2$ such that $F(A) + F(B) = \{(0,0)\}$, but $F(A+B) = \Delta_{\mathbb{R}}$.

From Theorem 2.5, by letting $A = \{x\}$ and $B = \{y\}$ for some $x, y \in D_F$, we can immediately get

Corollary 2.7. *If F is a linear relation on X to Y, then $F(\lambda x) = \lambda F(x)$ and $F(x+y) = F(x) + F(y)$ for all $\lambda \in K \setminus \{0\}$ and $x, y \in D_F$.*

Moreover, by using Theorem 2.5, we can also easily prove the following theorems.

Theorem 2.8. *If F is a linear relation on X to Y and G is a linear relation on Y into Z then $G \circ F$ is a linear relation on X to Z.*

Theorem 2.9. *If F is a linear relation on X to Y and A is a linear subspace of X, then $F(A)$ is a linear subspace of Y.*

Remark 2.10. Hence, by Corollary 2.4, it is clear that $D_F = F^{-1}(Y)$ is a linear subspace of X. Therefore, by considering a single linear relation F on X to Y, we may usually assume that $D_F = X$.

In addition to Corollary 2.4 and Theorem 2.8, it is also worth mentioning

Theorem 2.11. *If F and G are linear relations on X to Y and $\lambda \in K$, then λF and $F + G$ are also linear relations on X to Y.*

Remark 2.12. Note that, by Theorem 2.3, $\lambda \odot F = \{(\lambda x, \lambda y) : (x, y) \in F\}$ and $F \oplus G = \{(x + u, y + v) : (x, y) \in F,\ (u, v) \in F\}$ are also linear relations on X to Y.

3. Selections of linear relations

Theorem 3.1. *If F is a linear relation on X to Y, then*

$$F(x) = A + F(0)$$

for all $x \in X$ and $A \subset Y$ with $\emptyset \neq A \subset F(x)$.

Proof: If x and A are as above, then by the linearity of F it is clear that

$$A + F(0) \subset F(x) + F(0) \subset F(x + 0) = F(x).$$

Moreover, by choosing $y \in A$, we can also easily see that

$$F(x) = y - y + F(x) \subset A - F(x) + F(x) \subset A + F(-x + x) = A + F(0).$$

Therefore, the required equality is also true. □

Now, by letting $A = \{y\}$ for some $y \in F(x)$, we can also state

Corollary 3.2. *If F is a linear relation on X to Y, then $F(x) = y + F(0)$ for all $x \in X$ and $y \in F(x)$.*

Hence, we can at once see that a linear relation is nonmingled-valued.

Corollary 3.3. *If F is a linear relation on X to Y, then $F(x) \cap F(y) \neq \emptyset$ implies $F(x) = F(y)$ for all $x, y \in X$.*

Moreover, since $F(0)$ is a linear subspace of Y, from Corollary 3.2 it is clear that we also have

Corollary 3.4. *If F is a linear relation of X into Y, then $F(x) \in Y|F(0)$ for all $x \in X$.*

Hence, it is quite obvious that in particular we also have

Corollary 3.5. *A linear relation F on X to Y is a function if and only if $F(0) = \{0\}$.*

Moreover, as an immediate consequence of Theorem 3.1, we can also state

Theorem 3.6. *If F is a linear relation on X to Y and Φ is a selection relation of F, then*

$$F(x) = \Phi(x) + F(0)$$

for all $x \in X$.

Proof: In this case, $\emptyset \neq \Phi(x) \subset F(x)$ for all $x \in D_\Phi$. Therefore, by Theorem 3.1, $F(x) = \Phi(x) + F(0)$ for all $x \in D_\Phi$. Thus, since $D_\Phi = D_F$, the required equality is also true. □

Hence, by using Corollary 2.7, we can immediately get the following equivalent of assertions 2.02 of Arens [2] and I.2.11(b) of Cross [4].

Corollary 3.7. *If F is a linear relation on X to Y and Φ is a linear selection relation of F such that $\Phi(0) = F(0)$, then $\Phi = F$.*

Proof: Namely, by Theorem 3.6 and Corollary 2.7, we have

$$F(x) = \Phi(x) + F(0) = \Phi(x) + \Phi(0) = \Phi(x + 0) = \Phi(x)$$

for all $x \in X$. Therefore, $F = \Phi$ is also true. □

In addition to the above results, it is also worth proving the following

Theorem 3.8. *If Φ is a linear relation on X to Y, Z is a linear subspace of Y, and F is a relation on X to Y such that*

$$F(x) = \Phi(x) + Z$$

for $x \in X$, then F is the smallest linear relation on X to Y such that $\Phi \subset F$ and $Z \subset F(0)$.

Proof: By Theorem 2.11, it is clear that F is linear. Moreover, since $0 \in \Phi(0)$ and $0 \in Z$, we can at once see that

$$Z \subset \Phi(0) + Z = F(0) \quad \text{and} \quad \Phi(x) \subset \Phi(x) + Z = F(x)$$

for all $x \in X$. Therefore, $\Phi \subset F$ is also true.

On the other hand, if G is a linear relation of X to Y such that $\Phi \subset G$ and $Z \subset G(0)$, then it is clear that

$$F(x) = \Phi(x) + Z \subset G(x) + G(0) = G(x + 0) = G(x)$$

for all $x \in X$. Therefore, $F \subset G$ is also true. □

4. Linear equivalence relations

The first part of the following theorem is a very particular case of a proposition of Findlay [6]. While, the second part shows that linear equivalence relations are important particular cases of translation relations [19].

Theorem 4.1. *If F is a relation on X, then the following assertions are equivalent:*

(1) *F is a reflexive linear relation on X;*
(2) *F is a linear equivalence relation on X;*
(3) *$F(x) = x + F(0)$ for all $x \in X$ and $F(0)$ is a linear subspace of Y.*

Hint: If assertion (1) holds, then by Corollary 3.2 and Theorem 2.9, it is clear that assertion (3) also holds.

On the other hand, if assertion (3) holds, then by Theorem 3.8 it is clear that assertion (1) also holds. Hence, by Corollaries 2.4 and 3.2 and Theorem 2.8, it is clear that

$$F^{-1}(x) = x + F^{-1}(0) \quad \text{and} \quad (F \circ F)(x) = x + (F \circ F)(0)$$

for all $x \in X$. Therefore, to prove the symmetry and transitivity of F, it is now enough to show only that

$$F^{-1}(0) = -F(0) = F(0) \quad \text{and} \quad (F \circ F)(0) = F(0) + F(0) = F(0).$$

□

From Theorem 4.1, by Corollary 2.4 and Theorem 2.8, it is clear that in particular we also have

Corollary 4.2. *If F is a linear relation of X into Y, then $F^{-1} \circ F$ is a linear equivalence relation on X.*

Moreover, concerning the relation $F^{-1} \circ F$, we can also prove the following

Theorem 4.3. *If F is linear relation of X into Y, then*

$$F^{-1} \circ F = F^{-1} \circ \Phi$$

for any selection relation Φ of F.

Proof: Denote by $\mathcal{F}$ the family of all selection functions of F. Then, by the axiom of choice, for each $x \in X$ we have $F(x) = \bigcup_{f \in \mathcal{F}} \{f(x)\}$. Hence, it is clear that

$$\big(F^{-1} \circ F\big)(x) = F^{-1}\big(F(x)\big) = F^{-1}\Big(\bigcup_{f \in \mathcal{F}} \{f(x)\}\Big) = \bigcup_{f \in \mathcal{F}} F^{-1}\big(f(x)\big).$$

Moreover, if $f \in \mathcal{F}$, then we have $f(x) \in F(x)$, and hence $x \in F^{-1}\big(f(x)\big)$. Hence, by Corollary 3.2, it is clear that $F^{-1}\big(f(x)\big) = x + F^{-1}(0)$. Therefore, we have

$$\big(F^{-1} \circ F\big)(x) = \bigcup_{f \in \mathcal{F}} \big(x + F^{-1}(0)\big) = x + F^{-1}(0) = F^{-1}\big(f(x)\big) = (F^{-1} \circ f)(x)$$

for all $x \in X$ and $f \in \mathcal{F}$. Hence, it follows that $F^{-1} \circ F = F^{-1} \circ f$ for all $f \in \mathcal{F}$.

Now, if Φ is a selection relation of F, then by choosing a selection function f of Φ we can see that $F^{-1} \circ F = F^{-1} \circ f \subset F^{-1} \circ \Phi \subset F^{-1} \circ F$. Therefore, the required equality is also true. □

Now, as an immediate consequence of Theorem 4.3 and Corollary 4.2, we can also state

Corollary 4.4. *If F is linear relation of X into Y and Φ is a selection relation of F, then $F^{-1} \circ \Phi$ and $\Phi^{-1} \circ F$ are linear equivalence relations on X.*

Moreover, from Theorem 4.3, by using Theorem 4.1, we can also easily get

Theorem 4.5. *If F is linear relation of X into Y, then*

$$F^{-1}\big(F(x)\big) = x + F^{-1}(0)$$

for all $x \in X$.

Proof: Since $0 \in F(0)$, there exists a selection function f of F such that $f(0) = 0$. Hence, by Corollary 4.2 and Theorems 4.1 and 4.3, it is clear that

$$F^{-1}\big(F(x)\big) = \big(F^{-1} \circ F\big)(x) = x + \big(F^{-1} \circ F\big)(0) =$$
$$= x + \big(F^{-1} \circ f\big)(0) = x + F^{-1}\big(f(0)\big) = x + F^{-1}(0)$$

for all $x \in X$. □

Remark 4.6. From Theorem 4.5, by Corollary 2.4, it is clear that we also have $F\big(F^{-1}(y)\big) = y + F(0)$ for all $y \in F(X)$.

Therefore, by using Theorem 2.5, we can also easily prove

Corollary 4.7. *If F is linear relation of X into Y, then $F = F \circ F^{-1} \circ F$.*

Proof: By Theorems 4.5 and 2.5 and Remark 4.6, it is clear that

$$\big(F \circ F^{-1} \circ F\big)(x) = F\big(F^{-1}\big(F(x)\big)\big) = F\big(x + F^{-1}(0)\big) =$$
$$= F(x) + F\big(F^{-1}(0)\big) = F(x) + F(0) = F(x)$$

for all $x \in X$. Therefore, the required equality is also true. □

5. Relations with values in quotient spaces

Because of Corollary 3.4, it is also of some interest to prove the following

Theorem 5.1. *If Φ is a relation of a set E into Y and Z is a linear subspace of Y, then the following assertions are equivalent:*

(1) $\Phi(e) \in Y|Z$ *for all* $e \in E$;
(2) $\Phi(e) + Z = \Phi(e)$ *and* $\Phi(e) - \Phi(e) = Z$ *for all* $e \in E$;
(3) $\Phi(e) + Z \subset \Phi(e)$ *and* $\Phi(e) - \Phi(e) \subset Z$ *for all* $e \in E$.

Proof: If assertion (1) holds, then for each $e \in E$ there exists $d \in Y$ such that $\Phi(e) = d + Z$. Hence, it is clear that

$$\Phi(e) + Z = d + Z + Z = d + Z = \Phi(e) \quad \text{and} \quad \Phi(e) - \Phi(e) = d + Z - d - Z = Z.$$

Therefore, assertion (2) also holds.

Now, to complete the proof, we need only show that the implication (3) $\Rightarrow$(1) is also true. For this, note that for each $e \in E$ there exists $d \in \Phi(e)$. Therefore, if assertion (3) holds, then

$$d + Z \subset \Phi(e) + Z \subset \Phi(e) \quad \text{and} \quad \Phi(e) = d + \Phi(e) - d \subset d + \Phi(e) - \Phi(e) \subset d + Z.$$

Hence, it follows that $\Phi(e) = d + Z$. Therefore, assertion (1) also holds. $\square$

A simple application of Theorem 5.1 gives the following

Corollary 5.2. *If Φ is a relation of a set E into Y, then the following assertions are equivalent:*

(1) $\Phi(e) + \operatorname{lin}\big(\bigcup_{e\in E}\big(\Phi(e) - \Phi(e)\big)\big) = \Phi(e)$ *for all* $e \in E$;
(2) $\Phi(e) + \operatorname{lin}\big(\bigcup_{e\in E}\big(\Phi(e) - \Phi(e)\big)\big) \subset \Phi(e)$ *for all* $e \in E$;
(3) *there exists a linear subspace Z of Y such that $\Phi(e) \in Y|Z$ for all $e \in E$.*

Proof: To prove the implication (2) $\Rightarrow$(3), note that

$$Z = \operatorname{lin}\bigg(\bigcup_{e\in E}\big(\Phi(e) - \Phi(e)\big)\bigg)$$

is a linear subspace of Y such that $\Phi(e) - \Phi(e) \subset Z$ for all $e \in E$. Moreover, if assertion (2) holds, then we also have $\Phi(e) + Z \subset \Phi(e)$ for all $e \in E$. Hence, by Theorem 5.1, it follows that $\Phi(e) \in Y|Z$ for all $e \in E$. Therefore, assertion (3) also holds.

On the other hand, if assertion (3) holds, then by Theorem 5.1, we also have $\Phi(e) + Z = \Phi(e)$ and $\Phi(e) - \Phi(e) = Z$ for all $e \in E$. Hence, it is clear that

$$\Phi(e) + \operatorname{lin}\bigg(\bigcup_{e\in E}\big(\Phi(e) - \Phi(e)\big)\bigg) = \Phi(e) + Z = \Phi(e)$$

for all $e \in E$. Therefore, assertion (1) also holds. $\square$

6. The existence of linear extensions

We start with the following well-known theorem. The proof is sketched here only for the reader's convenience.

Theorem 6.1. *If E is a linearly independent subset of X and φ is a function of E into Y, then φ can be extended to a linear function f of X into Y.*

Hint: By [11, Theorem 1.1], E can be extended to a basis E' of X. Moreover, by the axiom of choice, φ can be extended to a function φ' of E' into Y. Therefore, we may assume, without loss of generality, that E is a basis of X.

If E is a basis of X, then for each $x \in X$ there exists a unique function $\hat{x}$ of E to $\mathbb{K}$ such that the set $E_x = \big\{ e \in E : \hat{x}(e) \neq \emptyset \big\}$ is finite and $x = \sum_{e \in E_x} \hat{x}(e)e$. Therefore, by defining $f(x) = \sum_{e \in E_x} \hat{x}(e)\varphi(e)$ for all $x \in X$, we can get a linear extension f of φ to X. □

Now, as a main result of this paper, we can also easily prove the following more general extension theorem.

Theorem 6.2. *If E is a linearly independent subset of X and Φ is a relation of E into Y, then the following assertions are equivalent:*

(1) *Φ can be extended to a linear relation F of X into Y;*
(2) *there exists a linear subspace Z of Y such that $\Phi(e) \in Y|Z$ for all $e \in E$.*

Proof: If assertion (1) holds, then by Corollary 2.9 $Z = F(0)$ is a linear subspace of Y. Moreover, by Corollary 3.2, we have

$$\Phi(e) = F(e) = d + F(0) = d + Z \in Y|Z$$

for all $e \in E$ and $d \in \Phi(e)$. Hence, since $\Phi(e) \neq \emptyset$ for all $e \in E$, it is clear that assertion (2) also holds.

To prove the converse implication, we first note that by the axiom of choice there exists a selection function φ of Φ. Moreover, by Theorem 6.1, φ can be extended to a linear function f of X to Y. Therefore, if assertion (2) holds, then by defining

$$F(x) = f(x) + Z$$

for all $x \in X$, we can get a linear extension F of Φ to X.

Namely, by Theorem 5.1, for each $e \in E$ we have

$$F(e) = f(e) + Z = \varphi(e) + Z \subset \Phi(e) + Z = \Phi(e).$$

Moreover, it is clear that we also have

$$\Phi(e) = \varphi(e) + \Phi(e) - \varphi(e) \subset \varphi(e) + \Phi(e) - \Phi(e) = \varphi(e) + Z = F(e).$$

Therefore, the equality $\Phi(e) = F(e)$ is also true. Moreover, by Theorem 3.8, F is linear. Therefore, assertion (1) also holds. □

Remark 6.3. Note that if Z is as in assertion (2), then by Theorem 5.1 we have $Z = \Phi(e) - \Phi(e)$ for all $e \in E$. Therefore, in contrast to the linear relation F, the subspace $Z = F(0)$ is already uniquely determined by the relation Φ.

From Theorem 6.2, by Corollary 5.2, it is clear that we also have

Corollary 6.4. *If E is a linearly independent subset of X and Φ is a relation of E into Y, then the following assertions are equivalent:*

(1) *Φ can be extended to a linear relation F of X into Y;*
(2) *$\Phi(e) + \operatorname{lin}\big(\bigcup_{e\in E}\big(\Phi(e) - \Phi(e)\big)\big) \subset \Phi(e)$ for all $e \in E$.*

7. The unicity of linear extensions

Theorem 7.1. *If E is a generator system of X and Φ is a relation of E into Y, then there exists at most one linear extension F of Φ to X.*

Proof: If F is a linear extension of Φ to X, then by Corollary 2.7 we have

$$F(0) = F(e - e) = F(e) + F(-e) = F(e) - F(e) = \Phi(e) - \Phi(e)$$

for all $e \in E$. Hence, since $E \neq \emptyset$, it is clear that the relation F is uniquely determined at the point 0.

Moreover, if $x \in X \setminus \{0\}$, then since $X = \operatorname{lin}(E)$, there exist finite families $(\lambda_i)_{i=1}^n$ in $K \setminus \{0\}$ and $(e_i)_{i=1}^n$ in $E \setminus \{0\}$ such that $x = \sum_{i=1}^n \lambda_i e_i$. Hence, by Corollary 2.7, it is clear that

$$F(x) = F\bigg(\sum_{i=1}^n \lambda_i e_i\bigg) = \sum_{i=1}^n F\big(\lambda_i e_i\big) = \sum_{i=1}^n \lambda_i F(e_i) = \sum_{i=1}^n \lambda_i \Phi(e_i).$$

Therefore, the relation F is uniquely determined at the point x too. □

In addition to Theorems 6.2 and 7.1, it is also worth proving the following

Theorem 7.2. *If Φ is a relation on X to Y, then $F = \operatorname{lin}\big(\Phi\big)$ is the smallest linear relation on X to Y such that $\Phi \subset F$. Moreover, $D_F = \operatorname{lin}\big(D_\Phi\big)$.*

Proof: By Theorem 2.3, it is clear that $F = \operatorname{lin}(\Phi)$ is the smallest linear relation on X to Y such that $\Phi \subset F$. Hence, since D_F is a linear subspace of X, it is clear that $\operatorname{lin}\big(D_\Phi\big) \subset \operatorname{lin}(D_F) = D_F$.

On the other hand, if $x \in D_F$, then there exists $y \in Y$ such that $(x, y) \in F = \operatorname{lin}\big(\Phi\big)$. Hence, it is clear that there exist finite families $(\lambda_i)_{i=1}^n$ in K and $\big((x_i, y_i)\big)_{i=1}^n$ in Φ such that $(x, y) = \sum_{i=1}^n \lambda_i(x_i, y_i)$. This, in particular, implies that $x = \sum_{i=1}^n \lambda_i x_i$. Hence, since $x_i \in D_\Phi$ for all $i = 1, 2, \ldots, n$, it is already clear that $x \in \operatorname{lin}\big(D_\Phi\big)$. Therefore, the equality $D_F = \operatorname{lin}\big(D_\Phi\big)$ is also true. □

Remark 7.3. Hence, we can at once see that $D_F = X$ if and only if D_Φ is a generator system of X.

Moreover, as a useful consequence of Theorems 7.1 and 7.2, we can also prove

Corollary 7.4. *If E is a generator system of X, Φ is a relation of E into Y and F is a linear extension of Φ to X, then $F = \operatorname{lin}\big(\Phi\big)$.*

Proof: Define $G = \operatorname{lin}\big(\Phi\big)$. Then, by Theorem 7.2, G is the smallest linear relation of X to Y such that $\Phi \subset G$. Therefore, since F is also a linear relation of X to Y such that $\Phi \subset F$, we necessarily have $G \subset F$. Hence, it is clear that $\Phi = \Phi|E \subset G|E \subset F|E = \Phi$, and thus $\Phi = G|E$. Therefore, G is also a linear extension of Φ to X. Hence, by Theorem 7.1, it follows that $G = F$. □

8. Linear selection functions of linear relations

Theorem 8.1. *If F is a linear relation of X to Y and D is a linearly independent subset of X, then each selection function φ of $F|D$ can be extended to a linear selection function f of F.*

Proof: By [11, Theorem 1.1], there exists a basis E of X such that $D \subset E$. Moreover, by the axiom of choice, there exists a selection function ψ of $F|(E \setminus D)$. Define $\lambda = \phi \cup \psi$. Then, by Theorem 6.1, λ can be extended to a linear function f of X into Y. Hence, by Corollary 7.4, it is clear that $f = \operatorname{lin}(\lambda) \subset \operatorname{lin}(F) = F$ is also true. □

Now, as some immediate consequences of Theorem 8.1, we can also state

Corollary 8.2. *If F is a linear relation of X to Y, $\xi \in X \setminus \{0\}$ and $\eta \in F(\xi)$, then there exists a linear selection function f of F such that $f(\xi) = \eta$.*

Proof: By letting $D = \{\xi\}$ and $\varphi = \{(\xi, \eta)\}$, Theorem 8.1 can be applied. □

Corollary 8.3. *If F is a linear relation of X to Y, then there exists a linear selection function f of F.*

Proof: If Corollary 8.2 cannot be applied, then $X = \{0\}$, and thus $f = \{(0, 0)\}$ is a linear selection function of F. □

Moreover, by Corollary 8.2, it is clear that we also have

Corollary 8.4. *If F is a linear relation of X to Y and $\mathcal{F}$ is the family of all selection functions of F, then $F = \big(\bigcup \mathcal{F}\big) \cup \big(\{0\} \times F(0)\big)$.*

Remark 8.5. Note that if $\mathcal{F}$ is a nonvoid family of linear functions of X into Y, then $F = \bigcup \mathcal{F}$ is, in general, only a sublinear relation of X into Y. Therefore, a hoped for characterization theorem for linear relations does not hold.

However, as an immediate consequence of Corollaries 8.3 and 2.9 and Theorems 3.6 and 3.8, we can at once state the following

Theorem 8.6. *If F is a relation of X into Y, then the following assertions are equivalent:*

(1) *F is linear;*
(2) *there exists a linear function f of X into Y and a linear subspace Z of Y such that $F(x) = f(x) + Z$ for all $x \in X$.*

Simple reformulations of assertion (2) give the following

Corollary 8.7. *If F is a relation of X into Y, then the following assertions are equivalent:*

(1) *F is linear;*
(2) *there exists a linear function f and a constant linear relation C of X into Y such that $F = f + C$;*
(3) *there exists a linear function f of X into Y and a linear equivalence relation E on Y such that $F = E \circ f$.*

Remark 8.8. The crucial fact that each linear relation has a linear selection function was first proved by Géza Száz. (See [20].)

At the same time, he also proved that the corresponding statement for additive relations is no longer true. (See also Godini [7] and Nikodem [12].)

9. Linear selection relations of linear relations

The following theorem has formerly been proved in [16] with the help of a Hahn-Banach type theorem. Now, we shall give a more direct proof by making use of the existence of linear selection functions and complementary subspaces.

Theorem 9.1. *If F is a linear relation of X into Y and Z is a linear subspace of X, then each linear selection relation Ψ of $F|Z$ can be extended to a linear selection relation Φ of F.*

PROOF: In this case, by Corollary 8.3, there exists a linear selection function f of F. Moreover, by [11, p. 5], there exist a linear subspace W of X and some unique functions p and q of X into Z and W, respectively, such that $\Delta_X = p + q$. In this case, the functions p and q are linear, moreover we have $\Delta_Z = p|Z$, $p(W) = \{0\}$, $q(Z) = \{0\}$ and $\Delta_W = q|W$. Therefore, by defining

$$\Phi = \Psi \circ p + f \circ q,$$

we can get a linear extension Φ of Ψ to X such that $\Phi \subset F$.

Namely, by Theorems 2.8 and 2.11, it is clear that Φ is a linear relation of X into Y. Moreover, we can also easily see that

$$\Phi(z) = \Psi\big(p(z)\big) + f\big(q(z)\big) = \Psi(z) + f(0) = \Psi(z)$$

for all $z \in Z$ and

$$\Phi(w) = \Psi\big(p(w)\big) + f\big(q(w)\big) = \Psi(0) + f(w) \subset F(0) + F(w) = F(w)$$

for all $w \in W$. Therefore, we also have

$$\Phi(x) = \Phi\big(p(x) + q(x)\big) = \Phi\big(p(x)\big) + \Phi\big(q(x)\big) \subset$$
$$\subset \Psi\big(p(x)\big) + F\big(q(x)\big) \subset F\big(p(x)\big) + F\big(q(x)\big) = F\big(p(x) + q(x)\big) = F(x)$$

for all $x \in X$. □

Remark 9.2. The particular case $Z \neq \{0\}$ of the above theorem can also be proved with the help of Theorem 6.2 and [11, Theorem 1.1].

For this, we can choose bases D and E of Z and X, respectively, such that $D \subset E$. Moreover, we can choose a selection function f of $F|(E \setminus D)$ and define

$$\Theta(e) = \Psi(e) \quad \text{for} \quad e \in D \quad \text{and} \quad \Theta(e) = f(e) + \Psi(0) \quad \text{for} \quad e \in E \setminus D.$$

Then, it is clear that $\Theta(e) \in Y|\Psi(0)$ for all $e \in E$. Therefore, by Theorem 6.2, Θ can be extended to a linear relation Φ of X into Y. Moreover, it is clear that $\Theta \subset F$. Therefore, by Corollary 7.4, we also have $\Phi = \operatorname{lin}(\Theta) \subset \operatorname{lin}(F) = F$. Now, it remains to note only that Ψ and $\Phi|Z$ are linear extensions of $\Psi|D$ to Z. Therefore, by Theorem 7.1, $\Psi = \Phi|Z$ is also true.

As a very particular case of Theorem 9.1, we can at once state

Corollary 9.3. *If Z is a linear subspace of X and Ψ is a linear relation of Z into Y, then Ψ can be extended to a linear relation Φ of X to Y.*

Proof: Note that, by letting $F = X \times Y$, Theorem 9.1 can be applied. □

Moreover, from Theorem 9.1 and Corollary 9.3, by using Corollary 3.5, we can immediately get the following two corollaries.

Corollary 9.4. *If F is a linear relation of X into Y and Z is a linear subspace of X, then each linear selection function φ of $F|Z$ can be extended to a linear selection function f of F.*

Corollary 9.5. *If Z is a linear subspace of X and ψ is a linear function of Z into Y, then ψ can be extended to a linear function φ of X to Y.*

Corollary 9.4 easily yields not only Corollary 8.3, but also the following

Corollary 9.6. *If F is a linear relation of X into Y, then there exists a linear selection function f of F such that $f\big(F^{-1}(0)\big) = \{0\}$.*

Proof: Note that, by Corollaries 2.4 and 2.9, $Z = F^{-1}(0)$ is a linear subspace of X. Moreover, $x \in Z$ implies $0 \in F(x)$. Therefore, $\varphi = Z \times \{0\}$ is a linear selection function of $F|Z$. Thus, by Corollary 9.4, φ can be extended to a linear selection function f of F. Now, it remains to note only that $f\big(F^{-1}(0)\big) = f(Z) = \{0\}$. □

10. Some further results on linear selection functions

Because of Corollary 9.6, it is also of some importance to prove the following

Theorem 10.1. *If F is a linear relation of X into Y and f is a linear selection function of F, then the following assertions are equivalent:*
(1) $f\big(F^{-1}(0)\big) = \{0\}$;
(2) $F^{-1} \circ F = f^{-1} \circ f$;
(3) $F^{-1} \circ f = f^{-1} \circ f$.

Proof: If $x \in X$, then by Theorem 4.5

$$\big(F^{-1} \circ F\big)(x) = x + F^{-1}(0).$$

Hence, we can already see that if $y \in \big(F^{-1} \circ F\big)(x)$, then $y - x \in F^{-1}(0)$. Therefore, if assertion (1) holds, then

$$f(y) - f(x) = f(y - x) \in f\big(F^{-1}(0)\big) = \{0\}.$$

Consequently, we have $f(y) = f(x)$, and hence $y \in \big(f^{-1} \circ f\big)(x)$. Therefore, $F^{-1} \circ F \subset f^{-1} \circ f$. Hence, since the converse inclusion is evidently true, it is clear that assertion (2) also holds.

Now, to complete the proof, we note that the equivalence (2) $\Leftrightarrow$ (3) is immediate from Theorem 4.3. Moreover, if assertion (3) holds, then

$$f\big(F^{-1}(0)\big) = f\big(F^{-1}(f(0))\big) = \big(f \circ \big(F^{-1} \circ f\big)\big)(0) =$$
$$= \big(f \circ \big(f^{-1} \circ f\big)\big)(0) = \big(f\big(f^{-1}(f(0))\big)\big) = \big(f\big(f^{-1}(0)\big)\big) = \{0\},$$

and thus assertion (1) also holds. $\square$

From Corollaries 9.6 and 3.3 and Theorem 10.1, it is clear we also have the following

Theorem 10.2. *If F is a linear relation on X to Y, then there exists a linear selection function f of F such that, for any $x, y \in X$, the following assertions are equivalent:*
(1) $f(x) = f(y)$;
(2) $F(x) = F(y)$;
(3) $f(x) \in F(y)$;
(4) $F(x) \cap F(y) \neq \emptyset$.

Hint: Note that the property $F^{-1} \circ F = f^{-1} \circ f$ in a detailed form means only that, for any $x, y \in X$, we have $y \in F^{-1}\big(F(x)\big)$ if and only if $y \in f^{-1}\big(F(x)\big)$. That is, for any $x, y \in X$, we have $F(y) \cap F(x) \neq \emptyset$ if and only if $f(y) = f(x)$. $\square$

Moreover, by Theorems 4.1 and 10.1 and Corollaries 9.6 and 4.2, it is clear that we also have the following

Theorem 10.3. *If F is a relation on X, then the following assertions are equivalent:*

(1) *F is a reflexive linear relation on X;*
(2) *there exists a linear function f of X into itself such that $F = f^{-1} \circ f$;*
(3) *there exists a linear function f of X into itself, with $f = f \circ f$, such that $F = f^{-1} \circ f$.*

Hint: Note that if f is a selection function of F and $F = f^{-1} \circ f$, then $f(x) \in f^{-1}\big(f(x)\big)$, and hence $f\big(f(x)\big) = f(x)$ for all $x \in X$. Therefore, the equality $f \circ f = f$ is also true. □

11. A further extension theorem for linear relations

To prove an important addition to Theorem 10.1, we shall need a consequence of the following two theorems.

Theorem 11.1. *If Z is a linear subspace of X and Φ is a linear relation of Z into itself, then Φ can be extended to a linear relation F of X into itself such that*

$$F\big(X \setminus Z\big) \subset X \setminus Z.$$

Proof: Let W, p and q be as in the proof of Theorem 9.1, and define

$$F = \Phi \circ p + q.$$

Then, from the proof of Theorem 9.1, it is clear that F is a linear extension of Φ to X.

Moreover, we can note that if $y \in F\big(X \setminus Z\big)$, then there exists $x \in X \setminus Z$ such that

$$y \in F(x) = \Phi\big(p(x)\big) + q(x) \subset Z + q(x).$$

Therefore, there exists $z \in Z$ such that $y = z + q(x)$, and hence $y - z = q(x)$. Hence, since $z \notin Z$, and thus $q(x) \neq 0$, it is already clear that $y \notin Z$. Therefore, the required inclusion is true. □

Theorem 11.2. *If Z is a subset of a set X, Φ is a relation of Z into X and F is an extension of Φ to X, then the following assertions are equivalent:*

(1) *$F\big(X \setminus Z\big) \subset X \setminus Z$;*
(2) *$\Phi^{-1}(z) = F^{-1}(z)$ for all $z \in Z$.*

Proof: If assertion (1) holds, then $F(x) \subset X \setminus Z$, and hence $F(x) \cap Z = \emptyset$ for all $x \in X \setminus Z$. Therefore, if $z \in Z$ and $x \in F^{-1}(z)$, i.e., $z \in F(x)$, then we necessarily have $x \in Z$. Hence, since $\Phi = F|Z$, it already follows that $\Phi(x) = F(x)$. Therefore, we also have $z \in \Phi(x)$, and hence $x \in \Phi^{-1}(z)$. Consequently, we have $F^{-1}(z) \subset \Phi^{-1}(z)$. Hence, since $\Phi \subset F$, and thus $\Phi^{-1} \subset F^{-1}$, it is already clear that assertion (2) also holds.

To prove the converse implication (2) $\Rightarrow$(1), note that if assertion (1) does not hold, then there exists $z \in Z$ such that $z \in F(X \setminus Z)$. Therefore, there exists $x \in X \setminus Z$ such that $z \in F(x)$, and hence $x \in F^{-1}(z)$. On the other hand, since $x \notin Z$, we also have $\Phi(x) = \emptyset$. Therefore, $z \notin \Phi(x)$, and hence $x \notin \Phi^{-1}(z)$. Therefore, assertion (2) does not hold either. $\square$

Remark 11.3. Note that the implication (2) $\Rightarrow$(1) does not require either F to be an extension of Φ or even to hold the inclusion $\Phi^{-1}(z) \subset F^{-1}(z)$ with $z \in Z$.

From Theorems 11.1 and 11.2, by Corollary 3.5, it is clear that in particular we also have

Corollary 11.4. *If Z is a linear subspace of X and φ is a linear function of Z into itself, then φ can be extended to a linear function f of X into itself such that $\varphi^{-1}(z) = f^{-1}(z)$ for all $z \in Z$.*

12. An important addition to Theorem 10.1

Now, by using Corollary 11.4, we can also prove the following

Theorem 12.1. *If F is a linear relation of X to Y and f is a linear selection function of F, then the following assertions are equivalent:*

(1) $f\big(F^{-1}(0)\big) = \{0\}$;
(2) $f \circ F^{-1}$ *is a function;*
(3) $f \circ F^{-1}$ *is a linear function;*
(4) *there exists a linear function g of Y to X such that $F = g^{-1} \circ f$.*

Proof: In this case, by Theorem 2.8 and Corollary 2.4, $f \circ F^{-1}$ is also a linear relation. Therefore, assertions (2) and (3) are equivalent. Moreover, if assertion (1) holds, then $\big(f \circ F^{-1}\big)(0) = \{0\}$. Therefore, by Corollary 3.5, assertion (2) also holds. Moreover, it is clear that the implication (3) $\Rightarrow$(1) is also true. Therefore, assertions (1) and (3) are also equivalent.

On the other hand, if assertion (3) holds, then $\varphi = f \circ F^{-1}$ is a linear function of $F(X)$ into itself. Therefore, by Corollary 11.4, φ can be extended to a linear function of X to itself such that

$$\varphi^{-1}(z) = g^{-1}(z)$$

for all $z \in F(X)$. Hence, since $f \subset F$, it follows that

$$\big(\varphi^{-1} \circ f\big)(x) = \varphi^{-1}\big(f(x)\big) = g^{-1}\big(f(x)\big) = \big(g^{-1} \circ f\big)(x)$$

for all $x \in X$, and thus $\varphi^{-1} \circ f = g^{-1} \circ f$. Now, by Corollary 4.7 and Theorem 10.1, it is clear that

$$F = F \circ F^{-1} \circ F = F \circ f^{-1} \circ f = \big(f \circ F^{-1}\big)^{-1} \circ f = \varphi^{-1} f = g^{-1} \circ f,$$

and thus assertion (4) also holds.

Finally, if assertion (4) holds, then it is clear that

$$\begin{aligned} f\big(F^{-1}(0)\big) &= \big(f \circ F^{-1}\big)(0) = \big(f \circ \big(g^{-1} \circ f\big)^{-1}\big)(0) = \\ &= \big(f \circ \big(f^{-1} \circ g\big)\big)(0) = f\big(f^{-1}\big(g(0)\big)\big) = f\big(f^{-1}(0)\big) = \{0\}, \end{aligned}$$

and thus assertion (1) also holds. □

Now, an immediate consequence of Corollary 9.6, Theorem 10.4 and Corollary 2.4 and Theorem 2.8, we can also state

Theorem 12.3. *If F is a relation of X into Y, then the following assertions are equivalent:*

(1) *F is linear;*
(2) *there exists a linear function f of X into Y and a linear function g of Y into itself such that $F = g^{-1} \circ f$;*
(3) *there exists a linear function f of X into Y and a linear function g of Y into itself, with $f = g \circ f$, such that $F = g^{-1} \circ f$.*

Hint: Note that if f is a selection function of F and $F = g^{-1} \circ f$, then $f(x) \in g^{-1}\big(f(x)\big)$, and hence $g\big(f(x)\big) = f(x)$ for all $x \in X$. Therefore, the equality $g \circ f = f$ is also true. □

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Institute of Mathematics and Informatics, University of Debrecen,
H-4010 Debrecen, Pf. 12, Hungary

E-mail: szaz@math.klte.hu

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