Products of k-spaces, and questions

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Abstract. As is well-known, every product of a locally compact space with a k-space is a k-space. But, the product of a separable metric space with a k-space need not be a k-space. In this paper, we consider conditions for products to be k-spaces, and pose some related questions.

Keywords: k-space, sequential space, strongly Fréchet space, bi-k-space, strongly sequential space, Tanaka space

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Definitions and preliminaries

We assume that all spaces are *regular* and T_1, and all maps are continuous surjections.

Let X be a space. For a (not necessarily open or closed) cover $\mathcal{P}$ of X, X *is determined by* a cover $\mathcal{P}$, if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$. Here, we can replace "open" by "closed". (Following [3], we shall use "X is determined by $\mathcal{P}$" instead of the usual "X has the *weak topology* with respect to $\mathcal{P}$".) Obviously, every space is determined by its open cover. As is well-known, a space is a *k-space* (resp. *sequential space*) if it is determined by the cover of *all* compact (resp. compact metric) subsets. We recall that every k-space (resp. sequential space) is characterized as a quotient space of a locally compact space (resp. metric space [2]). Every sequential space is a k-space, and the converse holds if each point is a G_δ-set ([7]).

We recall elementary facts which will be used later on. These are routinely shown, but Fact (4) is due to [15].

Facts: (1) Let X be a space determined by a cover $\mathcal{P}$, and let $\mathcal{C}$ be a cover of X. If each element of $\mathcal{P}$ is contained in some element of $\mathcal{C}$, then X is also determined by $\mathcal{C}$.

(2) Let X be a space determined by a cover $\{X_\alpha : \alpha\}$. If each X_α is determined by a cover $\mathcal{P}_\alpha$, then X is determined by the cover $\bigcup\{\mathcal{P}_\alpha : \alpha\}$.

(3) (i) Let $f : X \to Y$ be a quotient map. If X is determined by a cover $\mathcal{C}$, then Y is determined by the cover $\{f(C) : C \in \mathcal{C}\}$.

(ii) For a cover $\mathcal{P}$ of a space Y, Y is determined by the cover $\mathcal{P}$ if and only if the obvious map $f : \Sigma\, \mathcal{P} \to Y$ is quotient (where $\Sigma\, \mathcal{P}$ is the topological sum of $\mathcal{P}$).

(4) Let $f: X \to Y$ be a closed map. If Y is determined by a cover $\mathcal{P}$, then X is determined by the cover $\{f^{-1}(P) : P \in \mathcal{P}\}$.

A space X is *Fréchet* if for any $A \subset X$ and any $x \in \overline{A}$, there exist points $x_n \in A$ such that $\{x_n : n \in \mathbb{N}\}$ converges to x. Also, a space X is *strongly Fréchet* [12]; or *countably bi-sequential* [7], if for every decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then there exist points $x_n \in A_n$ ($n \in \mathbb{N}$) such that $\{x_n : n \in \mathbb{N}\}$ converges to the point x.

Let $f: X \to Y$ be a map. Then f is *bi-quotient* [6] (resp. *countably bi-quotient* [12]) if, whenever $y \in Y$ and $\mathcal{U}$ is a cover (resp. countable cover) of $f^{-1}(y)$ by open subsets of X, then finitely many $f(U)$, with $U \in \mathcal{U}$, cover a nbd of y in Y. Also, f is *hereditarily quotient* (or *pseudo-open*) if $f|f^{-1}(S) : f^{-1}(S) \to S$ is quotient for every $S \subset Y$ (equivalently, for any nbd U of $f^{-1}(y)$ in X, $\operatorname{int} f(U)$ is a nbd of y in Y (see [7]).

Obviously, we have the following implications: open (or perfect) map $\to$ bi-quotient map $\to$ countably bi-quotient map $\to$ hereditarily quotient map $\leftarrow$ closed map. Also, hereditarily quotient map $\to$ quotient map.

We recall the following characterizations by means of these maps. For these, and intrinsic definitions of related spaces, see [7]. Here, a space is an *M-space* if it is an inverse image of a metric space under a quasi-perfect map.

Characterizations: (1) A space X is bi-sequential (resp. countably bi-sequential; Fréchet; sequential) $\Leftrightarrow$ X is a bi-quotient (resp. countably bi-quotient; hereditarily quotient; quotient) image of a metric space.

(2) A space X is bi-k (resp. countably bi-k; singly bi-k; k) $\Leftrightarrow$ X is a bi-quotient (resp. countably bi-quotient; hereditarily quotient; quotient) image of a paracompact M-space.

(3) A space X is bi-quasi-k (resp. countably bi-quasi-k; singly bi-quasi-k; quasi-k) $\Leftrightarrow$ X is a bi-quotient (resp. countably bi-quotient; hereditarily quotient; quotient) image of an M-space.

We recall that a decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of sets is a *k-sequence* (resp. *q-sequence*) [7] if $K = \bigcap\{A_n : n \in \mathbb{N}\}$ is compact (resp. countably compact), and any open set $U \supset K$ contains some A_n. Recall that a space X is of *pointwise countable type* (resp. *q-space*) if each point has nbds $\{V_n : n \in \mathbb{N}\}$ which is a k-sequence (resp. q-sequence). Obviously, every first countable space is of pointwise countable type.

Recall that a space X is of pointwise countable type (resp. q-space) if and only if X an open image of a paracompact M-space (resp. M-space); see [7]. Thus, we can replace "paracompact M-space (M-space)" by "space of pointwise countable type (resp. q-space)" in Characterizations.

As weaker concepts than "strongly Fréchet spaces", let us recall Tanaka spaces and strongly sequential spaces defined by F. Mynard.

A space X is a *Tanaka space* (or *Tanaka topology*) [10] if it satisfies the following condition (C) in [16].

(C) Let $\{A_n : n \in \mathbb{N}\}$ be a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$. Then there exist $x_n \in A_n$ such that $\{x_n : n \in \mathbb{N}\}$ converges to some point $y \in X$.

Obviously, every sequentially compact space is precisely a countably compact Tanaka space. We note that every Tanaka space need not be sequential (not even a k-space).

A space X is *strongly sequential* [9] if, whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then the point x belongs to the (idempotent) *sequential closure* of the set A of limit points of convergent sequences $\{x_n : n \in \mathbb{N}\}$ with $x_n \in A_n$. Namely, a space X is strongly sequential if and only if it is a sequential space such that if $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then the point x belongs to the (usual) closure of the above set A.

A space X is *inner-closed* A [8] if, whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n - \{x\}}$ for any $n \in \mathbb{N}$, then there exist $F_n \subset A_n$ which are closed in X such that $\bigcup\{F_n : n \in \mathbb{N}\}$ is not closed in X. Among sequential spaces, we can assume that the F_n are singletons.

Let $S = \{\infty\} \cup \{p_n : n \in \mathbb{N}\} \cup \{p_{nm} : n, m \in \mathbb{N}\}$ be an infinite countable space such that each p_{nm} is isolated in S, $K = \{p_n : n \in \mathbb{N}\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in \mathbb{N}\}$ converges to $p_n \notin L_n$. The space S is called the *Arens' space* S_2, if for every finite $F_n \subset L_n$ $(n \in \mathbb{N})$, $\bigcup\{F_n : n \in \mathbb{N}\}$ is closed in S. The quotient space $S_2/(K \cup \{\infty\})$ is the *sequential fan* S_ω; that is, S_ω is the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points. The sequential spaces S_2, S_ω, and their modifications have played important roles in the theory of products of k-spaces; see [21], [22], [24], and [25], for example.

The following diagrams, etc., hold in view of Characterizations, or these are easily shown, but the first implication in Diagram (6) is shown in [7].

Diagrams: (1) First countable space → bi-sequential space → countably bi-sequential → Fréchet space → sequential space → k-space.

(2) Countably bi-sequential space (= strongly Fréchet space) → strongly sequential space → sequential space.

(3) Countably bi-sequential space → countably bi-k-space → countably bi-quasi-k-space ← bi-quasi-k-space ← bi-k-space ← bi-sequential space.

(4) Compact space → paracompact M-space → space of pointwise countable type → bi-k-space → countably bi-k-space → singly bi-k-space → k-space → quasi-k-space.

(5) Countably compact space → M-space → q-space → bi-quasi-k-space → countably bi-quasi-k-space → singly bi-quasi-k-space → quasi-k-space.

(6) Countably bi-quasi-k-space $\to$ inner-closed A-space $\to$ space which contains no closed copy of S_ω, and no S_2 $\leftarrow$ Tanaka space.

For a space, let us consider the following properties. Here, a cover $\mathcal{P}$ of a space X is called a *k-network* for X if, for any compact subset K of X and any open set $V \supset K$, $K \subset \bigcup \mathcal{P}' \subset V$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Every countably bi-$k$-space having property (P4) has a point-countable base, and every quotient Lindelöf image of a metric space has property (P4); see [3]. Here, a map is *Lindelöf* if every inverse-image of a point is Lindelöf. As for products of k-spaces having certain point-countable k-networks, see [24].

(P1) Fréchet space.
(P2) Space in which every point is a G_δ-set.
(P3) Hereditarily normal space.
(P4) Space with a point-countable k-network.
(P5) Closed image of a countably bi-k-space.
(P6) Closed image of an M-space.

Let us recall the following results. For (1) & (2); (3); (4); (5); and (6), see [20]; [4]; [25]; [16]: and [7] respectively.

Results: (1) For a space X having (P1), X is strongly Fréchet $\Leftrightarrow$ X contains no (closed) copy of S_ω.

(2) For a sequential space X having (P2) or (P3), X is strongly Fréchet $\Leftrightarrow$ X contains no (closed) copy of S_ω, and no S_2.

(3) For a k-space X having property (P4), X is first countable $\Leftrightarrow$ X contains no (closed) copy of S_ω, and no S_2.

(4) For a sequential space X having properties (P5) (resp. (P6)), X is a countably bi-k-space (resp. q-space) $\Leftrightarrow$ X contains no (closed) copy of S_ω.

(5) For a space X having (P1) or (P2), X is strongly Fréchet $\Leftrightarrow$ X is a Tanaka space.

(6) For a sequential space X which is a quotient Lindelöf image of a paracompact space S, if S is bi-sequential (resp. countably bi-sequential; bi-k; countably bi-k; bi-quasi-k; countably bi-quasi-k), then so is X respectively $\Leftrightarrow$ X is inner-closed A (equivalently, Tanaka space).

Results and questions

In [16], it is shown that, for a first countable space X, if $X \times Y$ is sequential, then X is locally countably compact, or Y is a sequential Tanaka space, and that the converse holds under some conditions on Y. F. Mynard [10] and [9] obtained Theorems 1 and 2 below respectively. Theorem 1 implies that every sequential countably bi-quasi-k-space is strongly sequential by Diagram (6).

Theorem 1. *For a space X, the following are equivalent.*

(a) *X is strongly sequential.*

(b) *X is sequential inner-closed A.*
(c) *X is a sequential Tanaka space.*

Theorem 2. *Let X be first countable, and Y be sequential. Then $X \times Y$ is sequential if and only if X is locally countably compact, or Y is a strongly sequential space.*

The following lemma holds in view of the proof of Lemma 6 in [19].

Lemma 3. *Let X be a space determined by a point-countable cover $\mathcal{C}$. Then, for a q-sequence $\{A_n : n \in \mathbb{N}\}$ in X, some A_n is contained in a finite union of elements of $\mathcal{C}$.*

Proposition 4. *Let X be a space determined by a point-countable cover $\mathcal{C}$. Then each point of X has a nbd which is contained in a finite union of elements of $\mathcal{C}$ if the following* (a), (b), *or* (c) *holds.*

(a) *X is countably bi-quasi-k, and $\mathcal{C}$ is closed.*
(b) *X is sequential and inner-closed A* (*equivalently, strongly sequential*).
(c) *X is inner-closed A, and $\mathcal{C}$ is countable, and closed* (*or increasing*).

Proof: Case (a): Suppose that some point x of X has no nbds which are contained in a finite union of elements of $\mathcal{C}$. Let $\{C \in \mathcal{C} : x \in C\} = \{C_n : n \in \mathbb{N}\}$, and let $B_n = \bigcup\{C_m : m \leq n\}$ for each $n \in \mathbb{N}$. Then, $x \in \overline{X - B_n}$ for each $n \in \mathbb{N}$. Since X is countably bi-quasi-k, there exists a q-sequence $\{A_n : n \in \mathbb{N}\}$ such that $x \in \overline{((X - B_n) \cap A_n)}$ for all $n \in \mathbb{N}$ ([7]). But, by Lemma 3, some A_m is contained in a union of finitely many closed sets F_n in $\mathcal{C}$. Let $V = X - \bigcup\{F_n : x \notin F_n\}$, then V is a nbd of x, so $x \in \overline{((V - B_n) \cap A_n)}$ for all $n \in \mathbb{N}$. But, some $(V - B_n) \cap A_n$ must be empty. This is a contradiction. Thus each point has a nbd which is contained in an element of the cover $\mathcal{C}$.

Case (b): Suppose that some point x of X has no nbds which are contained in a finite union of elements of $\mathcal{C}$. As is known, since X is sequential, if $x \in \overline{A}$, then $x \in \overline{B}$ for some countable $B \subset A$; see [7], for example. Then, there exists a sequence $\{B_n : n \in \mathbb{N}\}$ of countable subsets such that $B_1 = \{x\}$, $x \in \overline{B_n}$, and $B_n \cap C_i(B_j) = \emptyset$ whenever $i < n$ and $j < n$, here $\{C_i(B_j) : i \in \mathbb{N}\} = \{C \in \mathcal{C} : C \cap B_j \neq \emptyset\}$. Let $A_n = \bigcup\{B_k : k \geq n\}$. Then $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence such that $x \in \overline{A_n}$, but any $C \in \mathcal{C}$ meets only finitely many A_n. Since X is inner-closed A, there exist $F_n \subset A_n$ which are closed in X, but $A = \bigcup\{F_n : n \in \mathbb{N}\}$ is not closed in X. Since X is determined by $\mathcal{C}$ and A is not closed in X, some $C \in \mathcal{C}$ meets infinitely many closed sets F_n, so infinitely many A_n. This is a contradiction. Thus, each point has a nbd which is contained in a finite union of elements of $\mathcal{C}$.

Case (c): We can assume that $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ is increasing. Suppose that some point x of X has no nbds which are contained in some C_n. Then $x \in \overline{X - C_n}$

for each $n \in \mathbb{N}$. But, X is inner-closed A, so some C_m meets infinitely many $X - C_n$, a contradiction. $\square$

As is well-known, every product of a closed map with the identity map need not be quotient. Also, every product of a quasi-perfect map f with the identity map need not be closed ([13]).

Lemma 5. (1) *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be quasi-perfect maps. If X_1 is a k-space, and $Y_1 \times Y_2$ is sequential, then $f_1 \times f_2$ is quasi-perfect* ([13]).

(2) *Every product of bi-quotient maps is bi-quotient* ([6]).

(3) *Let $f : X \to Y$ be a countably bi-quotient map. If Z is first countable, then $f \times \mathrm{id}_Z$ is countably bi-quotient* ([7]).

We have the following sufficient conditions for products to be k-spaces. The result for case (c) or (e) is shown in [25]. For case (d) (resp. (e)), when X is countably bi-quasi-k or strongly sequential (resp. inner-closed A), X is locally compact by means of Proposition 4.

Theorem 6. *One of the following* (a)–(e) *implies that $X \times Y$ is a k-space.*

(a) *X is strongly sequential, and Y is a k-space which is bi-quasi-k.*
(b) *X is bi-k, and Y is a k-space which is countably bi-quasi-k.*
(c) *X is sequential, and Y is a k-space which is locally countably compact.*
(d) *X and Y are singly bi-quasi-k-spaces determined by a point-countable closed cover of locally compact subsets.*
(e) *X and Y are spaces determined by a countable, and closed* (*or increasing*) *cover of locally compact subsets.*

Proof: Case (a): Let Y be an image of an M-space S under a bi-quotient map f by Characterization (3). Let S be an inverse image of a metric space T under a quasi-perfect map g. Then $X \times T$ is a sequential space by Theorem 2. Hence, $X \times S$ is an inverse image of the sequential space $X \times T$ under a quasi-perfect map $\mathrm{id}_X \times g$ by Lemma 5(1). While, $X \times T$ is a k-space, so it is determined by the cover of all compact subsets. Thus, by Fact (1), $X \times T$ is determined by a cover $\{C \times T : C$ is compact in $X\}$. Thus, by Fact (4), $X \times S$ is determined by a cover $\{C \times S : C$ is compact in $X\}$. But, $X \times Y$ is an image of $X \times S$ under a quotient map $\mathrm{id}_X \times f$ by Lemma 5(2). Thus, $X \times Y$ is determined by a cover $\{C \times Y : C$ is compact in $X\}$ by Fact (3). But, as is well-known, each element $C \times Y$ is a k-space, for C is compact and Y is a k-space. Thus, $X \times Y$ is a k-space, for it is determined by the cover of all compact subsets by Facts (1) & (2).

Case (b): Let X be the image of a paracompact M-space S under a bi-quotient map f by Characterization (2). Let Y be the image of an M-space S' under a countably bi-quotient map g by Characterization (3). Let S be the inverse image of a metric space T under a perfect map p. Let S' be the inverse image of a metric space T' under a quasi-perfect map q. Then, $T \times S'$ is the inverse image of a metric space $T \times T'$ under a quasi-perfect map $\mathrm{id}_T \times q$. Then, $T \times S'$ is determined by a cover $\{C \times S' : C$ is a compact in $T\}$. While, $T \times Y$ is the image of $T \times S'$ under a quotient map $\mathrm{id}_T \times g$ by Lemma 5(3). Thus, $T \times Y$ is determined by a cover $\{C \times Y : C$ is a compact in $T\}$. But, $S \times Y$ is the inverse image of $T \times Y$ under a perfect map $p \times \mathrm{id}_Y$, so $S \times Y$ is determined by a cover $\{p^{-1}(C) \times Y : C$ is a compact in $T\}$ (by means of Fact (4)). Since $X \times Y$ is the image of $S \times Y$ under a quotient map $f \times \mathrm{id}_Y$, $X \times Y$ is determined by a cover $\{f(p^{-1}(C)) \times Y : C$ is a compact in $T\}$. But, each element $f(p^{-1}(C)) \times Y$ is a k-space since $f(p^{-1}(C))$ is compact in X. Thus, $X \times Y$ is also a k-space.

Case (d): Let X (resp. Y) be determined by a point-countable closed cover $\mathcal{C}$ (resp. $\mathcal{K}$) of locally compact subsets. We will show that $X \times Y$ is determined by a cover $\mathcal{L} = \{C \times K : C \in \mathcal{C}, K \in \mathcal{K}\}$, then $X \times Y$ is a k-space, because each element $C \times K$ is locally compact (hence a k-space). So, for $F \subset X \times Y$, suppose that $F \cap (C \times K)$ is closed in $C \times K$ for each $C \times K \in \mathcal{L}$. To show F is closed in $X \times Y$, let $A = X - F$, and let $(x, y) \in A$. Let $\{C \in \mathcal{C} : x \in C\} = \{C_n : n \in \mathbb{N}\}$, and $\{K \in \mathcal{K} : y \in K\} = \{K_n : n \in \mathbb{N}\}$. Here, we can assume that $C_n \subset C_{n+1}$ and $K_n \subset K_{n+1}$ since C_n and K_n are closed in X and Y respectively. Since $A \cap (C_n \times K_n)$ is open in $C_n \times K_n$ for each $n \in \mathbb{N}$, by induction, it is routine to show that there exist nbds U_n of x in C_n, and nbds V_n of y in K_n such that $(\overline{U}_n \times \overline{V}_n) \subset (U_{n+1} \times V_{n+1}) \subset (\overline{U}_{n+1} \times \overline{V}_{n+1}) \subset (C_{n+1} \times K_{n+1}) \cap A$, and all $\overline{U}_n$, $\overline{V}_n$ are compact sets. Let $U = \bigcup\{U_n : n \in \mathbb{N}\}$, and $V = \bigcup\{V_n : n \in \mathbb{N}\}$. Then $U \times V \subset A$. Also, U is a nbd of x in X. Indeed, suppose U is not a nbd of x in X. Then $x \in \overline{X - U}$. Since X is singly bi-quasi-k, there exists a q-sequence $\{A_n : n \in \mathbb{N}\}$ such that $x \in \overline{((X - U) \cap A_n)}$ for all $n \in \mathbb{N}$ ([7]). But, by Lemma 3, some A_m is contained in a finite union of elements of $\mathcal{C}$. Then, $x \in \overline{((X - U) \cap C)}$ for some $C \in \mathcal{C}$. Then, $x \in C$, so we can assume $C = C_k$ for some $k \in \mathbb{N}$. But, $U_k \cap (X - U) \supset U_k \cap ((X - U) \cap C_k) \neq \emptyset$. This is a contradiction to $U_k \cap (X - U) = \emptyset$. Hence U is a nbd of x in X. Similarly, V is a nbd of y in Y. Then A is open in $X \times Y$, thus F is closed in $X \times Y$. This shows that $X \times Y$ is a k-space. (For case (e), we can assume that the countable closed cover $\mathcal{C}$ is increasing by Fact (1), then $X \times Y$ is a k-space as in the first half of the proof).

□

Corollary 7. *The following* (a) *or* (b) *implies that $X \times Y$ is a k-space.*

(a) *X is sequential countably bi-quasi-k, and Y is a k-space which is bi-quasi-k.*

(b) *X is bi-k, and Y is countably bi-k* ([17]).

Lemma 8. *Let X and Y be sequential spaces. Then $X \times Y$ is sequential if and only if it is a k-space* ([14]).

Corollary 9. *Each of the following items* (a), (b), *or* (c) *implies that $X \times Y$ is a sequential space.*

(a) *X is strongly sequential, and Y is sequential bi-quasi-k.*

(b) *X is sequential countably bi-quasi-k, and Y is sequential bi-quasi-k.*
(c) *X is sequential, and Y is sequential locally countably compact* ([1]).

Corollary 10. *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be maps such that X_i are locally compact* (*resp. sequential locally compact*). *Then each of the following items* (a)–(e) *implies that $Y_1 \times Y_2$ is a k-space* (*resp. sequential space*).

(a) *f_i are quotient maps, and X_i are Lindelöf.*
(b) *f_i are quotient Lindelöf maps, X_i are paracompact, and Y_i are singly bi-quasi-k.*
(c) *f_i are hereditarily quotient Lindelöf maps, X_i are paracompact.*
(d) *f_i are closed maps, X_i are paracompact, and Y_i are locally Lindelöf.*
(e) *f_i are closed Lindelöf maps.*

Proof: For case (a), X_i are determined by a countable cover of compact subsets, then so are Y_i by Fact (3). Hence $Y_1 \times Y_2$ is a k-space by Theorem 6. For case (b), X_i are determined by a locally finite cover of compact subsets, Y_i are determined by a point-countable cover of compact subsets by Fact (3). But, Y_i are singly bi-quasi-k, then $Y_1 \times Y_2$ is a k-space by Theorem 6. Case (c) implies case (b), because every hereditarily quotient image of a locally compact space is singly bi-k ([7]), hence singly bi-quasi-k. For case (d), it is routine that Y_i are determined by a hereditarily closure-preserving cover of compact subsets. Then, since Y_i are locally Lindelöf, each point of Y_i has a nbd which is determined by a countable cover of compact subsets. Thus, $Y_1 \times Y_2$ is a locally k-space by Theorem 6. Hence, $Y_1 \times Y_2$ is a k-space by means of Fact (2). The result for case (e) is due to [18]. The parenthetic part holds by means of Lemma 8. □

Remark 11. (1) As is well-known, there exist a separable metric space X (or, closed image X of a metric locally compact space), and a closed image Y of a separable metric locally compact space such that $X \times Y$ is not a k-space; see [2], [18], for example.

(2) ($2^{\omega_0} < 2^{\omega_1}$). There exist countable, strongly Fréchet spaces X and Y such that $X \times Y$ is not a k-space ([11]).

(3) There exists a paracompact space X which is a quotient compact image of a metric locally compact space such that X^2 is not a k-space ([19]).

We note that every quotient Lindelöf image of a paracompact locally compact space is precisely a space determined by a point-countable cover of compact subsets by Fact (3). In view of Theorem 6, Corollary 10 and Remark 11(3), the author has a question whether every product of quotient Lindelöf images of paracompact locally compact spaces is a k-space if the images are Lindelöf. We recall the following general question. (1), (2) was posed in [21], [22] respectively.

Question 12. (1) Let X and Y be quotient Lindelöf images of paracompact locally compact spaces. What is a necessarily and sufficient condition for $X \times Y$ to be a k-space ?

(2) Let X and Y be closed images of paracompact bi-k-spaces. What is a necessarily and sufficient condition for $X \times Y$ to be a k-space ?

Let us review partial answers to Question 12. First, we recall some related matters. For a cardinal number α, a space is a *k_α-space* if it is determined by a cover $\mathcal{C}$ of compact subsets with $|\mathcal{C}| \leq \alpha$. A space X is *locally* $< k_\alpha$ if each point $x \in X$ has a nbd whose closure is a k_{β_x}-space, $\beta_x < \alpha$. Every locally compact space is locally k_ω (i.e., locally $< k_{\omega_1}$), and so is every space determined by a countable closed cover of locally compact subsets ([24]).

For products of k-spaces, we recall the following Hypotheses (H) and (H*); see [21] and [22] (or [24]). (A pair of spaces X and Y is said to have *Tanaka's condition* in [5], if (a'), (b), or (c) in (H) holds, where (a') X and Y are first countable.)

(H): Let X and Y be k-spaces. Then $X \times Y$ is a k-space if and only if (a), (b) or (c) below holds. (The "if" part of (H) is valid).

(a) X and Y are bi-k.
(b) X or Y is locally compact.
(c) X and Y are locally k_ω.

(H*): Same as (H), but change (c) to (c'): One of X and Y is locally k_ω, and another is locally $< k_{\mathfrak{c}}$, $\mathfrak{c} = 2^\omega$.

Let F be the collection of all functions from $\mathbb{N}$ to $\mathbb{N}$. The set-theoretic axiom BF(ω_2) means that if whenever $A \subset F$ with $|A| < \omega_2$, there exists $g \in F$ such that $f \leq g$ for all $f \in A$, here $f \leq g$ means $\{n \in \mathbb{N} : f(n) > g(n)\}$ is finite. (CH) implies BF(ω_2) is false, and Martin's axiom (MA) + $\neg$ CH implies BF(ω_2).

Then, for example, we have the following partial answers to Question 12. (1) holds by means of [24, Theorem 1.1] and [23, Theorem 2.3], and (2), (3), and (4) are due to [22, Theorem 1.1].

Theorem 13. (1) *Let X and Y be Fréchet spaces which are quotient Lindelöf images of metric spaces. Then Hypothesis* (H) *holds.*

(2) *Let X and Y be sequential spaces which are closed Lindelöf images of paracompact bi-k-spaces. Then Hypothesis* (H) *holds.*

(3) *BF(ω_2) is false if and only if the assertion* $(*)$ *below is valid. When $X = Y$,* $(*)$ *is valid without any set-theoretic axiom.*

$(*)$*: Let X and Y be sequential spaces which are closed images of paracompact bi-k-spaces. Then Hypothesis* (H) *holds.*

(4) *Let X and Y be sequential spaces which are closed images of paracompact bi-k-spaces. Then the "only if" part of Hypothesis* (H*) *holds. Also, under* (MA)*, Hypothesis* (H*) *holds if all compact sets in X and Y are metric* (*in particular, X and Y are closed images of metric spaces*).

However, Hypothesis (H) does not suggest an answer to Question 12. Indeed, under BF(ω_2), there exist spaces X and Y which are quotient finite-to-one (or closed) images of metric locally compact spaces such that $X \times Y$ is a k-space, but none of the properties (a), (b), and (c) holds ([5] or [24]).

Now, the author does not know whether every product of countably compact k-spaces is a k-space, more generally he has the following question in view of Theorem 6. When X is sequential, this is affirmative by Corollary 7.

Question 14. Let X be a k-space which is bi-quasi-k (or countably bi-quasi-k), and let Y be a k-space which is bi-quasi-k. Is $X \times Y$ a k-space ?

Lemma 15. *Let X be a bi-k-space, and let Y be sequential. If $X \times Y$ is a k-space, then X is locally countably compact, or Y is a Tanaka space* ([25]).

The following holds by means of Lemma 15, and Theorems 1 & 6.

Theorem 16. *Let X be a bi-k-space, and let Y be sequential. Then $X \times Y$ is a k-space if and only if X is locally countably compact, or Y is a Tanaka space.*

Question 17. In the previous theorem, is it possible to replace "bi-k-space" by "k-space which is a bi-quasi-k-space (or M-space)" ?

The "if" part of Theorem 16, under Y being a k-space which is bi-quasi-k, remains true by Theorems 1 & 6. Thus, Question 17 is reduced to the question whether the replacement in Lemma 15 remains valid.

The following holds by means of Theorem 16, Corollary 7, and Results.

Corollary 18. *Let X be a bi-k-space. Let Y be a sequential space having one of the properties* (P1)–(P6) *in the previous section. Then the following* (a), (b), *and* (c) *are equivalent* ([25]).

(a) *$X \times Y$ is a k-space.*
(b) *X is locally countably compact, or Y is a Tanaka space.*
(c) *X is locally countably compact, or Y contains no* (*closed*) *copy of S_ω, and no S_2.*

Question 19. Let X be a bi-k-space, and let Y be a sequential space. Is it true that $X \times Y$ is a k-space if and only if X is locally countably compact, or Y contains no (closed) copy of S_ω, and no S_2 ?

The "only if" part holds by Theorem 16. Question 19 is reduced to the question whether every sequential is a Tanaka space if it contains no (closed) copy of S_ω, and no S_2.

Question 19 is affirmative if Y is a quotient Lindelöf image of a metric space by Corollary 18. But, the author does not know whether Question 19 is also affirmative if Y is a sequential space which is a quotient Lindelöf image of a paracompact, and M-space (or bi-k-space) in view of Results.

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