Iterates of a class of discrete linear operators via contraction principle

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Abstract. In this paper we are concerned with a general class of positive linear operators of discrete type. Based on the results of the weakly Picard operators theory our aim is to study the convergence of the iterates of the defined operators and some approximation properties of our class as well. Some special cases in connection with binomial type operators are also revealed.

Keywords: linear positive operators, contraction principle, weakly Picard operators, delta operators, operators of binomial type

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1. Introduction

In the late decades the second author developed the theory of weakly Picard operators, see e.g. [9], [10], [11]. For the convenience of the reader, the basic features of the theory will be presented in the next section. Here we also comprised elements about delta operators and their basic polynomials. Further on, in Section 3 we construct a general class of linear positive operators acting on the space $C([a,b])$ and we study the convergence of their iterates. Simultaneously, approximation properties of our family of operators are investigated.

The focus of Section 4 is to present concrete examples of our approach. They are in connection with approximation operators of binomial type. This way, results regarding the iterates of Bernstein, Stancu and respectively Cheney-Sharma operators are obtained and a "bridge" between the contraction principle and approximation of functions by binomial polynomials is built up.

2. Notation and preliminaries

Definition 1 ([9])**.** Let (X,d) be a metric space. The operator $A : X \to X$ is a *weakly Picard operator* (WPO) if the sequence of iterates $(A^m(x))_{m\geq 1}$ converges for all $x \in X$ and the limit is a fixed point of A.

If the operator A is WPO and $\mathcal{F}_A = \{x^*\}$ then A is called a *Picard operator* (PO).

Here $\mathcal{F}_A := \{x \in X |\ A(x) = x\}$ stands for the fixed point set of A and, as usually, we put $A^0 = I_X$, $A^{m+1} = A \circ A^m$, $m \in \mathbb{N}$.

Moreover we have the following characterization of the WPOs.

Theorem 1 ([9]). *Let (X, d) be a metric space. The operator $A : X \to X$ is WPO if and only if a partition of X exists, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that for every $\lambda \in \Lambda$ one has*

(i) $X_\lambda \in \mathcal{I}(A)$,

(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,*

where $\mathcal{I}(A) := \{Y \subset X|\ A(Y) \subset Y\}$ represents the family of all non-empty invariant subsets of A.

Further on, if A is WPO we consider $A^\infty \in X^X$ defined by

$$A^\infty(x) := \lim_{m\to\infty} A^m(x), \quad x \in X. \tag{1}$$

Clearly, we have $A^\infty(X) = \mathcal{F}_A$. Also, if A is WPO, then ([11]) the identities $\mathcal{F}_{A^m} = \mathcal{F}_A \neq \emptyset$, $m \in \mathbb{N}$, hold true.

In what follows, some elements regarding the delta operators are presented. For any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we denote by Π_n the linear space of polynomials of degree less or equal to n and by Π_n^* the set of polynomials of degree n. We set $\Pi := \bigcup_{n\geq 0} \Pi_n$. A sequence $b = (b_n)_{n\geq 0}$, $b_n \in \Pi_n^*$ for every $n \in \mathbb{N}_0$, is called of *binomial type* if for any $(x, y) \in \mathbb{R} \times \mathbb{R}$ the following identities are satisfied

$$b_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(y), \quad n \in \mathbb{N}. \tag{2}$$

An operator $T \in \mathcal{L} := \{L : \Pi \to \Pi|\ L \text{ linear}\}$ which commutes with all shift operators E^a, $a \in \mathbb{R}$, is called a *shift-invariant operator* and the set of these polynomials will be denoted by $\mathcal{L}_s$. We recall: $(E^a p)(x) = p(x + a)$, $p \in \Pi$. Throughout the paper e_n stands for monomials, $e_0 = 1$ and $e_n(x) = x^n$, $n \in \mathbb{N}$.

Definition 2. An operator Q is called a *delta operator* if $Q \in \mathcal{L}_s$ and Qe_1 is a non-zero constant. $\mathcal{L}_\delta$ denotes the set of all delta operators.

A polynomial sequence $p = (p_n)_{n\geq 0}$ is called the *sequence of basic polynomials associated to Q* if one has $p_0 = e_0$, $p_n(0) = 0$ and $Qp_n = np_{n-1}$, for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

It was proved [8, Proposition 3] that every delta operator has a unique sequence of basic polynomials. We point out that the first rigorous version of the so-called *umbral calculus* belongs to Gian-Carlo Rota and his collaborators, see, e.g., [8]. Among the most recent survey papers dedicated to this subject we quote [1], [5] and in what follows we gather some classical results and formulas concerning this symbolic calculus.

Theorem 2. (a) *If $p = (p_n)_{n\geq 0}$ is a basic sequence for some delta operator Q, then it is a sequence of binomial type. Reciprocally, if p is a sequence of binomial type, then it is a basic sequence for some delta operator.*

(b) *Let $T \in \mathcal{L}_s$ and $Q \in \mathcal{L}_\delta$ with its basic sequence $p = (p_n)_{n \geq 0}$. One has*

$$T = \sum_{k \geq 0} \frac{(Tp_k)(0)}{k!} Q^k. \tag{3}$$

(c) *An isomorphism Ψ exists from $(\mathcal{F}, +, \cdot)$, the ring of the formal power series over $\mathbb{R}$ field, onto $(\mathcal{L}_s, +, \cdot)$ such that*

$$\Psi(\phi(t)) = T, \quad \text{where } \ \phi(t) = \sum_{k \geq 0} \frac{a_k}{k!} t^k \ \text{ and } \ T = \sum_{k \geq 0} \frac{a_k}{k!} Q^k.$$

(d) *An operator $P \in \mathcal{L}_s$ is a delta operator if and only if it corresponds under the isomorphism defined by (3), to a formal power series $\phi(t)$ such that $\phi(0) = 0$ and $\phi'(0) \neq 0$.*

(e) *Let $Q \in \mathcal{L}_\delta$ with $p = (p_n)_{n \geq 0}$ its sequence of basic polynomials. Let $\phi(D) = Q$ and $\varphi(t)$ be the inverse formal power series of $\phi(t)$, where D represents the derivative operator. Then one has*

$$\exp(x\varphi(t)) = \sum_{n \geq 0} \frac{p_n(x)}{n!} t^n, \tag{4}$$

where $\varphi(t)$ has the form $c_1 t + c_2 t^2 + \ldots$ $(c_1 \neq 0)$.

We accompany this brief exposition with the following

Examples. The symbol I stands for the identity operator on the space Π.
1. The *derivative operator* D has its basic sequence given by $(e_n)_{n \geq 0}$.
2. The *forward difference operator* $\Delta_h := E^h - I$ has its basic sequence $(x^{[n,h]})_{n \geq 0}$ where $x^{[0,h]} := 1$ and $x^{[n,h]} := x(x-h)\ldots(x - \overline{n-1}h)$ represent the generalized factorial power with the step h. Analogously, $((x + \overline{n-1}h)^{[n,h]})_{n \geq 0}$ is the sequence of basic polynomials associated to $\nabla_h := I - E^{-h}$, the *backward difference operator*. It is evident that $\nabla_h = \Delta_h E^{-h}$.
3. *Abel operator* $A_a := DE^a$, $a \neq 0$, is also a delta operator. For every $p \in \Pi$, $(A_a p)(x) = \dfrac{dp}{dx}(x+a)$ or, symbolically, we also can write $A_a = D(e^{aD})$. The Abel sequence of polynomials $\widetilde{a} = (a_n^{\langle a \rangle})_{n \geq 0}$, where $a_0^{\langle a \rangle} := 1$, $a_n^{\langle a \rangle}(x) := x(x - na)^{n-1}$, $n \in \mathbb{N}$, forms the sequence of basic polynomials associated to A_a.

It is not complicate to prove that all the above polynomial sequences verify relation (2).

3. A sequence of operators in study

At first we construct an approximation process of discrete type acting on the space $C([a,b])$ endowed with the Chebyshev norm $\|\cdot\|_\infty$. For each integer $n \geq 1$ we consider the following.

(i) A net on $[a,b]$ named Δ_n is fixed $(a = x_{n,0} < x_{n,1} < \cdots < x_{n,n} = b)$.
(ii) A system $(\psi_{n,k})_{k=\overline{0,n}}$ is given, where every $\psi_{n,k}$ belongs to $C([a,b])$. We assume that it is a blending system with a certain connection with Δ_n, more precisely the following conditions hold:

$$\psi_{n,k} \geq 0 \ (k = \overline{0,n}), \quad \sum_{k=0}^{n} \psi_{n,k} = e_0, \quad \sum_{k=0}^{n} x_{n,k}\psi_{n,k} = e_1. \tag{5}$$

Now we define the operators

$$L_n : C([a,b]) \to C([a,b]), \quad (L_n f)(x) = \sum_{k=0}^{n} \psi_{n,k}(x) f(x_{n,k}). \tag{6}$$

Remark 1. L_n, $n \in \mathbb{N}$, are positive linear operators and consequently they become monotone. Taking into account (5) we have $L_n e_0 = e_0$, $L_n e_1 = e_1$. Moreover $\|L_n\| := \sup_{\|f\|_\infty \leq 1} \|L_n f\|_\infty = 1$, for every $n \in \mathbb{N}$. We indicate the necessary and sufficient condition which offers to $(L_n)_{n\geq 1}$ the attribute of approximation process.

Theorem 3. *Let L_n, $n \in \mathbb{N}$, be defined by (6).*

(i) *If $\lim_{n\to\infty} \sum_{k=1}^{n} x_{n,k}^2 \psi_{n,k} = e_2$ uniformly on $[a,b]$ then for every $f \in C([a,b])$ one has $\lim_{n\to\infty} L_n f = f$ uniformly on $[a,b]$.*

(ii) *For every $f \in C([a,b])$, $x \in [a,b]$ and $\delta > 0$, one has*

$$|(L_n f)(x) - f(x)| \leq \Big(1 + \delta^{-1}\Big(\sum_{k=0}^{n} x_{n,k}^2 \psi_{n,k}(x) - x^2\Big)^{1/2}\Big)\omega_1(f,\delta),$$

where $\omega_1(f,\cdot)$ represents the modulus of continuity of f.

Proof: The first statement results directly from the theorem of Bohman-Korovkin and relations (5) as well. The second statement holds true by virtue of the classical results regarding the rate of convergence, see e.g. the monograph [2, Theorem 5.1.2]. $\square$

Our main objective is to study the convergence of the iterates of our operators. We state and prove

Theorem 4. *Let L_n, $n \in \mathbb{N}$, be defined by* (6) *such that $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$. Let us denote $u_n := \min_{x\in[a,b]}(\phi_{n,0}(x) + \phi_{n,n}(x))$.*

If $u_n > 0$ then the iterates sequence $(L_n^m)_{m\geq 1}$ verifies

$$\lim_{m\to\infty}(L_n^m f)(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a), \quad f \in C([a,b]), \tag{7}$$

uniformly on $[a,b]$.

Proof: At first we define

$$X_{\alpha,\beta} := \{f \in C([a,b]) \mid f(a) = \alpha,\ f(b) = \beta\}, \quad (\alpha,\beta) \in \mathbb{R}\times\mathbb{R}.$$

Clearly, every $X_{\alpha,\beta}$ is a closed subset of $C([a,b])$ and the system $X_{\alpha,\beta}$, $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$, makes up a partition of this space.

Since $\psi_{n,0}(a) = 1$ and $\psi_{n,n}(b) = 1$, the relations (5) imply $(L_n f)(a) = f(a)$ and $(L_n f)(b) = f(b)$, in other words for all $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ and $n \in \mathbb{N}$, $X_{\alpha,\beta}$ is an invariant subset of L_n.

Further on, we prove that $L_n|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$ is a contraction for every $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ and $n \in \mathbb{N}$. Indeed, if f and g belong to $X_{\alpha,\beta}$ then, for every $x \in [a,b]$, we can write

$$\begin{aligned}
|(L_n f)(x) - (L_n g)(x)| &= \left|\sum_{k=1}^{n-1} \psi_{n,k}(x)(f-g)(x_{n,k})\right| \\
&\leq \sum_{k=1}^{n-1} \psi_{n,k}(x)\|f-g\|_\infty \\
&= (1 - \phi_{n,0}(x) - \phi_{n,n}(x))\|f-g\|_\infty \\
&\leq (1-u_n)\|f-g\|_\infty,
\end{aligned}$$

and consequently, $\|L_n f - L_n g\|_\infty \leq (1-u_n)\|f-g\|_\infty$. The assumption $u_n > 0$ guarantees our statement.

On the other hand, the function $p^*_{\alpha,\beta} := \alpha + ((\beta-\alpha)/(b-a))(e_1 - a)$ belongs to $X_{\alpha,\beta}$ and since L_n reproduces the affine functions, $p^*_{\alpha,\beta}$ is a fixed point of L_n.

For any $f \in C([a,b])$ one has $f \in X_{f(a),f(b)}$ and, by using the contraction principle, we get $\lim_{m\to\infty} L_n^m f = p^*_{f(a),f(b)}$.

We obtained the desired result (7). □

Remark 2. Following the lines of the familiar Bohman-Korovkin arguments we indicate a necessary and sufficient condition for the iterates of our sequence $(L_n)_{n\geq 1}$ to converge to the identity operator. Considering $(k_n)_{n\geq 1}$ an increasing sequence of positive integer numbers tending to infinity, we enunciate:

$\lim_{n\to\infty} \|L_n^{k_n} f - f\|_\infty = 0$ *for each* $f \in C([a,b])$ *if and only if the same limit relation holds for* $f = e_2$.

Taking the advantage of Definition 1, Theorem 1 and relation (1) as well, for $X = C([a,b])$ we obtain

Corollary. *Under the hypothesis of Theorem* 4, *the operator* L_n *is WPO for every* $n \in \mathbb{N}$ *and*

$$L_n^\infty f = c_1(f)e_1 + c_2(f), \quad f \in C([a,b]),$$

where $c_1(f) := (f(b) - f(a))/(b-a)$ *and* $c_2(f) := (bf(a) - af(b))/(b-a)$.

Actually, $(L_n)_{n\geq 1}$ is a wide class of discrete operators and in the next section we show that it includes the so-called binomial operators.

4. Application

Keeping in mind the datum of Section 2, let Q be a delta operator and $p = (p_n)_{n\geq 0}$ be its sequence of basic polynomials under the additional assumption that $p_n(1) \neq 0$ for every $n \in \mathbb{N}$. Also, according to Theorem 2 we shall keep the same meaning of the functions ϕ and φ. For every $n \geq 1$ we consider the operator $L_n^Q : C([0,1]) \to C([0,1])$ defined as follows:

$$(L_n^Q f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}. \tag{8}$$

They are called (cf. e.g., P. Sablonniere [12]) *Bernstein-Sheffer operators*, but as D.D. Stancu and M.R. Occorsio motivated in [14], these operators can be named *Popoviciu operators*. In 1931 Tiberiu Popoviciu [7] indicated the construction (8), in front of the sum appearing the factor d_n^{-1} from the identities

$$(1 + d_1 t + d_2 t^2 + \dots)^x = \exp(x\varphi(t)) = \sum_{n=0}^{\infty} p_n(x) t^n / n!,$$

see (4). If we choose $x = 1$ it becomes obvious that $d_n = p_n(1)/n!$. In what follows we prove that the operators L_n^Q, $n \in \mathbb{N}$, are particular cases of the operators defined by (6). Firstly, in (6) we choose $a = 0$, $b = 1$, and

$$x_{n,k} = \frac{k}{n}, \quad \psi_{n,k}(x) = \binom{n}{k} p_k(x) p_{n-k}(1-x)/p_n(1), \quad k = \overline{0,n}. \tag{9}$$

Choosing in (2) $y := 1 - x$, from (8) we obtain $L_n^Q e_0 = e_0$.

The positivity of these operators are given by the sign of the coefficients of the series $\varphi(t) = c_1 + c_2 t + \dots$ $(c_1 \neq 0)$. More precisely, T. Popoviciu [7] and later P. Sablonniere [12, Theorem 1] have established that

Lemma 1. *L_n^Q is a positive operator on $C([0,1])$ for every $n \geq 1$ if and only if $c_1 > 0$ and $c_n \geq 0$ for all $n \geq 2$.*

Among the most significant results concerning L_n^Q operators (under the assumptions of Lemma 1) we recall:

(i)

$$L_n^Q e_1 = e_1,\ n \geq 1, \ \text{ and } \ L_n^Q e_2 = e_2 + a_n(e_1 - e_2),\ n \geq 2, \tag{10}$$

where $na_n = 1 + (n-1) r_{n-2}(1)/p_n(1)$, the sequence $(r_n(x))_{n\geq 0}$ being generated by

$$\varphi''(t)\exp(x\varphi(t)) = \sum_{n\geq 0} r_n(x) t^n/n!;$$

(ii) $L_n^Q f$ converges uniformly to $f \in C([0,1])$ if and only if the condition

$$\lim_{n\to\infty} (r_{n-2}(1)/p_n(1)) = 0$$

holds.

These results allow us to state

Lemma 2. *Let L_n^Q, $n \in \mathbb{N}$, be defined by (8). Under the hypothesis of Lemma 1 these operators satisfy the requirements (5) and consequently they are particular cases of L_n defined on the space $C([0,1])$ by formula (6).*

Since the basic polynomial p_0 satisfies $p_0 = e_0$, from (9) we get $\psi_{n,0}(0) = \psi_{n,n}(1) = 1$. Examining Theorem 4 we easily deduce

Theorem 5. *Let L_n^Q, $n \in \mathbb{N}$, be defined by (8) such that the assumptions of Lemma 1 are fulfilled. Let us consider the polynomials q_n, $n \in \mathbb{N}$, where $q_n(x) := p_n(1-x) + p_n(x)$. If the polynomials q_n, $n \in \mathbb{N}$, have no zeros on $[0,1]$ then the iterates sequence $((L_n^Q)^m)_{m\geq 1}$ verifies*

$$\lim_{m\to\infty} (L_n^Q)^m f = f(0) + (f(1) - f(0)) e_1, \tag{11}$$

uniformly on $[0,1]$.

At the end, choosing concrete delta operators Q we reobtain some classical linear positive operators. Practically we come back to Examples given in Section 2 and we apply Theorem 5.

1. If $Q := D$, then L_n^D becomes the *Bernstein operator* B_n,

$$(L_n^D f)(x) \equiv (B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]. \tag{12}$$

We have $q_n(x) = (1-x)^n + x^n \geq 1/2^{n-1}$, $x \in [0,1]$, and the identity (11) holds true, in accordance with a result due to R.P. Kelisky and T.J. Rivlin [4, Equation (2.4)].

2. If $Q := \dfrac{1}{\alpha}\nabla_\alpha$, $\alpha > 0$, then $L_n^{\alpha^{-1}\nabla_\alpha}$ becomes the *Stancu operator* [13] $P_n^{[\alpha]}$,

$$(L_n^{\alpha^{-1}\nabla_\alpha} f)(x) \equiv (P_n^{[\alpha]} f)(x) = \sum_{k=0}^{n} w_{n,k}(x;\alpha) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $w_{n,k}(x;\alpha) = \binom{n}{k} x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}/1^{[n,-\alpha]}$, α being a positive parameter which may depend only on the natural number n. For the above Q the basic polynomials are given by $p_n(x) = (x + (n-1)\alpha)^{[n,\alpha]}$ and consequently, $q_n(x) \geq (1-x)^n + x^n \geq 1/2^{n-1}$, $x \in [0,1]$. Once more, our identity (11) harmonizes with a result established in 1978 by G. Mastroianni and Mario Rosario Occorsio [6].

3. We choose $Q := A_a$ (Abel operator) with its basic sequence $\widetilde{a}$. Assuming that the parameter a is non positive and depends on n, $a := -t_n$, one obtains the *Cheney-Sharma operator* named G_n^*, see [3] or the monograph [2, Equation (5.3.16)]. It is known: if the sequence $(nt_n)_{n\geq 1}$ converges to zero then $\lim_{n\to\infty} \|G_n^* f - f\|_\infty = 0$ for every $f \in C([0,1])$. This time we have $p_0(x) = 1$ and $p_n(x) = x(x+nt_n)^{n-1}$, $n \in \mathbb{N}$. Thus, the polynomials q_n, $n \in \mathbb{N}_0$, have no zeros on $[0,1]$ and (11) holds true.

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