Semilinear elliptic problems with nonlinearities depending on the derivative

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Abstract. We deal with the boundary value problem

$$\begin{aligned} -\Delta u(x) &= \lambda_1 u(x) + g(\nabla u(x)) + h(x), \quad && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is an smooth bounded domain, λ_1 is the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions on Ω, $h \in L^{\max\{2,N/2\}}(\Omega)$ and $g : \mathbb{R}^N \longrightarrow \mathbb{R}$ is bounded and continuous. Bifurcation theory is used as the right framework to show the existence of solution provided that g satisfies certain conditions on the origin and at infinity.

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1. Introduction

We consider the semilinear elliptic boundary value problem:

$$\begin{aligned} -\Delta u(x) &= \lambda_1 u(x) + g(\nabla u(x)) + h(x), \quad && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $h \in L^{\max\{2,N/2\}}(\Omega)$, $g : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous nonlinearity and λ_1 is the first eigenvalue of the Laplace operator with zero Dirichlet boundary condition on Ω.

In contrast with the case in which the nonlinearity g depends on the solution u (instead of the derivatives of u), the literature on this kind of problems is not very large. Specifically, it has been studied in [2], [6], [7], [9], [10], [15], [16], [17], [20], [23]. Indeed, all of them except [6], [10], [23] are only for the one-dimensional case, either with Dirichlet boundary value conditions [7], [9], [16], [17] or Neumann and periodic boundary conditions [2], [7], [15], [20]. In [7] it was considered the problem

$$\begin{aligned} -u''(x) &= u(x) + g(u'(x)) + h(x), \quad && x \in [0,\pi] \\ u(0) &= u(\pi) = 0. \end{aligned} \tag{2}$$

Splitting $h(x) = \overline{h}\sin(x) + \widetilde{h}(x)$, $\int_0^\pi \widetilde{h}(x)\sin(x)dx = 0$ and assuming that the nonlinearity g has finite limits $g(+\infty) = \lim_{\xi\to+\infty} g(\xi)$ and $g(-\infty) = \lim_{\xi\to-\infty} g(\xi)$ with $g(+\infty) + g(-\infty) = 0$ (without loss of generality), the authors show that for every fixed $\widetilde{h} \in C[0,\pi]$ there exist $a, b \in \mathbb{R}$, $a \le 0 \le b$ such that (2) has no solution if $\overline{h} \notin [a,b]$ while (2) has at least one solution if $\overline{h} \in (a,b) \cup (\{a,b\} - \{0\})$. This result was improved in [16] where existence of at least two solutions is proved provided that $\overline{h} \in (a,b) - \{0\}$. However, we have to remark that in the above results nothing is said about the nondegeneration of the interval $[a,b]$. In principle, it could be a single point. In order to overcome this possibility, following [17], Habets and Sanchez [16] prove that $a < 0 < b$ provided that, in addition, g is C^1 with $g(0) = 0 \ne g'(0)$.

The scope of this paper is to make clear how bifurcation theory gives a right framework which enables us to study the P.D.E. case, i.e. problem (1). Specifically, we see that the existence of solution of this problem is heavily supported on sufficient conditions that means a balance between the linearized problem at infinity

$$\begin{aligned} -\Delta w(x) &= \lambda w(x), && x \in \Omega \\ w(x) &= 0, && x \in \partial\Omega \end{aligned}$$

and the linearized problem at zero

$$\begin{aligned} -\Delta w(x) &= \lambda w(x) + \nabla g(0)\cdot\nabla w(x), && x \in \Omega \\ w(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

Indeed, it is worthwhile noting that the role of the linearized problem at zero seems to be not very well understood in the previous references. We can illustrate easily the meaning of the asymptotic behaviour at infinity of the nonlinearity g in (1) by considering firstly a simpler model with bounded nonlinearity g depending on x and on $\frac{\partial u}{\partial x_1}$. More clearly, let us assume that Ω is a symmetric domain in the Steiner sense (i.e. symmetric with respect to the hyperplane $x_1 = 0$ and convex in the variable x_1) and consider for $\lambda \in \mathbb{R}$, the boundary value problem

$$(3) \qquad \begin{aligned} -\Delta u(x) &= \lambda u(x) + g(x, \frac{\partial u}{\partial x_1}(x)) + h(x), && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

It is well-known ([13]) that in this case a positive eigenfunction φ_1 associated to λ_1 is symmetric with respect to $x_1 = 0$ with $\frac{\partial\phi_1}{\partial x_1} < 0$ for $x_1 > 0$ (and $\frac{\partial\phi_1}{\partial x_1} > 0$ for $x_1 < 0$). Using φ_1 as test function in (3) we get that

$$(4) \qquad (\lambda_1 - \lambda)\int_\Omega u\varphi_1 = \int_\Omega g(x, \frac{\partial u}{\partial x_1})\varphi_1 + \int_\Omega h\varphi_1.$$

Therefore, if we assume that there exists the limits $g_{\pm\infty}(x) = \lim_{s\to\pm\infty} g(x,s)$ (unif. in $x \in \Omega$) with

$$\begin{aligned}(5)\qquad \int_{x_1>0} g_{+\infty}(x)\varphi_1 + \int_{x_1<0} g_{-\infty}(x)\varphi_1 < -\int_\Omega h\varphi_1 < \int_{x_1>0} g_{-\infty}(x)\varphi_1 \\ + \int_{x_1<0} g_{+\infty}(x)\varphi_1\end{aligned}$$

then, observing that every sequence (λ_n, u_n) of solutions of (3) with λ_n tending to λ_1 and $\|u_n\|$ tending to $+\infty$ satisfies that the normalized sequence $u_n/\|u_n\|$ converges either to φ_1 or to $-\varphi_1$, we easily deduce that for such a sequence we have

$$\left(\int_\Omega u_n\varphi_1\right)\left(\int_\Omega g(x, \frac{\partial u_n}{\partial x_1})\varphi_1 + \int_\Omega h\varphi_1\right) > 0, \ \forall n >> 0.$$

Hence, identity (4) implies that the existing bifurcation from infinity at $\lambda = \lambda_1$ has to be to the left, or equivalently, there exists an *a priori* estimate of the norm of every solution u of (3) for $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$ ($\varepsilon > 0$ small enough). From this a priori bound, the existence of a sequence (λ, u_n) of solutions of (3) with $\lambda_n \downarrow \lambda_1$ and u_n converging to a solution of the resonant problem, i.e. of problem (3) for $\lambda = \lambda_1$, follows.

In this way, the hypothesis (5) (or the dual hypothesis obtained by reversing the inequalities) can be considered as an extension of the classical Landesman-Lazer condition [18] for nonlinearities $g(x,u)$. See [6], [10], [23] for similar results in this direction.

However, notice that if we suppose that g is autonomous (i.e., $g = g(\frac{\partial u}{\partial x_1})$) or, more general, if the limits at $\pm\infty$ do not depend on x (for instance they are zero) then the previous existence result is meaningless. Precisely, we study here the case in which $\lim_{|\xi|\to+\infty} g(\xi) = 0$ and obtain the following extension of the results in [7], [16].

Theorem 1.1. *If $g(0) = 0$, g is differentiable at zero with $\nabla g(0) \neq 0$ and $\lim_{|\xi|\to+\infty} g(\xi) = 0$ then there exists $\varepsilon > 0$ for which*

- (i) *problem (1) has at least one solution provided that $\int_\Omega h\varphi_1 = 0$ and $\|h\|_2 < \varepsilon$,*
- (ii) *problem (1) has at least two solutions provided that $\int_\Omega h\varphi_1 \neq 0$ and $\|h\|_2 < \varepsilon$.*

To prove this theorem, as above and following [4], [21], we embed (1) into a one-parameter family of b.v.p., namely

$$(P_\lambda^h)\qquad \begin{aligned} -\Delta u(x) &= \lambda u(x) + g(\nabla u(x)) + h(x), && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

We study the problem (P_λ^h) in the next section. We have to point out that the main difficulty for this bifurcation problem consists in the fact that, on the contrary with the previous results for nonlinearities g depending either on u or only on the first derivatives of u, here it is not possible to determine the side of the bifurcation from infinity at $\lambda = \lambda_1$ of (P_λ^h). Certainly, we shall prove that in our case the bifurcation from infinity occurs on both sides of $\lambda = \lambda_1$. Thus, we have no a priori estimates on the norm of solutions u of (P_λ^h) for λ near λ_1. Consequently, to show that the continuum emanating from infinity at λ_1 crosses the line $\lambda = \lambda_1$, some additional arguments based on the Leray-Schauder degree theory are needed. Specifically, we prove Theorem 2.6 below which gives an estimate from below of the number of solutions of (P_λ^h) as $\lambda < \lambda_1 + |\nabla g(0)|^2/16$. In this way, Theorem 1.1 is a direct corollary of Theorem 2.6.

Finally, in Section 3 we also show (see Theorem 3.1 below) that problem (1) has no solution provided that $\left|\int_\Omega h\varphi_1\right|$ is large enough.

2. A one-parameter family of problems

As it has been mentioned in the previous section, we devote this one to the study of problem (P_λ^h). A few words about some notation are in order: we consider the Sobolev space $H_0^1(\Omega)$, equipped with the norm $\|v\| = \|\nabla v\|_2$, $(v \in H_0^1(\Omega))$, where $\|\cdot\|_2$ is the usual norm of $L^2(\Omega)$. On the other hand, it is well-known that λ_1 is simple and its eigenspace is spanned by a positive function $\varphi_1 \in H_0^1(\Omega)$ which will be chosen so that $\|\varphi_1\| = 1$. To study (P_λ^h), firstly, we transform it into the problem of looking for the zeros of a suitable operator F_λ defined on $H_0^1(\Omega)$ by

$$F_\lambda(u) = u - \lambda Lu - L(g(\nabla u) + h) = 0, \; u \in H_0^1(\Omega),$$

where we denote by $L := (-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ the inverse of the Laplace operator with homogeneous Dirichlet boundary conditions.

Now we recall a result about the regularity of solutions which will be useful in the sequel. It is based on a result of Brézis and Kato [5] together with an application of the Calderon-Zygmund inequality.

Proposition 2.1. *Let us assume that the boundary $\partial\Omega$ is of class C^2. Then every solution u of (P_λ^h) satisfies $u \in C^{1,\alpha}(\overline{\Omega})$ for all $0 \le \alpha < 1$. Moreover, there exists $K = K(\Omega, N, \alpha)$ such that*

$$\|u\|_{1,\alpha} \le K\|u\|$$

where $\|\cdot\|_{1,\alpha}$ denotes the usual norm of $C^{1,\alpha}(\overline{\Omega})$.

Proof: Let us consider the problem

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) + g(\nabla u(x)) + h(x), \quad && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega \end{aligned} \tag{6}$$

and observe that $v(x) := g(\nabla u(x)) + h(x) \in L^{\max\{2,N/2\}}(\Omega)$. If we set

$$\widetilde{g}(x,u) = \lambda u(x) + v(x)$$

then

$$\widetilde{g}(x,u) = a(x)(1+|u|)$$

where $a(x) = \frac{\lambda u(x)+v(x)}{1+|u(x)|} \in L^{\max\{2,N/2\}}(\Omega)$. Taking into account [5] (see also Lemma B.3 in [26]) we have that $u \in L^q(\Omega)$ for all $q < \infty$ with

$$\|u\|_q \leq C_1 \|u\|$$

for a certain constant $C_1 = C_1(\Omega, N, q)$, where $\|\cdot\|_q$ denotes the usual norm of $L^q(\Omega)$. Thus, we use that $\partial\Omega \in C^2$ together with the Calderon-Zygmund inequality (see [14]) to prove that $u \in W^{2,q}(\Omega) \cap H_0^1(\Omega)$ with $\|u\|_{2,q} \leq C_2\|u\|_q$ for a certain constant $C_2 = C_2(\Omega, N, q)$, where $\|\cdot\|_{2,q}$ denotes the usual norm of the Sobolev space $W^{2,q}(\Omega)$. Finally, using the classical embedding theorems for the Sobolev spaces we can guarantee that $u \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha < 1$ and certain values of the parameter q. Thus, if we fix a suitable value for q, we have that $\|u\|_{1,\alpha} \leq C_3\|u\|_{2,q} \leq C_4\|u\|$, where $C_4 = C_4(\Omega, N, \alpha)$. This concludes the proof. □

Next, we give two lemmas. In the first, we prove an a priori estimate of the norm of a solution. In the second, we show that the problem (P_λ^0) has only the trivial solution provided that $\lambda < 0$.

Lemma 2.2. *If $\lambda < \lambda_1$ then every solution $u \in H_0^1(\Omega)$ of (P_λ^h) satisfies*

$$\|u\| \leq \frac{\sqrt{\lambda_1}}{\lambda_1 - \lambda}\left[\|g\|_\infty |\Omega|^{1/2} + \|h\|_2\right].$$

Proof: Suppose that $u \in H_0^1(\Omega)$ is a solution of (P_λ^h). Using u as test function, we obtain

$$\|u\|^2 = \lambda \int_\Omega u^2 + \int_\Omega g(\nabla u)u + \int_\Omega hu.$$

By using the variational characterization of the first eigenvalue λ_1 and Hölder inequality we deduce that

$$\begin{aligned}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 &\leq \int_\Omega g(\nabla u)u + \int_\Omega hu \\ &\leq \|g(\nabla u)\|_2\|u\|_2 + \|h\|_2\|u\|_2 \\ &\leq \frac{\|u\|}{\sqrt{\lambda_1}}\left[\|g\|_\infty|\Omega|^{1/2} + \|h\|_2\right],\end{aligned}$$

from which the assertion of the lemma clearly follows. □

Lemma 2.3. *If $\lambda < 0$ then the unique solution of (P^0_λ) is the zero solution.*

Proof: Let us assume that u is a solution of

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) + g(\nabla u(x)), \quad && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega \end{aligned} \tag{7}$$

for some $\lambda < 0$. It follows from Proposition 2.1 that $u \in C^1(\overline{\Omega})$. It suffices prove that $\min_{\overline{\Omega}} u = 0 = \max_{\overline{\Omega}} u$. We just show here that $\min_{\overline{\Omega}} u = 0$ (the proof of the equality $\max_{\overline{\Omega}} u = 0$ is similar, so that we leave the details to the reader). Let us assume that, on the contrary,

$$\min_{\overline{\Omega}} u = u(x_1) < 0, \; x_1 \in \Omega.$$

In this case, we claim that there exists $r > 0$ such that

$$\int_\Omega \nabla u \nabla \varphi > 0 \; \text{ for all } \; \varphi \in C_0^\infty(B(x_1, r)) \; \text{ and } \; \varphi \geq 0 \tag{8}$$

where $B(x_1, r)$ denotes the ball of center x_1 and radius r. Indeed, otherwise, for every $r > 0$ there would be a function $\varphi_r \in C_0^\infty(B(x_1, r))$ with $\varphi_r \geq 0$ and satisfying

$$\int_\Omega \nabla u \nabla \varphi_r \leq 0.$$

Without loss of generality we may assume that

$$\frac{1}{|B(x_1, r)|} \int_\Omega \varphi_r = 1$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$. It follows from the fact that u is a solution of (P^0_λ) that

$$\lambda \int_\Omega u\varphi_r + \int_\Omega g(\nabla u)\varphi_r = \int_\Omega \nabla u \nabla \varphi_r \leq 0.$$

Thus, if we divide by $|B(x_1, r)|$, we obtain

$$\lambda \frac{1}{|B(x_1, r)|} \int_\Omega u\varphi_r + \frac{1}{|B(x_1, r)|} \int_\Omega g(\nabla u)\varphi_r \leq 0.$$

Taking into account that since $u \in C^1(\overline{\Omega})$ and $\nabla u(x_1) = 0$,

$$\begin{aligned} &\lim_{r \to 0} \frac{1}{|B(x_1, r)|} \int_\Omega u\varphi_r = \lim_{r \to 0} [u(x_1) + \frac{1}{|B(x_1, r)|} \int_\Omega (u - u(x_1))\varphi_r] = u(x_1) \\ &\lim_{r \to 0} \frac{1}{|B(x_1, r)|} \int_\Omega g(\nabla u)\varphi_r = \lim_{r \to 0} \frac{1}{|B(x_1, r)|} \int_\Omega (g(\nabla u) - g(\nabla u(x_1)))\varphi_r = 0, \end{aligned} \tag{9}$$

we get from the above inequality that $\lambda u(x_1) \le 0$. This contradicts the fact that $\lambda < 0$ by hypothesis and $u(x_1) < 0$.

Therefore (8) holds and we can now use the Strong Maximun Principle (Theorem 8.19 of [14]) for the Laplace operator in the ball $B(x_1, r)$ to obtain

$$u(x) = u(x_1) \quad \text{for all} \quad x \in \overline{B(x_1, r)}.$$

Hence, for all $\varphi \in C_0^\infty(B(x_1, r))$ which satisfies $\varphi \ge 0$ we have

$$\int_\Omega \nabla u \nabla \varphi = 0,$$

which contradicts (8). This ends the proof. □

Lemma 2.4. *For every $\overline{\lambda} < \lambda_1 + \frac{|\nabla g(0)|^2}{16}$ there exist $\varepsilon, \delta > 0$ such that if $t \in \left[\frac{1}{2}, 1\right]$, $\lambda \le \overline{\lambda}$ and $\|h\|_2 < \varepsilon$, then the problem*

$$u = t\lambda Lu + L\left(tg(\nabla u) + (2t-1)h\right) \tag{10}$$

has no solution $u \in H_0^1(\Omega)$ with $\|u\| = \delta$.

Proof: We split the proof in two steps.

Step 1: *There exists $\delta > 0$ such that the unique solution $u \in H_0^1(\Omega)$ of* (10) *with $h \equiv 0$ and $\|u\| \le \delta$ is the zero constant.*

Step 2: *Conclusion for a general term h.*

Proof of Step 1: Take $h \equiv 0$. By applying Lemma 2.3 we can suppose without loss of generality that $0 \le \lambda \le \overline{\lambda}$. We argue by contradiction assuming that there exists sequences $\{t_n\} \subset \left[\frac{1}{2}, 1\right]$, $\{\lambda_n\} \subset [0, \overline{\lambda}]$ and $\{u_n\} \subset H_0^1(\Omega)$ such that for $h \equiv 0$ and $t = t_n$, u_n is a solution of (10) satisfying $0 < \|u_n\| \le \frac{1}{n}$. Denote $w_n = u_n/\|u_n\|$ and observe that it verifies the equation

$$w_n = t_n\lambda_n L w_n + L(t_n \nabla g(0) \cdot \nabla w_n) + L\left(t_n \frac{g(\nabla u_n)}{\|u_n\|} - t_n \nabla g(0) \cdot \nabla w_n\right).$$

By using that g is differentiable at zero and the Lebesgue dominated convergence theorem, we derive that (passing to a subsequence if necessary),

$$\lim_{n\to+\infty} \left\| \frac{g(\nabla u_n)}{\|u_n\|} - \nabla g(0) \cdot \nabla w_n \right\|_2 = 0$$

and it is deduced from the continuity of L that

$$\lim_{n\to+\infty} \left\| L\left(t_n \frac{g(\nabla u_n)}{\|u_n\|} - t_n \nabla g(0) \cdot \nabla w_n\right) \right\| = 0.$$

This together to the compactness of L, Proposition 2.1 and the boundedness of w_n in $H_0^1(\Omega)$ implies that, passing again to a subsequence if necessary, we can assume that

$$\lim_{n\to+\infty}(t_n,\lambda_n)=(t_*,\lambda_*)\in\left[\frac{1}{2},1\right]\times[0,\overline{\lambda}],$$
$$\lim_{n\to+\infty}\|w_n-w\|=0,$$

with (t_*,λ_*,w) satisfying the b.v.p.

$$\begin{aligned}-\Delta w&=t_*\lambda_* w+t_*\nabla g(0)\cdot\nabla w, && x\in\Omega\\ w&=0, && x\in\partial\Omega.\end{aligned}$$

By using [3] we find that necessarily $\overline{\lambda}\geq t_*\lambda_*\geq\lambda_1+\frac{|t_*\nabla g(0)|^2}{4}\geq\lambda_1+\frac{|\nabla g(0)|^2}{16}$, a contradiction. Thus, Step 1 is proved.

Proof of Step 2: Consider the number δ given by Step 1. To prove the assertion of this step, we use again a contradiction argument. Let us suppose that there exists sequences $\{t_n\}\subset\left[\frac{1}{2},1\right]$, $\{\lambda_n\}\subset(-\infty,\overline{\lambda}]$, $\{u_n\}\subset H_0^1(\Omega)$ and $\{h_n\}\subset L^2(\Omega)$ such that u_n is a solution of (10) with $t=t_n$, $\lambda=\lambda_n$, $h=h_n$ and

$$\|u_n\|=\delta,\quad \|h_n\|_2\leq\frac{1}{n}\,.$$

Then, from Lemma 2.2, the condition $\|u_n\|=\delta$ implies that λ_n is bounded from below and, consequently, by the compactness of the operator L and applying Proposition 2.1 (passing to a subsequence if necessary) we can assume that there exist $(t_*,\lambda_*)\in[\frac{1}{2},1]\times(-\infty,\overline{\lambda}]$ and $u\in H_0^1(\Omega)$ for which

$$\begin{aligned}&\lim_{n\to+\infty}(t_n,\lambda_n)=(t_*,\lambda_*),\\ &\lim_{n\to+\infty}\|u_n-u\|=0,\\ &-\Delta u=t_*\lambda_* u+t_*g(\nabla u),\quad x\in\Omega.\end{aligned}$$

By observing that necessarily $\|u\|=\delta$ and $t_*\lambda_*\leq\overline{\lambda}$, we obtain, from Step 1, a contradiction proving Step 2 and hence the lemma. □

In order to prove our existence result (Theorem 2.6 below), we use in addition to the above lemmas the following classical result about continuation property of the topological degree:

Theorem 2.5 (Leray-Schauder [19] (see also [11], [24])). *Consider $\underline{\lambda} < \overline{\lambda}$, U an open bounded subset of a Banach space X and, for $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, let $T(\lambda, \cdot) : \overline{U} \subset X \longrightarrow X$ be a family of compact operators such that the equation*

$$x = T(\lambda, x)$$

has no solution in the boundary ∂U for any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. If the degree

$$\deg(I - T(\underline{\lambda}, \cdot), U, 0) \neq 0,$$

then there exists a continuum (connected and closed) C in the solution set $\Sigma \equiv \{(\lambda, x) \in [\underline{\lambda}, \overline{\lambda}] \times X \,:\, x = T(\lambda, x)\}$ such that

$$C \cap (\{\underline{\lambda}\} \times U) \neq \emptyset, \quad C \cap (\{\overline{\lambda}\} \times U) \neq \emptyset.$$

□

Theorem 2.6. *If $g(0) = 0$, g is differentiable at zero with $\nabla g(0) \neq 0$ and $\lim_{|\xi| \to +\infty} g(\xi) = 0$ then for every $\overline{\lambda} < \lambda_1 + \frac{|\nabla g(0)|^2}{16}$ there exists $\varepsilon > 0$ such that for $\|h\|_2 < \varepsilon$ and $\lambda \leq \overline{\lambda}$, problem (P_λ^h) has at least one solution. If, in addition, $\int_\Omega h\varphi_1 \neq 0$, then there is also $\eta > 0$ such that (for $\|h\|_2 < \varepsilon$)*

(i) *problem (P_λ^h) has at least three solutions provided that $\lambda_1 - \eta < \lambda < \lambda_1$,*
(ii) *problem $(P_{\lambda_1}^h)$ has at least two solutions,*
(iii) *problem (P_λ^h) has at least three solutions provided that $\lambda_1 < \lambda < \lambda_1 + \eta$.*

Proof: Clearly, to show the theorem we can suppose that $\lambda_1 < \overline{\lambda} < \lambda_1 + \frac{|\nabla g(0)|^2}{16}$. Let us consider the positive numbers ε and δ given by Lemma 2.4. We assume in all the proof that $\|h\|_2 < \varepsilon$. We split again the proof in two steps:

Step 1: *For $\lambda \leq \overline{\lambda}$, there exists at least one solution of (P_λ^h) with norm smaller than δ.*

Step 2: *Multiplicity of solutions provided that $\int_\Omega h\varphi_1 \neq 0$.*

Proof of Step 1: It suffices to prove that for every $\underline{\lambda} < 0$, if $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ the problem (P_λ^h) has a solution. Hence, fix $\underline{\lambda} < 0$. Take $U = \left\{u \in H_0^1(\Omega) : \|u\| < \delta\right\}$ and $T(\lambda, u) = \lambda Lu + L(g(\nabla u) + h)$. In order to verify the hypotheses of Theorem 2.5, we observe that by Lemma 2.4 if $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ the equation

$$u = T(\lambda, u)$$

has no solution in $\partial U = \left\{u \in H_0^1(\Omega) : \|u\| = \delta\right\}$. Moreover, we claim that

$$\deg(I - T(\underline{\lambda}, \cdot), U, 0) = 1.$$

Indeed, to show this, we define $H : [0,1] \times U \longrightarrow H_0^1(\Omega)$ and $y : [0,1] \longrightarrow H_0^1(\Omega)$ by setting

$$H(t,u) = tT(\underline{\lambda}, u), \quad t \in [0,1],\ u \in H_0^1(\Omega)$$

$$y(t) = \begin{cases} -tLh, & t \in \left[0, \frac{1}{2}\right) \\ (t-1)Lh, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Since $L : \overline{U} \subset H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ is compact, H is also a compact map. In addition, y is continuous and, from Lemmas 2.3 and 2.4, it satisfies

$$y(t) \notin (I - H(t,\cdot))(\partial U), \quad \forall t \in [0,1].$$

Hence, by the homotopy invariance of the degree,

$$\begin{aligned} \deg(I - T(\underline{\lambda}, \cdot), U, 0) &= \deg(I - H(1,\cdot), U, y(1)) \\ &= \deg(I - H(0,\cdot), U, y(0)) \\ &= \deg(I, U, 0) = 1. \end{aligned}$$

This proves the claim.

Therefore, by Theorem 2.5, there exists a solution in U of problem (P_λ^h) for every $\underline{\lambda} \le \lambda \le \overline{\lambda}$, provided that $\|h\|_2 < \varepsilon$.

Proof of Step 2: It is well-known [25] that a continuum of solutions of (P_λ^h) emanates from infinity at $\lambda = \lambda_1$ with the next global property: either it contains another bifurcation point from infinity (so greater than λ_1) or its projection on the axis λ is unbounded (recall that, since $\int_\Omega h\varphi_1 \neq 0$, $u = 0$ is not a bifurcation point from zero). Now we need to study the local behaviour of the bifurcation from infinity. Thus, consider a sequence (λ_n, u_n) with $F_{\lambda_n}(u_n) = 0$, $\|u_n\| \to \infty$ and $\lambda_n \to \lambda_1$. We may set

$$w_n = \frac{u_n}{\|u_n\|}$$

for all n big enough. We observe that by taking $v = \varphi_1$ as test function in $(P_{\lambda_n}^h)$, we deduce that

$$(\lambda_1 - \lambda_n) \int_\Omega u_n \varphi_1 = \int_\Omega g(\nabla u_n)\varphi_1 + \int_\Omega h\varphi_1. \tag{11}$$

Taking subsequences (if necessary) we may assume that w_n is converging in $H_0^1(\Omega)$ to some w, which, by taking limits in the equation $(P_{\lambda_n}^h)$ divided by $\|u_n\|$, satisfies

$$\begin{aligned} -\Delta w &= \lambda_1 w, \quad x \in \Omega \\ w &= 0, \qquad x \in \partial\Omega, \end{aligned}$$

i.e. $w = \pm\varphi_1$. Let us remark explicitly that both possibilities, either $+\varphi_1$ or $-\varphi_1$, may occur provided that the sequence (λ_n, u_n) is properly chosen. Indeed, similar arguments to those in [1] prove that $\lambda = \lambda_1$ is a bifurcation point from infinity for the problem

$$\begin{aligned} -\Delta u &= \lambda u^+ + g(\nabla u) + h, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

For this problem, it is easily seen that every sequence (λ_n, u_n) with a solution u_n for $\lambda = \lambda_n$ such that $\|u_n\| \to \infty$ and $\lambda_n \to \lambda_1$ has to satisfy

$$\frac{u_n}{\|u_n\|} \to +\varphi_1.$$

Now, Proposition 2.1 implies that u_n solves $(P^h_{\lambda_n})$ provided that n is large enough and, consequently, the first possibility occurs. On the other hand, the second possibility is showed by considering the problem

$$\begin{aligned} -\Delta u &= \lambda u^- + g(\nabla u) + h, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

Passing again to subsequences (if necessary) we can claim that

$$\frac{\nabla u_n(x)}{\|u_n\|} = \nabla w_n(x) \to \pm\nabla\varphi_1(x) \ a.e.\ x \in \Omega.$$

On the other hand, the following unique continuation property is true: the measure of the set $\{x \in \Omega : \nabla\varphi_1(x) = 0\}$ is equal to zero. Indeed, for every $i = 1, 2, \ldots, N$, the derivative $\frac{\partial\varphi_1}{\partial x_i}$ is a solution of $-\Delta v = \lambda_1 v$ in Ω and, by [12, Theorem 1.2], $\frac{\partial\varphi_1}{\partial x_i}$ is a locally A_p-weight of Muckenhoupt [22] (see also [8]); hence they cannot vanish on a set of positive measure. If we take into consideration that $\lim_{|\xi|\to+\infty} g(\xi) = 0$ we obtain

$$g(\nabla u_n(x)) \to 0 \ a.e.\ x \in \Omega$$

and we can use the Lebesgue dominated convergence theorem to prove that

$$\lim_{n\to\infty} \int_\Omega g(\nabla u_n)\varphi_1 = 0,$$

and, consequently, by (11) we have that

$$\lim_{n\to\infty} (\lambda_1 - \lambda_n) \int_\Omega u_n\varphi_1 = \int_\Omega h\varphi_1.$$

Thus, if we denote $c := \int_\Omega h\varphi_1$ we conclude that

- if $c > 0$ then $w_n = \frac{u_n}{\|u_n\|}$ converges to $+\varphi_1$ ("bifurcation from $+\infty$") if and only if $\lambda_n < \lambda_1$ for n large enough, and w_n converges to $-\varphi_1$ ("bifurcation from $-\infty$") if and only if $\lambda_n > \lambda_1$ for n large enough;
- if $c < 0$ then $w_n = \frac{u_n}{\|u_n\|}$ converges to $+\varphi_1$ ("bifurcation from $+\infty$") if and only if $\lambda_n > \lambda_1$ for n large enough, and w_n converges to $-\varphi_1$ ("bifurcation from $-\infty$") if and only if $\lambda_n < \lambda_1$ for n large enough.

We claim that in both cases, either $c > 0$ or $c < 0$, the continuum emanating from infinity to the left of $\lambda = \lambda_1$ cross the line $\lambda = \lambda_1$ in a point (λ_1, u) with $\|u\| > \delta$ (by Lemma 2.4). Otherwise, the global nature of the continuum means that its projection on the λ-axis is $(-\infty, \lambda_1)$. In this case, Lemma 2.2 and a connectedness argument imply the existence in the continuum of a solution (λ, u) of (P_λ^h) with norm $\|u\| = \delta$. This contradicts Lemma 2.4.

In conclusion, taking $\eta > 0$ sufficiently small we have:

- existence of at least three solutions if $\lambda_1 - \eta < \lambda < \lambda_1$, one with norm smaller than δ in the continuum given in Step 1 and other two solutions with norm greater than δ in the continuum emanating from infinity to the left of λ_1;
- existence of at least two solutions if $\lambda = \lambda_1$, one with norm smaller than δ in the continuum given in Step 1, and the other with norm greater than δ in the continuum emanating from infinity to the left of λ_1.
- existence of at least three solutions if $\lambda_1 < \lambda < \lambda_1 + \eta$, one with norm smaller than δ in the continuum given in Step 1, and other two solutions with norm greater than δ in the continuum emanating from infinity to the right of λ_1.

□

Remark 2.7. With respect to the existence of at least one solution of (P_λ^h), the above proof can be slightly modified in order to handle the more general case $\overline{\lambda} < \lambda_1 + \frac{|\nabla g(0)|^2}{4}$. Indeed, if $\overline{\lambda} < \lambda_1 + \frac{|\nabla g(0)|^2}{4}$ we choose $0 < r < 1$ such that $\overline{\lambda} < \lambda_1 + \frac{r^2|\nabla g(0)|^2}{4}$. We can prove in a way similar to that of Lemma 2.4 that there exist $\varepsilon, \delta > 0$ such that if $t \in [r, 1]$, $\lambda \leq \overline{\lambda}$ and $\|h\|_2 < \varepsilon$, then the problem

$$u = t\lambda Lu + L\left(tg(\nabla u) + \left(\frac{t-1}{r} + 1\right)h\right)$$

has no solution $u \in H_0^1(\Omega)$ with $\|u\| = \delta$, so that the proof of Step 1 in Theorem 2.6 can be extended to cover the case $\overline{\lambda} < \lambda_1 + \frac{r^2|\nabla g(0)|^2}{4}$ instead of the previous one $\overline{\lambda} < \lambda_1 + \frac{|\nabla g(0)|^2}{16}$.

3. Final remark

Now we give a result about nonexistence of solutions. More precisely, we prove that problem (1) has no solution provided that $\left|\int_\Omega h\varphi_1\right|$ is large enough.

Theorem 3.1. *If* $\left|\int_\Omega h\varphi_1\right| > \|g\|_\infty \int_\Omega \varphi_1$ *then problem* (1) *has no solution.*

PROOF: Let us suppose that there exists a solution $u \in H_0^1(\Omega)$ of (1). Using φ_1 as test function we get

$$0 = \int_\Omega g(\nabla u)\varphi_1 + \int_\Omega h\varphi_1,$$

and we deduce that $\left|\int_\Omega h\varphi_1\right| = \left|\int_\Omega g(\nabla u)\varphi_1\right| \leq \|g\|_\infty \int_\Omega \varphi_1$, which concludes the proof. □

References

[1] Ambrosetti A., Hess P., *Positive solutions of asymptotically linear elliptic eigenvalue problems*, J. Math. Anal. Appl. **73** (2) (1980), 411–422.

[2] Almira J.M., Del Toro N., *Some remarks on certains semilinear problems with nonlinearities depending on the derivative*, Electron. J. Differential Equations **2003** (2003), no. 18, 1–11.

[3] Anane A., Chakrone O., Gossez J.P., *Spectre d'ordre supérieur et problèmes de non-résonance*, C.R. Acad. Sci. Paris **325** Série I (1997), 33–36.

[4] Arcoya D., Gámez J.L., *Bifurcation theory and related problems: anti-maximum principle and resonance*, Comm. Partial Differential Equations **26** (2001), no. 9–10, 1879–1911.

[5] Brezis H., Kato T., *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. IX, **58** (1979), 137–151.

[6] Cañada A., *Nonselfadjoint semilinear elliptic boundary value problems*, Ann. Mat. Pura Appl. **CXLVIII** (1987), 237–250.

[7] Cañada A., Drábek P., *On semilinear problems with nonlinearities depending only on derivatives SIAM J. Math. Anal.* **27** (1996), 543–557.

[8] Coifman R.R., Fefferman C.L., *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.

[9] Drábek P., Girg P., Roca F., *Remarks on the range properties of certain semilinear problems of Landesman-Lazer type*, J. Math. Anal. Appl. **257** (2001), 131–140.

[10] Drábek P., Nicolosi F., *Semilinear boundary value problems at resonance with general nonlinearities*, Differential Integral Equations **5**-2 (1992), 339–355.

[11] De Figuereido D.G., Lions P.L., Nussbaum R.D., *A priori estimates and existence of positive solutions for semi-linear elliptic equations*, J. Math. Pures Appl. **61** (1982), 41–63.

[12] Garofalo N., Lin F.H., *Unique continuation for elliptic operators: a geometric-variational approach*, Comm. Pure Appl. Math. **XL** (1987), 347–366.

[13] Gidas B., Ni W.M., Nirenberg L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[14] Gilbarg D., Trudinger N., *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.

[15] Girg P., *Neumann and periodic boundary-value problems for quasilinear ordinary differential equations with a nonlinearity in the derivative*, Electron. J. Differential Equations **63** (2000), 1–28.

[16] Habets P., Sanchez L., *A two-point problem with nonlinearity depending only on the derivative*, SIAM J. Math. Anal. **28** (1997), 1205–1211.

[17] Kannan R., Nagle R.K., Pothoven K.L., *Remarks on the existence of solutions of* $x'' + x + \arctan(x') = p(t)$; $x(0) = x(\pi) = 0$, Nonlinear Anal. **22** (1994), 793–796.

[18] Landesman E.M., Lazer A.C., *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1970), 609–623.
[19] Leray J., Schauder J., *Topologie et équations fonctionelles*, Ann. Scient. Éc. Norm. Sup. **51** (1934), 45–78.
[20] Mawhin J., *Some remarks on semilinear problems at resonance where the nonlinearity depends only on the derivatives*, Acta Math. Inform. Univ. Ostraviensis **2** (1994), 61–69.
[21] Mawhin J., Schmitt K., *Landesman-Lazer type problems at an eigenvalue of odd multiplicity*, Results Math. **14** (1988), 138–146.
[22] Muckenhoupt B., *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
[23] Nagle R.K., Pothoven K., Singkofer K., *Nonlinear elliptic equations at resonance where the nonlinearity depends essentially on the derivatives J. Diff. Equations* **38** (1980), 210–225.
[24] Nussbaum R., *Uniqueness and nonuniqueness for periodics solutions of* $x'(t) = -g(x(t-1))$, J. Differential Equations **34** (1979), 24–54.
[25] Rabinowitz P.H., *On bifurcation from infinity*, J. Differential Equations **14** (1973), 462–475.
[26] Struwe M., *Variational Methods. Application to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, 1990.

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