Cancellative actions

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Abstract. The following problem is considered: when can the action of a cancellative semigroup S on a set be extended to a simply transitive action of the universal group of S on a larger set.

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Introduction

The following problem arose in [4]. Let S be a cancellative semigroup and $G(S)$ be its universal group. Assume that S can be embedded in $G(S)$. When can the action of S on a set X be extended to a simply transitive action of $G(S)$ on some set $Y \supseteq X$? When S is commutative the solution of this problem is easy but leads to concepts that are of great importance for finitely generated commutative semigroups [4].

Here we consider the general case of an arbitrary semigroup S which acts on a set X. In Section 1 we use the universal group $G(S)$ of S, and the canonical homomorphism $\gamma : S \longrightarrow G(S)$, to construct a set Y, a mapping $\iota : X \longrightarrow Y$, and an action of $G(S)$ on Y which extends the action of S in the sense that $\iota(s \,.\, x) = \gamma(s) \,.\, \iota(x)$ for all s and x, and has a universal property. This leads in Section 2 to necessary and sufficient conditions for extending the action of S on X to a simply transitive action of $G(S)$ on some set $Z \supseteq X$, or to a simply transitive action of $G(S)$ on Y. A later article will show that the latter conditions are equivalent to explicit sets of implications.

We do not assume that S is a monoid. But, if S is a monoid, then we may assume that it acts on X as a monoid ($1 \,.\, x = x$ for all $x \in X$), since otherwise its action cannot be extended to a group action.

Recall that a *semigroup* is a set with an associative operation, which we write as a multiplication. A semigroup S is *cancellative* when $xz = yz$ implies $x = y$, and $zx = zy$ implies $x = y$ (for all $x, y, z \in S$). A *left semigroup action* $\,.\,$ of a semigroup S on a set X is a mapping $(s, x) \longmapsto s \cdot x$ of $S \times X$ into X. Then S acts *simply* on X when $s \,.\, x = t \,.\, x$ implies $x = t$; S acts *transitively* on X when, for every $x, y \in X$, there exists some $s \in S$ such that $s \,.\, x = y$.

1. Universal actions

This section takes place in the category *Act* of semigroup acts. The objects of *Act* are all ordered triples $(S, X, .)$ of a semigroup S, a set X, and a left semigroup action $.$ of S on X; then X is an S-set and $(S, X, .)$ is an S-*act*. In *Act*, a morphism from $(S, X, .)$ to $(T, Y, .)$ is an ordered pair (φ, f) of a semigroup homomorphism $\varphi : S \longrightarrow T$ and a mapping $f : X \longrightarrow Y$ such that $f(s \,.\, x) = \varphi(s) \,.\, f(x)$ for all $s \in S$ and $x \in X$; if φ and f are injective, then the action of T on Y extends the action of S on X. Composition and identity morphisms are componentwise.

1. When $\varphi : S \longrightarrow T$ is a semigroup homomorphism, every S-act has a universal T-act:

Proposition 1.1. *Let $(S, X, .)$ be a semigroup act and $\varphi : S \longrightarrow T$ be a homomorphism. There exist a set Y, an action $.$ of T on Y, and a mapping $\iota : X \longrightarrow Y$ such that $(\varphi, \iota) : (S, X, .) \longrightarrow (T, Y, .)$ is a morphism and, for every morphism $(\varphi, \alpha) : (S, X, .) \longrightarrow (T, Z, .)$, there exists a unique action-preserving mapping $\beta : Y \longrightarrow Z$ such that $\beta \circ \iota = \alpha$.*

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & Y \arrow[d, "\beta"] \\ & Z \end{tikzcd}$$

Proof: We construct Y as a tensor product of S-sets (as introduced in [5]): namely, $Y = T^1 \otimes_S X$, where S acts on T^1 on the right by $t \,.\, s = t\ \varphi(s)$. The details are as follows. Let $\sim$ be the smallest equivalence relation on the set $T^1 \times X$ such that

(1) for all $t, u, v \in T^1$ and $x, y \in X$, $(u, x) \sim (v, y)$ implies $(tu, x) \sim (tv, y)$; and
(2) for all $s \in S$ and $x \in X$, $(\varphi(s), x) \sim (1, s \,.\, x)$.

This exists since an intersection of equivalence relations with properties (1) and (2) again has properties (1) and (2). A more detailed description of $\sim$ is given in Lemma 1.2 below.

We show that $Y = \big(T^1 \times X\big)/{\sim}$ serves. Let $\mathrm{cls}(t, x)$ denote the $\sim$-class of (t, x). The mapping $\iota : X \longrightarrow Y$ is given by

$$\iota(x) \ = \ \mathrm{cls}(1, x).$$

By (1), an action $.$ of T^1 on Y is well defined by

$$t \,.\, \mathrm{cls}(u, x) \ = \ \mathrm{cls}(tu, x).$$

This is a monoid action since $1 \,.\, \mathrm{cls}(u,x) = \mathrm{cls}(u,x)$ and

$$t \,.\, \big(u \,.\, \mathrm{cls}(v,x)\big) \;=\; t \,.\, \mathrm{cls}(uv,x) \;=\; \mathrm{cls}(tuv,x) \;=\; tu \,.\, \mathrm{cls}(v,x).$$

In particular, T acts on Y. Also

$$\iota(s \,.\, x) \;=\; \mathrm{cls}(1, s \,.\, x) \;=\; \mathrm{cls}(\varphi(s), x) \;=\; \varphi(s) \,.\, \iota(x)$$

by (2). Thus $(T, Y, \,.\,)$ is an object of *Act* and (φ, ι) is a morphism.

Let $(\varphi, \alpha) : (S, X, \,.\,) \longrightarrow (T, Z, \,.\,)$ be a morphism. The mapping α induces a mapping $\overline{\alpha} : T^1 \times X \longrightarrow Z$ defined by

$$\overline{\alpha}(t,x) \;=\; t \,.\, \alpha(x)$$

(with $\overline{\alpha}(1,x) = \alpha(x)$ if $t = 1 \in T^1$). If $\overline{\alpha}(u,x) = \overline{\alpha}(v,y)$, then $u \,.\, \alpha(x) = v \,.\, \alpha(y)$ and

$$\begin{aligned} \overline{\alpha}(tu,x) \;&=\; tu \,.\, \alpha(x) \;=\; t \,.\, \big(u \,.\, \alpha(x)\big) \\ &=\; t \,.\, \big(v \,.\, \alpha(y)\big) \;=\; tv \,.\, \alpha(y) \;=\; \overline{\alpha}(tv,y). \end{aligned}$$

Also

$$\overline{\alpha}(\varphi(s),x) \;=\; \varphi(s) \,.\, \alpha(x) \;=\; \alpha(s \,.\, x) \;=\; 1 \,.\, \alpha(s \,.\, x) \;=\; \overline{\alpha}(1,\; s \,.\, x)$$

by the choice of α. Thus the equivalence relation induced by $\overline{\alpha}$ satisfies (1) and (2). It therefore contains $\sim$: $(t,x) \sim (u,y)$ implies $\overline{\alpha}(t,x) = \overline{\alpha}(u,y)$. Hence a mapping $\beta : Y \longrightarrow Z$ is well defined by

$$\beta\big(\mathrm{cls}(t,x)\big) \;=\; \overline{\alpha}(t,x) \;=\; t \,.\, \alpha(x).$$

In particular $\beta(\iota(x)) = \beta\big(\mathrm{cls}(1,x)\big) = 1 \,.\, \alpha(x) = \alpha(x)$ and $\beta \circ \iota = \alpha$. If moreover $y = \mathrm{cls}(u,x) \in Y$, then

$$\beta(t \,.\, y) \;=\; \beta\big(\mathrm{cls}(tu,x)\big) \;=\; tu \,.\, \alpha(x) \;=\; t \,.\, \big(u \,.\, \alpha(x)\big) \;=\; t \,.\, \beta(y).$$

Thus β is action-preserving.

If conversely $\beta' : Y \longrightarrow Z$ is action-preserving and $\beta' \circ \iota = \alpha$, then

$$\begin{aligned} \beta'\big(\mathrm{cls}(t,x)\big) \;&=\; \beta'\big(t \,.\, \mathrm{cls}(1,x)\big) \;=\; \beta'\big(t \,.\, \iota(x)\big) \\ &=\; t \,.\, \beta'(\iota(x)) \;=\; t \,.\, \alpha(x) \;=\; \beta\big(\mathrm{cls}(t,x)\big); \end{aligned}$$

hence β is unique. □

We give a more precise description of $\sim$ (which would work more generally in any tensor product of S-sets). For this it is convenient to regard the elements of $T^1 \times X$ as the vertices of a directed graph, in which there is a labelled edge $(t, s \,.\, x) \xrightarrow{s} (t\varphi(s), x)$ for every $(t,x) \in T^1 \times X$ and $s \in S^1$. In particular there is an identity edge $(t,x) \xrightarrow{1} (t,x)$ for every $(t,x) \in T^1 \times X$. We note two properties:

If $a \xrightarrow{s'} b \xrightarrow{s''} c$, then $a \xrightarrow{s's''} c$: indeed, if $a = (t, -)$ and $c = (-, x)$, then $b = (t\varphi(s'), s'' \,.\, x)$, so that $a = (t,\; s' \,.\, (s'' \,.\, x))$, $c = (t\varphi(s')\varphi(s''), x)$, and $a \xrightarrow{s's''} c$.

If $(u, s \,.\, x) \xrightarrow{s} (u\varphi(s), x)$, then $(tu, s \,.\, x) \xrightarrow{s} (tu\varphi(s), x)$.

Lemma 1.2. *In $T^1 \times X$, $a \sim b$ if and only if*

$$a = a_0 \xleftarrow{s_1} a_1 \xrightarrow{s_2} a_2 \ \cdots \ a_{2n-2} \xleftarrow{s_{2n-1}} a_{2n-1} \xrightarrow{s_{2n}} a_{2n} = b$$

for some $n \geq 0$, $a_0, \dots, a_{2n} \in T^1 \times X$, and $s_1, s_2, \dots, s_{2n} \in S^1$.

Proof: Let $a \,\mathcal{C}\, b$ if and only if

$$a = a_0 \xleftarrow{s_1} a_1 \xrightarrow{s_2} a_2 \ \cdots \ a_{2n-2} \xleftarrow{s_{2n-1}} a_{2n-1} \xrightarrow{s_{2n}} a_{2n} = b$$

for some $n \geq 0$, $a_0, \dots, a_{2n} \in T^1 \times X$, and $s_1, \dots, s_{2n} \in S^1$. It is immediate that $\mathcal{C}$ is reflexive (let $n = 0$), symmetric, and transitive. Also $(u, x)\,\mathcal{C}\,(v, y)$ implies $(tu, x)\,\mathcal{C}\,(tv, y)$, since $(u, s \,.\, x) \xrightarrow{s} (u\varphi(s), x)$ implies $(tu, s \,.\, x) \xrightarrow{s} (tu\varphi(s), x)$; and $(\varphi(s), x)\,\mathcal{C}\,(1, s \,.\, x)$, since $(\varphi(s), x) \xleftarrow{s} (1, s \,.\, x) \xrightarrow{1} (1, s \,.\, x)$. Thus $\mathcal{C}$ is an equivalence relation with properties (1) and (2).

If conversely $\mathcal{A}$ is an equivalence relation with properties (1) and (2), then $(t\varphi(s), x)\,\mathcal{A}\,(t, s \,.\, x)$ for all t, s, and x; hence $(t, s \,.\, x) \xrightarrow{s} (t\varphi(s), x)$ implies $(t, s \,.\, x)\,\mathcal{A}\,(t\varphi(s), x)$, and $a \,\mathcal{C}\, b$ implies $a \,\mathcal{A}\, b$. Therefore $\mathcal{C}$ coincides with $\sim$. $\square$

2. Proposition 1.1 implies that every semigroup act has a universal group act in *Act*. First recall that every semigroup S has a universal group in the category of semigroups and homomorphisms: that is, there exist a group $G(S)$ and a homomorphism $\gamma : S \longrightarrow G(S)$, such that every homomorphism φ of S into a group G factors uniquely through γ ($\varphi = \psi \circ \gamma$ for some unique homomorphism $\psi : G(S) \longrightarrow G$). For instance let F be the free monoid on the set $S \cup S'$, where S' is disjoint from S and comes with a bijection $s \longmapsto s'$ of S onto S'. Let $\iota : S \cup S' \longrightarrow F$ be the canonical mapping. Let $\mathcal{C}$ be the smallest congruence on F such that $\iota(st)\,\mathcal{C}\,\iota(s)\ \iota(t)$, $\iota(s)\ \iota(s')\,\mathcal{C}\,1$, and $\iota(s')\ \iota(s)\,\mathcal{C}\,1$, for all $s, t \in S$; then $F/\mathcal{C}$ and the canonical mapping $S \longrightarrow F \longrightarrow F/\mathcal{C}$ serve as $G(S)$ and γ. The existence of a universal group also follows from the Adjoint Functor Theorem.

Proposition 1.3. *Let $(S, X, \,.\,)$ be a semigroup act. Let $G(S)$ be the universal group of S and $\gamma : S \longrightarrow G(S)$ be the canonical homomorphism. The universal $G(S)$-set Y of X and its canonical morphism $(\gamma, \iota) : (S, X, \,.\,) \longrightarrow (G(S), Y, \,.\,)$ have the following universal property: for every morphism $(\varphi, \alpha) : (S, X, \,.\,) \longrightarrow (G, Z, \,.\,)$, where G is a group, there exists a unique morphism $(\psi, \beta) : (G(S), Y, \,.\,) \longrightarrow (G, Z, \,.\,)$ such that $(\psi, \beta) \circ (\gamma, \iota) = (\varphi, \alpha)$.*

$$\begin{tikzcd} S \arrow[r, "\gamma"] \arrow[dr, "\varphi"'] & G(S) \arrow[d, "\psi"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & Y \arrow[d, "\beta"] \\ & Z \end{tikzcd}$$

PROOF: By Proposition 1.1, $(\gamma, \iota) : (S, X, \,.\,) \longrightarrow (G(S), Y, \,.\,)$ is a morphism and, for every morphism $(\gamma, \alpha) : (S, X, \,.\,) \longrightarrow (G(S), Z, \,.\,)$, there exists a unique action-preserving mapping $\beta : Y \longrightarrow Z$ such that $\beta \circ \iota = \alpha$. We now prove the stronger universal property in the statement.

Let G be a group and $(\varphi, \alpha) : (S, X, \,.\,) \longrightarrow (G, Z, \,.\,)$ be a morphism. Since $G(S)$ is the universal group of S there exists a unique homomorphism $\psi : G(S) \longrightarrow G$ such that $\psi \circ \gamma = \varphi$. The action of G on Z then induces an action of $G(S)$ on Z, given by

$$g \,.\, z \;=\; \psi(g) \,.\, z$$

for all $g \in G(S)$ and $z \in Z$. Then

$$\alpha(s \,.\, x) \;=\; \varphi(s) \,.\, \alpha(x) \;=\; \psi(\gamma(s)) \,.\, \alpha(x) \;=\; \gamma(s) \,.\, \alpha(x)$$

for all $s \in S$ and $x \in X$, and $(\gamma, \alpha) : (S, X, \,.\,) \longrightarrow (G(S), Z, \,.\,)$ is a morphism of acts. Hence there is a unique mapping $\beta : Y \longrightarrow Z$ such that $\beta \circ \iota = \alpha$ and β preserves the action of $G(S)$. This last condition states that

$$\beta(g \,.\, y) \;=\; g \,.\, \beta(y) \;=\; \psi(g) \,.\, \beta(y)$$

for all $g \in G(S)$ and $y \in Y$, i.e. $(\psi, \beta) : (G(S), Y, \,.\,) \longrightarrow (G, Z, \,.\,)$ is a morphism of acts. Thus there is a unique morphism $(\psi, \beta) : (G(S), Y, \,.\,) \longrightarrow (G, Z, \,.\,)$ such that $(\psi, \beta) \circ (\gamma, \iota) = (\varphi, \alpha)$. $\square$

3. In Proposition 1.3 (up to isomorphism of acts) $Y = \big(G(S) \times X\big)/{\sim}$, where $\sim$ is the smallest equivalence relation on the set $G(S) \times X$ such that (1) $(g, x) \sim (h, y)$ implies $(kg, x) \sim (kh, y)$ and (2) $(\gamma(s), x) \sim (1,\ s \,.\, x)$, for all $g, h, k \in G(S)$, $x, y \in X$, and $s \in S$; then $g \,.\, \mathrm{cls}(h, x) = \mathrm{cls}(gh, x)$ and $\iota(x) = \mathrm{cls}(1, x)$. Lemma 1.2 then leads to a better description of $\sim$.

When $x, y \in X$, a *connected sequence* from x to y is a triple of sequences $x_0, x_1, \dots, x_n \in X$, $s_1, \dots, s_n \in S^1$, $t_1, \dots, t_n \in S^1$ (where $n \geq 0$) such that $x = x_0$, $x_n = y$, and

$$t_1 \,.\, x_0 = s_1 \,.\, x_1,\ t_2 \,.\, x_1 = s_2 \,.\, x_2,\ \dots,\ t_n \,.\, x_{n-1} = s_n \,.\, x_n$$

holds in X (with $1 \,.\, x = x$ in case S is not a monoid). The *group value* of a connected sequence $x_0, x_1, \dots, x_n \in X$, $s_1, \dots, s_n \in S^1$, $t_1, \dots, t_n \in S^1$ is

$$\gamma(t_1)^{-1}\ \gamma(s_1)\ \gamma(t_2)^{-1}\ \gamma(s_2)\ \dots\ \gamma(t_n)^{-1}\ \gamma(s_n) \in G(S)$$

(with $\gamma(1) = 1 \in G(S)$ in case S is not a monoid).

Lemma 1.4. *$(g,x) \sim (h,y)$ if and only if there exists a connected sequence from x to y with group value $g^{-1}h$.*

PROOF: Assume $(g,x) \sim (h,y)$. By Lemma 1.2 there exist $n \geq 0$, (g_0,x_0), (g_1,x_1), ..., $(g_{2n},x_{2n}) \in G(S) \times X$, and $s_1,\dots,s_{2n} \in S^1$ such that $(g_0,x_0) = (g,x)$, $(g_{2n},x_{2n}) = (h,y)$, and

$$\begin{aligned}(g_0,x_0) \xleftarrow{s_1} (g_1,x_1) \xrightarrow{s_2} (g_2,x_2) \xleftarrow{s_3} \cdots \\ \xrightarrow{s_{2n-2}} (g_{2n-2},x_{2n-2}) \xleftarrow{s_{2n-1}} (g_{2n-1},x_{2n-1}) \xrightarrow{s_{2n}} (g_{2n},x_{2n}).\end{aligned}$$

Then $g_0 = g_1\gamma(s_1)$, $s_1 \cdot x_0 = x_1 = s_2 \cdot x_2$, and $g_2 = g_1\gamma(s_2) = g_0\ \gamma(s_1)^{-1}\ \gamma(s_2)$. Similarly $s_3 \cdot x_2 = s_4 \cdot x_4$, $g_4 = g_2\ \gamma(s_3)^{-1}\ \gamma(s_4)$, ..., $s_{2n-1} \cdot x_{2n-2} = s_{2n} \cdot x_{2n}$, and $g_{2n} = g_{2n-2}\ \gamma(s_{2n-1})^{-1}\ \gamma(s_{2n})$. Hence $x_0,x_2,\dots,x_{2n} \in X$, $s_2,s_4,\dots,s_{2n} \in S^1$, $s_1,s_3,\dots,s_{2n-1} \in S^1$, is a connected sequence from x to y, whose group value is $g_0^{-1}\ g_{2n} = g^{-1}h$, since

$$g_{2n} = g_0\ \gamma(s_1)^{-1}\ \gamma(s_2)\ \gamma(s_3)^{-1}\ \gamma(s_4)\ \cdots\ \gamma(s_{2n-1})^{-1}\ \gamma(s_{2n}).$$

The converse is similar. Let $x_0,x_1,\dots,x_n \in X$, $s_1,\dots,s_n \in S^1$, $t_1,\dots,t_n \in S^1$ be a connected sequence from x to y with group value $g^{-1}h$. Let

$$y_1 = t_1 \cdot x_0 = s_1 \cdot x_1,\ y_2 = t_2 \cdot x_1 = s_2 \cdot x_2,\ \dots,\ y_n = t_n \cdot x_{n-1} = s_n \cdot x_n$$

and $g_0 = g$, $h_1 = g_0\ \gamma(t_1)^{-1}$, $g_1 = h_1\gamma(s_1)$, $h_2 = g_1\ \gamma(t_2)^{-1}$, $g_2 = h_2\gamma(s_2)$, ..., $h_n = g_{n-1}\ \gamma(t_n)^{-1}$, $g_n = h_n\gamma(s_n)$. Then $g_n = h$, since

$$g_n\ =\ g_0\ \gamma(t_1)^{-1}\ \gamma(s_1)\ \gamma(t_2)^{-1}\ \gamma(s_2)\ \dots\ \gamma(t_n)^{-1}\ \gamma(s_n)\ =\ g\ g^{-1}h.$$

Moreover,

$$\begin{aligned}(g_0,x_0) \xleftarrow{t_1} (h_1,y_1) \xrightarrow{s_1} (g_1,x_1) \xleftarrow{t_2} \cdots \\ \cdots \xrightarrow{s_{n-1}} (g_{n-1},x_{n-1}) \xleftarrow{t_n} (h_n,y_n) \xrightarrow{s_n} (g_n,x_n).\end{aligned}$$

Hence $(g,x) \sim (h,y)$. □

It is convenient to write $x \xrightarrow{g} y$ when $x,y \in X$ and there is a connected sequence from x to y with group value $g \in G(S)$. We note the following properties.

Lemma 1.5. *$x \xrightarrow{1} x$ and $s \,.\, x \xrightarrow{\gamma(s)} x$ for every $x \in X$ and $s \in S$. If $x \xrightarrow{g} y$, then $y \xrightarrow{g^{-1}} x$. If $x \xrightarrow{g} y$ and $y \xrightarrow{h} z$, then $x \xrightarrow{gh} z$.*

PROOF: When $x \in X$, then $x \xrightarrow{1} x$ since there is a connected sequence with $n = 0$ (also, $(1,x) \sim (1,x)$). More generally, when $s \in S^1$, then $s \,.\, x = x_0$, $x_1 = x \in X$, $s_1 = s \in S^1$, $t_1 = 1 \in S^1$ is a connected sequence from $s \,.\, x$ to x with group value $\gamma(s)$; hence $s \,.\, x \xrightarrow{\gamma(s)} x$.

If $x = x_0, x_1, \dots, x_n = y \in X$, $s_1, \dots, s_n \in S^1$, $t_1, \dots, t_n \in S^1$ is a connected sequence from x to y with group value

$$g \;=\; \gamma(t_1)^{-1}\, \gamma(s_1)\, \gamma(t_2)^{-1}\, \gamma(s_2)\, \dots\, \gamma(t_n)^{-1}\, \gamma(s_n),$$

then $y = x_n, x_{n-1}, \dots, x_0 = x \in X$, $t_n, \dots, t_1 \in S^1$, $s_n, \dots, s_1 \in S^1$ is a connected sequence from y to x with group value

$$\gamma(s_n)^{-1}\, \gamma(t_n)\, \gamma(s_{n-1})^{-1}\, \gamma(t_{n-1})\, \dots\, \gamma(s_1)^{-1}\, \gamma(t_1) \;=\; g^{-1}.$$

Hence $x \xrightarrow{g} y$ implies $y \xrightarrow{g^{-1}} x$.

If finally $x = x_0, x_1, \dots, x_m = y \in X$, $s_1, \dots, s_m \in S^1$, $t_1, \dots, t_m \in S^1$ is a connected sequence from x to y with group value

$$g \;=\; \gamma(t_1)^{-1}\, \gamma(s_1)\, \gamma(t_2)^{-1}\, \gamma(s_2)\, \dots\, \gamma(t_m)^{-1}\, \gamma(s_m),$$

and $y = y_0, y_1, \dots, y_n = z \in X$, $u_1, \dots, u_n \in S^1$, $v_1, \dots, v_n \in S^1$ is a connected sequence from y to z with group value

$$h \;=\; \gamma(v_1)^{-1}\, \gamma(u_1)\, \gamma(v_2)^{-1}\, \gamma(u_2)\, \dots\, \gamma(v_n)^{-1}\, \gamma(u_n),$$

then $x = x_0, x_1, \dots, x_m = y_0,\ y_1, \dots, y_n \in X$, $s_1, \dots, s_m,\ u_1, \dots, u_n \in S^1$, $t_1, \dots, t_n,\ v_1, \dots, v_n \in S^1$ is a connected sequence from x to z with group value

$$\begin{aligned}&\gamma(t_1)^{-1}\, \gamma(s_1)\, \gamma(t_2)^{-1}\, \gamma(s_2)\, \dots\, \gamma(t_n)^{-1}\, \gamma(s_n)\\ &\quad \gamma(v_1)^{-1}\, \gamma(u_1)\, \gamma(v_2)^{-1}\, \gamma(u_2)\, \dots\, \gamma(v_n)^{-1}\, \gamma(u_n) \;=\; gh.\end{aligned}$$

Hence $x \xrightarrow{g} y$ and $y \xrightarrow{h} z$ implies $x \xrightarrow{gh} z$. □

When S acts on X, the relation

$$x \equiv y \text{ if and only if there exists a connected sequence from } x \text{ to } y$$

is by Lemma 1.5 an equivalence relation on X; we call its equivalence classes the *connected components* of X. We write the quotient set $X/{\equiv}$ (the set of all connected components of X) as a family $(C_i)_{i\in I}$.

We say that the action of S on X is *connected* when there is only one connected component (when for every $x, y \in X$ there exists a connected sequence from x to y); we also say that the S-set X is connected. This is weaker than the usual transitivity conditions in, say, [3]. The connected components of any S-set are themselves connected S-sets.

4. We now give an alternate construction of the universal group act, in which the orbits of Y (under the action of $G(S)$) are constructed from the connected components of X. First we note:

Proposition 1.6. *In the universal group act $(G(S), Y, \,.\,)$ of $(S, X, \,.\,)$, $\iota(x)$ and $\iota(y)$ lie in the same orbit if and only if x and y lie in the same connected component of S.*

Proof: Let $x, y \in X$. If $\iota(x)$ and $\iota(y)$ lie in the same orbit, then $\mathrm{cls}(1, x) = g \,.\, \mathrm{cls}(1, y) = \mathrm{cls}(g, y)$ for some $g \in G(S)$ and there exists a connected sequence from x to y, by Lemma 1.4. If conversely there exists a connected sequence from x to y, and $g \in G(S)$ is its group value, then $\iota(x) = \mathrm{cls}(1, x) = \mathrm{cls}(g, y) = g \,.\, \iota(y)$, by Lemma 1.4, so that $\iota(x)$ and $\iota(y)$ lie in the same orbit. □

Stabilizers and orbits in Y can be retrieved from X as follows.

Lemma 1.7. *Let C be a connected component of X and $c \in C$. Then*

$$H(C) \;=\; \{\, h \in G(S) \mid c \xrightarrow{h} c \,\}$$

is a subgroup of $G(S)$; for every $x \in C$, $\{\, g \in G(S) \mid x \xrightarrow{g} c \,\}$ is a left coset of $H(C)$.

Proof: $H = H(C)$ is a subgroup of $G(S)$ by Lemma 1.5. Let $x \xrightarrow{g} c$. If $c \xrightarrow{h} c$, then $x \xrightarrow{gh} c$. If conversely $x \xrightarrow{g'} c$, then $c \xrightarrow{g^{-1}} x \xrightarrow{g'} c$, $g^{-1}g' \in H$, and $g' \in gH$; thus $\{\, g' \in G(S) \mid x \xrightarrow{g'} c \,\} = gH$. □

Recall that, when H is a subgroup of a group G, then the left cosets of H constitute a set G/H, on which G acts by left multiplication: $g' \,.\, gH = g'gH$.

Proposition 1.8. *Let $(S, X, \,.\,)$ be a semigroup act, $(G(S), Y, \,.\,)$ be its universal group act, and $(C_i)_{i\in I}$ be its connected components. For any cross-section $(c_i)_{i\in I}$ of $\equiv$, Y is (up to an isomorphism of $G(S)$-acts) the disjoint union*

$$Y \;=\; \bigcup_{i\in I} \big(G(S)/H(C_i) \times \{i\} \big),$$

with $g\,.\,(g'H(C_i),i) = (gg'H(C_i),i)$ *and* $\iota(x) = (gH(C_i),i)$ *when* $x \in C_i$ *and* $x \xrightarrow{g} c_i$.

PROOF: We need a cross-section $(c_i)_{i\in I}$ of $\equiv$ (with $c_i \in C_i$) to define $H(C_i) = \{\, h \in G(S) \mid c_i \xrightarrow{h} c_i \,\}$. Let

$$Z \;=\; \bigcup_{i\in I} \big(G(S)/H(C_i) \times \{i\}\big),$$

with $g\,.\,(g'H(C_i),i) = (gg'H(C_i),i)$ as in the statement. Then Z is a $G(S)$-set. By Lemma 1.7, a mapping $\alpha : X \longrightarrow Z$ is well-defined by

$$\alpha(x) \;=\; (gH(C_i),i) \text{ when } x \in C_i \text{ and } x \xrightarrow{g} c_i.$$

Let $x \in X$, $x \in C_i$, and $s \in S$. Then $s\,.\,x \xrightarrow{\gamma(s)} x$ by Lemma 1.5, in particular $s\,.\,x \in C_i$. If $x \xrightarrow{g} c_i$, so that $\alpha(x) = (gH(C_i),i)$, then $s\,.\,x \xrightarrow{\gamma(s)g} c_i$ and

$$\alpha(s\,.\,x) \;=\; (\gamma(s)gH(C_i),i) \;=\; \gamma(s)\,.\,\alpha(x).$$

Thus $(\gamma,\alpha) : (S,X,\,.\,) \longrightarrow (G(S),Z,\,.\,)$ is a morphism of acts.

By Proposition 1.1, there exists an action-preserving mapping $\beta : Y \longrightarrow Z$ such that $\beta\circ\iota = \alpha$. We show that β is bijective. Since β is action-preserving, we have

$$\beta(\mathrm{cls}(g,x)) \;=\; \beta(g\,.\,\mathrm{cls}(1,x)) \;=\; \beta(g\,.\,\iota(x)) \;=\; g\,.\,\beta(\iota(x)) \;=\; g\,.\,\alpha(x)$$

for all $x \in X$ and $g \in G(S)$. Now $\alpha(c_i) = (H(C_i),i)$, since $c_i \xrightarrow{1} c_i$; hence $(gH(C_i),i) = g\,.\,\alpha(c_i)$ and β is surjective.

Now assume that $\beta(\mathrm{cls}(g,x)) = \beta(\mathrm{cls}(h,y))$. Let $x \in C_i$, $x \xrightarrow{a} c_i$ and $y \in C_j$, $y \xrightarrow{b} c_j$, so that $\alpha(x) = (aH(C_i),i)$ and $\alpha(y) = (bH(C_j),j)$. We have

$$(gaH(C_i),i) \;=\; g\,.\,\alpha(x) \;=\; h\,.\,\alpha(y) \;=\; (hbH(C_j),j),$$

so that $i = j$ (x and y lie in the same connected component) and $gaH(C_i) = hbH(C_i)$. Hence $aH(C_i) = g^{-1}hbH(C_i)$, there exists $x \xrightarrow{g^{-1}hb} c_i$ by Lemma 1.7, and $x \xrightarrow{g^{-1}hb} c_i \xrightarrow{b^{-1}} y$ yields $x \xrightarrow{g^{-1}h} y$ and $\mathrm{cls}(g,x) = \mathrm{cls}(h,y)$, by Lemma 1.4. Thus β is injective. $\square$

5. A notable particular case occurs when S acts on itself by left multiplication.

Proposition 1.9. *When a semigroup S acts on itself by left multiplication, the connected components of S are left ideals, and $\equiv$ is the smallest congruence $\mathcal{C}$ on S such that $S/\mathcal{C}$ is a right zero semigroup.*

PROOF: For all $x, y \in S$ the equality $x \,.\, y = xy = 1 \,.\, xy$ shows that $y \equiv xy$; hence the $\equiv$-classes $(L_i)_{i \in I}$ are left ideals. In particular $L_i L_j \subseteq L_j$ for all i, j; hence $\equiv$ is a congruence and $S/\equiv$ is a right zero semigroup ($ab = b$ for all $a, b \in S/\equiv$). Conversely let $\mathcal{C}$ be a right zero semigroup congruence on S. If $x, y \in S$ and $s, t \in S^1$, then $sx = ty$ implies $x \,\mathcal{C}\, sx \,\mathcal{C}\, ty \,\mathcal{C}\, y$; therefore $\equiv$ is contained in $\mathcal{C}$. □

Proposition 1.9 goes back to Dubreil [2]. A semigroup S may be called *left connected* when S, as an S-set under left multiplication, has only one connected component. For example, every monoid is left connected ($s \sim 1$ for every s since $1 \,.\, s = s \,.\, 1$). (On the other hand, nontrivial right zero bands, and free semigroups with two or more generators, are not left connected.) Proposition 1.9 implies that every semigroup is a right zero band of left-connected semigroups. Additional results on band decompositions, including right zero band decompositions, can be found in [1].

Lemma 1.10. *When S acts on itself by left multiplication, every connected sequence from x to y has group value $\gamma(x)\ \gamma(y)^{-1}$.*

PROOF: As in the proof of Lemma 1.4, let $x_0, x_1, \ldots, x_n \in X$, $s_1, \ldots, s_n \in S$, $t_1, \ldots, t_n \in S$ be a connected sequence from x to y with group value g. Then

$$t_1 x_0 = s_1 x_1,\ t_2 x_1 = s_2 x_2,\ \ldots,\ t_n x_{n-1} = s_n x_n$$

and $\gamma(t_1)^{-1}\ \gamma(s_1)\ \gamma(t_2)^{-1}\ \gamma(s_2)\ \ldots\ \gamma(t_n)^{-1}\ \gamma(s_n) = g$. In $G(S)$ we have

$$\gamma(t_1)^{-1}\ \gamma(s_1) = \gamma(x_0)\ \gamma(x_1)^{-1},\ \gamma(t_2)^{-1}\ \gamma(s_2) = \gamma(x_1)\ \gamma(x_2)^{-1},$$
$$\ldots,\ \gamma(t_n)^{-1}\ \gamma(s_n) = \gamma(x_{n-1})\ \gamma(x_n)^{-1}.$$

Hence

$$\begin{aligned} g &= \gamma(t_1)^{-1}\ \gamma(s_1)\ \gamma(t_2)^{-1}\ \gamma(s_2)\ \gamma(t_n)^{-1}\ \gamma(s_n) \\ &= \gamma(x_0)\ \gamma(x_n)^{-1} = \gamma(x)\ \gamma(y)^{-1}. \end{aligned}$$

□

Lemma 1.10 implies that $H(C) = \{1\}$ for every connected component C of S. Proposition 1.8 then yields:

Corollary 1.11. *Let S act on itself by left multiplication. The universal group act of S is isomorphic to a disjoint union of copies of $G(S)$, one for every connected component of S, on which $G(S)$ acts by left multiplication.*

If in particular S is left connected (e.g. if S is a monoid), then the universal group act of S is isomorphic to $G(S)$, acting on itself by left multiplication.

2. Simply transitive actions

1. We now turn to the general problem posed in the beginning: can the action of S on a set X be extended to a simply transitive action of $G(S)$? that is, is there an action-preserving injection $\alpha : X \longrightarrow Z$, where $G(S)$ acts simply and transitively on Z?

We note some necessary conditions.

Proposition 2.1. *Let $(S, X, \,.\,)$ be a semigroup act, $(G(S), Y, \,.\,)$ be its universal group act, and $(\gamma, \alpha) : (S, X, \,.\,) \longrightarrow (G(S), Z, \,.\,)$ be a morphism of acts, so that $\alpha = \beta \circ \iota$. If α is injective, then ι is injective. If $G(S)$ acts simply on Z, then $G(S)$ acts simply on Y. If $X \neq \emptyset$ and $G(S)$ acts simply and transitively on Z, then $\beta : Y \longrightarrow Z$ is surjective; moreover, for every $z \in Z$, $\beta^{-1}(z)$ contains a single element of every orbit of Y.*

PROOF: By Proposition 1.1 there is a unique action-preserving mapping $\beta : Y \longrightarrow Z$ such that $\alpha = \beta \circ \iota$. If α is injective, then so is ι.

If $G(S)$ acts simply on Z and $g \,.\, y = h \,.\, y$ for some $y \in Y$, then $g \,.\, \beta(y) = \beta(g \,.\, y) = \beta(h \,.\, y) = h \,.\, \beta(y)$ and $g = h$; thus $G(S)$ acts simply on Y.

If $X \neq \emptyset$ and $G(S)$ acts simply and transitively on Z, then $Y \neq \emptyset$ and, for any $z \in Z$ and $y \in Y$, we have $z = g \,.\, \beta(y) = \beta(g \,.\, y)$ for some unique $g \in G(S)$; thus β is surjective, and $\beta^{-1}(z)$ contains exactly one element $g \,.\, y$ of the orbit of y. $\square$

When α is injective and $G(S)$ acts simply and transitively on Z, Proposition 2.1 implies that β is made of bijections from every orbit of Y onto Z.

As in Section 1, let $(C_i)_{i \in I}$ be the family of connected components of X, and let $(c_i)_{i \in I}$ be a cross-section of $\equiv$ (with $c_i \in C_i$). Let V_i be the set of all $g \in G(S)$ such that g is the group value of a connected sequence from some $x \in C_i$ to c_i:

$$V_i \;=\; \{\, g \in G(S) \;\big|\; x \xrightarrow{g} c_i \text{ for some } x \in C_i \,\}.$$

By Lemma 1.7, V_i is a union of left cosets of $H(C_i)$.

Lemma 2.2. *In Proposition 2.1, let α be injective and $G(S)$ act simply and transitively on Z. Let $p \in Z$. Then*

$$\alpha(x) \;=\; \delta(x) \,.\, p$$

defines an injective mapping $\delta : X \longrightarrow G(S)$. Moreover

$$\delta(C_i) \;=\; V_i \, \delta(c_i)$$

for every connected component C_i of X and $c_i \in C_i$.

PROOF: δ is well-defined: since $G(S)$ act simply and transitively on Z there is for every $x \in X$ a unique $\delta(x) \in G(S)$ such that $\alpha(x) = \delta(x) \,.\, p$. Then δ is injective, since α is injective.

If $x \xrightarrow{g} c_i$, then $(1,x) \sim (g,c_i)$ by Lemma 1.4 and $\iota(x) = g\,.\,\iota(c_i)$. Applying β yields

$$\alpha(x) \;=\; \beta(\iota(x)) \;=\; \beta(g\,.\,\iota(c_i)) \;=\; g\,.\,\beta(\iota(c_i)) \;=\; g\,.\,\alpha(c_i).$$

Hence $\delta(x)\,.\,p = g\delta(c_i)\,.\,p$ and $\delta(x) = g\delta(c_i)$. Therefore $\delta(C_i) = V_i\ \delta(c_i)$. □

We say that the connected components of S *have disjoint images in* $G(S)$ (relative to a cross-section of $\equiv$) if there exist $g_i \in G(S)$ such that the sets $V_i g_i$ are disjoint. If the action of S on a set X can be extended to a simply transitive action of $G(S)$, then (relative to any cross-section of $\equiv$) the connected components of S have disjoint images in $G(S)$, by Lemma 2.2.

Theorem 2.3. *Let $(S, X, .\,)$ be a semigroup act and $(G(S), Y, .\,)$ be its universal group act. The action of S on X can be extended to a simply transitive action of $G(S)$ on some set $Z \supseteq X$ if and only if ι is injective, $G(S)$ acts simply on Y, and, relative to some cross-section of $\equiv$, the connected components of S have disjoint images in $G(S)$.*

Proof: These conditions are necessary by Proposition 2.1 and Lemma 2.2. Conversely, assume that ι is injective, $G(S)$ acts simply on Y, and, relative to a cross-section $(c_i)_{i \in I}$ of $\equiv$, the connected components of S have disjoint images in $G(S)$: the sets $V_i g_i$ are disjoint for some $g_i \in G(S)$.

Construct $\alpha : X \longrightarrow G(S)$ as follows. Let $x \in C_i$. When $x \xrightarrow{g} c_i$, then $(1,x) \sim (g,c_i)$ by Lemma 1.4 and $\iota(x) = \mathrm{cls}(1,x) = \mathrm{cls}(g,c_i) = g\,.\,\iota(c_i)$. Since $G(S)$ acts simply on Y, g depends only on x (all connected sequences from x to c_i have the same group value). Therefore a mapping $\alpha : X \longrightarrow G(S)$ is well-defined by

$$\alpha(x) \;=\; gg_i \text{ when } x \in C_i \text{ and } x \xrightarrow{g} c_i.$$

Let $x, y \in X$. If x and y lie in different connected components C_i and C_j, then $\alpha(x) \neq \alpha(y)$, since the sets $V_i g_i$ and $V_j g_j$ are disjoint. Now let x and y lie in the same connected component C_i. Let $x \xrightarrow{g} c_i$ and $y \xrightarrow{h} c_i$. If $\alpha(x) = \alpha(y)$, then $g = h$,

$$\iota(x) \;=\; g\,.\,\iota(c_i) \;=\; \iota(y)$$

by Lemma 1.4, and $x = y$ since ι is injective. Thus α is injective.

Now $G(S)$ acts simply and transitively on itself by left multiplication. We show that $(\gamma, \alpha) : (S, X, .\,) \longrightarrow (G(S), Z, .\,)$ is a morphism of acts. Let $x \in X$ and $s \in S$. Let $x \in C_i$ and $x \xrightarrow{g} c_i$. By Lemma 1.5, $s\,.\,x \xrightarrow{\gamma(s)} x$, $s\,.\,x \xrightarrow{\gamma(s)g} c_i$, and

$$\alpha(s\,.\,x) \;=\; \gamma(s)gg_i = \gamma(s)\alpha(x).$$

□

2. The following results complete Theorem 2.3.

Proposition 2.4. *In the universal group act* $(G(S), Y, \mathbin{.})$ *of* $(S, X, \mathbin{.})$, $\iota(x) = \iota(y)$ *if and only if there exists a connected sequence from* x *to* y *with group value* 1. *If* ι *is injective, then* S *acts by injections.*

Proof: $\iota(x) = \iota(y)$ if and only if $(1, x) \sim (1, y)$, so the first part of the statement follows from Lemma 1.4. Now assume that ι is injective. If $s \mathbin{.} x = s \mathbin{.} y$, then $x_0 = x$, $x_1 = y$, $s_1 = s$, $t_1 = s$ is a connected sequence from x to y with group value 1; hence $x = y$; thus S acts by injections. □

Proposition 2.5. *In the universal group act* $(G(S), Y, \mathbin{.})$ *of* $(S, X, \mathbin{.})$, $G(S)$ *acts simply on* Y *if and only if, for every* $x \in X$, *every connected sequence from* x *to* x *has group value* 1.

Proof: This follows from Proposition 1.8, but we give a direct proof. If $x \xrightarrow{g} x$, then $(1, x) \sim (g, x)$ by Lemma 1.4 and $1 \mathbin{.} \mathrm{cls}(1, x) = g \mathbin{.} \mathrm{cls}(1, x)$; if $G(S)$ acts simply on Y this implies $g = 1$. Conversely let $g \mathbin{.} \mathrm{cls}(k, x) = h \mathbin{.} \mathrm{cls}(k, x)$. Then $(gk, x) \sim (hk, x)$; by Lemma 1.4, there is a connected sequence from x to x with group value $(gk)^{-1}\,(hk)$. If all such sequences have group value 1, then $gk = hk$ and $g = h$; thus $G(S)$ acts simply on Y. □

Propositions 2.4 and 2.5 will be made more explicit in Section 4.

If X is connected, then the universal $G(S)$-set Y of X serves in Theorem 2.3:

Proposition 2.6. *In the universal group action* $(G(S), Y, \mathbin{.})$ *of* $(S, X, \mathbin{.})$, $G(S)$ *acts transitively on* Y *if and only if* X *is connected.*

Proof: This follows from Proposition 1.6, and from Proposition 1.8, but can be shown directly as follows. Let $x, y \in X$. If $G(S)$ acts transitively on Y, then $\mathrm{cls}(1, x) = g \mathbin{.} \mathrm{cls}(1, y) = \mathrm{cls}(g, y)$ for some $g \in G(S)$ and there exists a connected sequence from x to y, by Lemma 1.4; thus X is connected. Conversely let $\mathrm{cls}(h, x)$, $\mathrm{cls}(k, y) \in Y$. If X is connected, there exists a connected sequence from x to y and $(h, x) \sim (g, y)$ for some $g \in G(S)$, by Lemma 1.4; then $kg^{-1} \mathbin{.} \mathrm{cls}(h, x) = kg^{-1} \mathbin{.} \mathrm{cls}(g, y) = \mathrm{cls}(k, y)$. Thus $G(S)$ acts transitively on Y. □

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