Relative normality and product spaces

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Abstract. Arhangel'skiĭ defines in [Topology Appl. 70 (1996), 87–99], as one of various notions on relative topological properties, strong normality of A in X for a subspace A of a topological space X, and shows that this is equivalent to normality of X_A, where X_A denotes the space obtained from X by making each point of $X \setminus A$ isolated. In this paper we investigate for a space X, its subspace A and a space Y the normality of the product $X_A \times Y$ in connection with the normality of $(X \times Y)_{(A \times Y)}$. The cases for paracompactness, more generally, for γ-paracompactness will also be discussed for $X_A \times Y$. As an application, we prove that for a metric space X with $A \subset X$ and a countably paracompact normal space Y, $X_A \times Y$ is normal if and only if $X_A \times Y$ is countably paracompact.

Keywords: strongly normal in, normal, γ-paracompact, product spaces, weak C-embedding

Classification: Primary 54B10; Secondary 54B05, 54C20, 54C45, 54D15, 54D20

1. Introduction

Throughout this paper all spaces are assumed to be Hausdorff. Let γ denote an infinite cardinal, and $\mathbb{N}$ the set of natural numbers.

Let X be a space and A a subspace of X.

As is known, A is said to be C^*-*embedded* (respectively C-*embedded*) in X if every bounded real-valued (respectively real-valued) continuous function on A can be extended to a continuous function over X.

Next we recall some relative topological properties in Arhangel'skiĭ [2]. We say that A is *strongly normal in* X if for every pair E, F of disjoint closed subsets of A there exist disjoint open subsets U and V of X such that $E \subset U$ and $F \subset V$. The subspace A is *weakly* C-*embedded* in X if for every real-valued continuous function f on A there exists a real-valued function on X which is an extension of f and continuous at each point of Y.

For a space X and a subspace A of X let X_A denote the space obtained from the space X, with the topology generated by $\{\, U \,|\, U$ is open in X or $U \subset X \setminus A\}$. Hence A is a closed subspace of X_A and points in $X \setminus A$ are isolated. As is seen in [2], the space X_A is often useful to describe several relative topological properties. Indeed, the following are shown in [2]: (1) X_A is normal if and only if A is strongly normal in X if and only if A is normal itself, and is weakly C-embedded in X, (2) A is weakly C-embedded in X if and only if A is C^*-embedded in X_A.

On the other hand, in a joint paper [9] of the first author with Yamazaki the notion of weak C-embedding was characterized by extending disjoint cozero-sets of a subspace to disjoint open sets of the whole space. And it was applied there for a space X, a subspace A of X and a space Y to describe weak C-embedding of $A \times Y$ in the product $X_A \times Y$; actually, it was shown that if Y is compact Hausdorff, $A \times Y$ is C^*-embedded in $X_A \times Y$ if and only if $A \times Y$ is C^*-embedded in $(X \times Y)_{(A \times Y)}$, that is, $A \times Y$ is weakly C-embedded in $X \times Y$. Being motivated by this result, our main concern in this paper is to study normality of the product $X_A \times Y$ in relation to normality of $(X \times Y)_{(A \times Y)}$ (or, equivalently, strong normality of $A \times Y$ in $X \times Y$). Namely we prove

Theorem 1.1. *For a space X, a subspace A of X and a space Y, the product $X_A \times Y$ is normal if and only if $(X \times Y)_{(A \times Y)}$ is normal and the following condition $(*)$ holds:*

- $(*)$ *for every closed subset E of $X_A \times Y$ disjoint from $A \times Y$ there exists an open subset U of $X_A \times Y$ such that $E \subset U$ and $\overline{U} \cap (A \times Y) = \emptyset$.*

As a corollary to this result we have that for a space X, a subspace A of X and a compact Hausdorff space Y, $X_A \times Y$ is normal if and only if $(X \times Y)_{(A \times Y)}$ is normal. Moreover, using condition $(*)$ above we prove analogous results for γ-collectionwise normality or γ-paracompactness. In particular, the case $\gamma = \omega$ is applied to obtain further the following theorem; putting $A = X$, we have the well-known theorem due to Morita [14] (for the proof see [10]) and Rudin and Starbird [16].

Theorem 1.2. *Let X be a metric space, A a subspace of X and Y a normal and countably paracompact space. Then $X_A \times Y$ is normal if and only if $X_A \times Y$ is countably paracompact.*

For undefined notation and terminology see Engelking's book [6].

2. Preliminaries

The following theorem due to Arhangel'skiĭ [2] mentioned in the introduction is useful.

Theorem 2.1 ([2]). *For a subspace A of a space X, the following statements are equivalent:*

(1) *X_A is normal,*
(2) *A is strongly normal in X,*
(3) *A is normal and A is weakly C-embedded in X.*

Weak C-embedding was characterized in [9] as follows.

Theorem 2.2 ([9]). *Let A be a subspace of a space X. Then A is weakly C-embedded in X if and only if for every pair G_0, G_1 of disjoint cozero-sets in A there exist disjoint open subsets H_0, H_1 of X such that $G_i \subset H_i$ $(i = 0, 1)$.*

By this result we see that if either A is dense in X or A is z-embedded in X, then A is weakly C-embedded in X ([5], [9]); a subspace A of a space X is said to be *z-embedded* in X if every zero-set Z of A can be written as $Z = Z' \cap A$ with a zero-set Z' of X. It is known that every cozero-set of a space or a Lindelöf subspace of a Tychonoff space is z-embedded. Also, observe the following implications:

$$C^*\text{-embedding} \quad \Rightarrow \quad z\text{-embedding} \quad \Rightarrow \quad \text{weak } C\text{-embedding.}$$

The next two results show when a subspace $A \times Y$ is weakly C-embedded in $X \times Y$ for a space X, a subspace A of X and a metric space Y. The first one is essentially due to Kodama [11].

Theorem 2.3 ([11]). *Let X be a normal space, A a closed subspace of X and Y a metric space. If $A \times Y$ is normal and countably paracompact, then $A \times Y$ is z-embedded in $X \times Y$, hence, weakly C-embedded in $X \times Y$.*

In case $A \times Y$ is not assumed to be normal, we have the following.

Theorem 2.4. *Let A be an arbitrary subspace of a hereditarily normal space X, and Y a metric space. Then $A \times Y$ is weakly C-embedded in $X \times Y$.*

Proof: We show that any two disjoint open sets of $A \times Y$ are separated by disjoint open sets of $X \times Y$, which implies weak C-embedding of $A \times Y$ in $X \times Y$ by Theorem 2.2. Let G_0 and G_1 be disjoint open sets of $A \times Y$. Let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ be a σ-locally finite open base for Y, where each $\mathcal{B}_n$ is locally finite. Let $\mathcal{B}_n = \{B_{n\lambda} \mid \lambda \in \Lambda_n\}$. Define for $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$

$$H^0_{n\lambda} = \bigcup \{\, O \mid O \text{ is open in } A,\ O \times \overline{B_{n\lambda}} \subset G_0 \}.$$

Then $H^0_{n\lambda}$ and $p_A\big((A \times \overline{B_{n\lambda}}) \cap G_1\big)$ are disjoint open subsets of A. Since X is hereditarily normal, there exists an open set $W^0_{n\lambda}$ of X such that

$$H^0_{n\lambda} \subset W^0_{n\lambda}, \quad \overline{W^0_{n\lambda}} \cap p_A\big((A \times \overline{B_{n\lambda}}) \cap G_1\big) = \emptyset.$$

For each $n \in \mathbb{N}$ let us put $U^0_n = \bigcup \{\, W^0_{n\lambda} \times B_{n\lambda} \mid \lambda \in \Lambda_n \}$. Then U^0_n is an open set of $X \times Y$ and we have $G_0 \subset \bigcup_{n \in \mathbb{N}} U^0_n$ and $\overline{U^0_n} \cap G_1 = \emptyset$ for every $n \in \mathbb{N}$. Similarly, we can find an open set U^1_n of $X \times Y$ for each $n \in \mathbb{N}$ so that $G_1 \subset \bigcup_{n \in \mathbb{N}} U^1_n$ and $\overline{U^1_n} \cap G_0 = \emptyset$ for every $n \in \mathbb{N}$. Hence, as is well-known, G_0 and G_1 are separated by open sets of $X \times Y$. This completes the proof. □

It was shown in [9] that every subspace of a space X is weakly C-embedded in X if and only if X is hereditarily normal.

In connection with Theorems 2.3 and 2.4, let us observe the following two examples.

Example 2.5. (1) (Michael [12]) Let $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{P}$ be the real line, the set of rationals and the set of irrationals, respectively. Then $\mathbb{R}_{\mathbb{Q}}$ is known as the Michael line, and it is hereditarily normal. Since $\mathbb{Q} \times \mathbb{P}$ is Lindelöf, it is z-embedded in $\mathbb{R}_{\mathbb{Q}} \times \mathbb{P}$, but is not C^*-embedded as was shown by Morita [15].

(2) (Vaughan [17]) Let $D(\omega_1)$ denote the set ω_1 with the discrete topology. Let $\widehat{D}(\omega_1)$ denote the space obtained from the space $\omega_1 + 1$ with the usual order topology by letting all points except ω_1 be isolated. That is, $\widehat{D}(\omega_1) = (\omega_1+1)_{\{\omega_1\}}$.

Let $X = \Box_\omega \widehat{D}(\omega_1)$ denote the box product of countably many copies of $\widehat{D}(\omega_1)$, and $Y = D(\omega_1)^\omega$ denote the usual product of countably many copies of $D(\omega_1)$.

Then X is hereditarily paracompact and Y is metrizable. Put

$$A = X \setminus Y, \quad \Delta(Y) = \{\langle x, x\rangle \mid x \in Y\}.$$

Then $A \times Y$ and $\Delta(Y)$ are disjoint closed sets of $X \times Y$ and cannot be separated by open sets, which shows $X \times Y$ is not normal ([17]).

By Theorem 2.4 we see that $A \times Y$ is weakly C-embedded in $X \times Y$. Since A contains a closed subset homeomorphic to X, $A \times Y$ is not normal. Hence, in view of Theorem 2.3, it may be of interest to see whether $A \times Y$ is z-embedded in $X \times Y$, but this is unknown to the authors. However, we can show further that $A \times Y$ is not C^*-embedded in $X \times Y$. To prove this, first note that $Y \cong$ (is homeomorphic to) Y^2. Hence, if we show the fact below, by the same argument of Morita [15] we can conclude that $A \times Y$ is not C^*-embedded in $X \times Y$.

Fact. $\Delta(Y)$ is a zero-set of $X \times Y$.

Proof: Since the box topology is stronger than the usual topology, it suffices to show that $\Delta(Y)$ is a zero-set of $\widehat{D}(\omega_1)^\omega \times D(\omega_1)^\omega$.

For each point $\langle x, y\rangle \in \widehat{D}(\omega_1)^\omega \times D(\omega_1)^\omega \setminus \Delta(Y)$, define

$$n(x, y) = \min\{k \mid x_k \neq y_k\}.$$

Put

$$H_m = \{\langle x, y\rangle \in \widehat{D}(\omega_1)^\omega \times D(\omega_1)^\omega \setminus \Delta(Y) \mid n(x, y) = m\}.$$

Then we have

$$\widehat{D}(\omega_1)^\omega \times D(\omega_1)^\omega \setminus \Delta(Y) = \textstyle\bigcup_{m\in\mathbb{N}} H_m,$$
$$m \neq m' \;\Rightarrow\; H_m \cap H_{m'} = \emptyset.$$

Claim. *H_m is an open and closed subset of $\widehat{D}(\omega_1)^\omega \times D(\omega_1)^\omega$.*

Proof of Claim: Let $\langle x, y\rangle \in H_m$. Since $n(x, y) = m$, we have $x_1 = y_1, \dots,$ $x_{m-1} = y_{m-1} < \omega_1$.

Case (i). $x_m > y_m$. Put

$$\begin{aligned} U &= \{x_1\} \times \cdots \times \{x_{m-1}\} \times (y_m, \omega_1] \times \widehat{D}(\omega_1) \times \cdots, \\ V &= \{y_1\} \times \cdots \times \{y_{m-1}\} \times \{y_m\} \quad \times D(\omega_1) \times \cdots. \end{aligned}$$

Then $\langle x, y\rangle \in U \times V \subset H_m$.

Case (ii). $x_m < y_m$. Put

$$\begin{aligned} U &= \{x_1\} \times \cdots \times \{x_{m-1}\} \times \{x_m\} \times \widehat{D}(\omega_1) \times \cdots, \\ V &= \{y_1\} \times \cdots \times \{y_{m-1}\} \times \{y_m\} \times D(\omega_1) \times \cdots. \end{aligned}$$

Then $\langle x, y\rangle \in U \times V \subset H_m$.
Hence, in either case H_m is open in $\widehat{D}(\omega_1)^\omega \times D(\omega_1)^\omega$.

For each $\langle y, y\rangle \in \Delta(Y)$, put

$$\begin{aligned} U &= \{y_1\} \times \cdots \times \{y_m\} \times \widehat{D}(\omega_1) \times \cdots, \\ V &= \{y_1\} \times \cdots \times \{y_m\} \times D(\omega_1) \times \cdots. \end{aligned}$$

Then $(U \times V) \cap H_m = \emptyset$. Hence, $\Delta(Y) \cap \overline{H_m} = \emptyset$, which shows that H_m is closed in $X \times Y$.

It follows that H_m is a cozero-set, therefore, $\bigcup_{m\in\mathbb{N}} H_m$ is a cozero-set of $\widehat{D}(\omega_1)^\omega \times D(\omega_1)^\omega$. Hence $\Delta(Y)$ is a zero-set of $X \times Y$. This completes the proof. $\square$

3. Proof of Theorem 1.1

First we prove

Lemma 3.1. *Let X be a space, A a subspace of X and Y a space. If $X_A \times Y$ is normal, then $(X \times Y)_{(A\times Y)}$ is normal.*

Proof: Let E and F be disjoint closed subsets of $A \times Y$. Then they are closed also in $X_A \times Y$ and disjoint. Hence, there exist disjoint open subsets U and V of $X_A \times Y$ such that $E \subset U$ and $F \subset V$. Define $U' = \mathrm{Int}_{(X\times Y)} U$ and $V' = \mathrm{Int}_{(X\times Y)} V$, where $\mathrm{Int}_Z W$ denotes the interior of W in the space Z. Then U' and V' are disjoint open in $X \times Y$ and so in $(X \times Y)_{(A\times Y)}$, and we have $E \subset U'$ and $F \subset V'$. Hence, $A \times Y$ is strongly normal in $X \times Y$. Hence by Theorem 2.1 $(X \times Y)_{(A\times Y)}$ is normal. This completes the proof. $\square$

Remark. $(\mathbb{R} \times \mathbb{P})_{(\mathbb{Q}\times\mathbb{P})}$ is normal, but $\mathbb{R}_{\mathbb{Q}} \times \mathbb{P}$ is not normal. The converse of the lemma, therefore, need not hold.

Proof of Theorem 1.1: From Lemma 3.1 the "only if" part easily follows. To prove the "if" part, assume that $(X \times Y)_{(A\times Y)}$ is normal and condition $(*)$ holds. Let E, F be a pair of disjoint closed subsets of $X_A \times Y$. Since $A \times Y$ is strongly normal in $X \times Y$ by Theorem 2.1, there exist disjoint open subsets U and V of $X \times Y$ such that $E \cap (A \times Y) \subset U$ and $F \cap (A \times Y) \subset V$. Put $D = (E \setminus U) \cup (F \setminus V)$. Then D is a closed subset of $X \times Y$ and $D \cap (A \times Y) = \emptyset$. Then by $(*)$, there exists an open subset W of $X_A \times Y$ such that $A \times Y \subset W$ and $\overline{W} \cap D = \emptyset$.

Put $U_1 = U \cap W$ and $V_1 = V \cap W$. Then we have

$$(A \times Y) \cap E \subset U_1,\ \overline{U_1} \cap F = \emptyset, \quad \text{and} \quad (A \times Y) \cap F \subset V_1,\ \overline{V_1} \cap E = \emptyset.$$

Then $E \setminus U_1$ and $F \setminus V_1$ are disjoint closed subsets of $(X_A \setminus A) \times Y$. Since $(X_A \setminus A) \times Y$ is normal, there exist disjoint open subsets U_2 and V_2 of $(X_A \setminus A) \times Y$ such that $E \setminus U_1 \subset U_2$ and $F \setminus V_1 \subset V_2$. Therefore, $U_1 \cup \big(U_2 \setminus \overline{V_1}\big)$ and $V_1 \cup \big(V_2 \setminus \overline{U_1}\big)$ are disjoint open subsets of $X_A \times Y$, which satisfy $E \subset U_1 \cup \big(U_2 \setminus \overline{V_1}\big)$ and $F \subset V_1 \cup \big(V_2 \setminus \overline{U_1}\big)$. Hence $X_A \times Y$ is normal. This completes the proof. $\square$

The following is proved in Burke and Pol [4].

Theorem 3.2 ([4]). *Let A and X be subsets of $\mathbb{R}$ with $A \subset X$ and let Y be a metric space. Then $X_A \times Y$ is normal if and only if condition $(*)$ holds.*

Since $X \times Y$ is a metric space, $(X \times Y)_{(A\times Y)}$ is normal. Therefore, this theorem immediately follows from Theorem 1.1.

The following result was formulated in [9] without proof.

Theorem 3.3 ([9]). *Let A be a subset of a space X and Y be a compact Hausdorff space. Then $X_A \times Y$ is normal if and only if $(X \times Y)_{(A\times Y)}$ is normal.*

Proof: Since the projection $p_{X_A}\colon X_A \times Y \to X_A$ is a closed map, condition $(*)$ in Theorem 1.1 is easily satisfied. Hence the theorem follows. $\square$

Recall that a space X is γ-*collectionwise normal* if for every discrete collection $\{E_\alpha \,|\, \alpha < \gamma\}$ of closed subsets there exists a disjoint collection $\{G_\alpha \,|\, \alpha < \gamma\}$ of open subsets such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$.

A subspace A of a space X is said to be *strongly γ-collectionwise normal in X* if for every discrete collection $\{E_\alpha \,|\, \alpha < \gamma\}$ of closed subsets of A there is a disjoint collection $\{U_\alpha \,|\, \alpha < \gamma\}$ of open subsets of X such that $E_\alpha \subset U_\alpha$ for each $\alpha < \gamma$ ([9]).

It was proved in [9] that X_A is γ-collectionwise normal if and only if A is strongly γ-collectionwise normal in X. With this result similarly to Theorem 1.1 we can prove the following.

Theorem 3.4. *For a space X, a subspace A of X and a space Y, $X_A \times Y$ is γ-collectionwise normal if and only if $(X \times Y)_{(A\times Y)}$ is γ-collectionwise normal and condition $(*)$ in Theorem 1.1 holds.*

A space X is *γ-paracompact* if every open cover of X of cardinality not greater than γ has a locally finite open refinement.

Theorem 3.5. *If $X_A \times Y$ is γ-paracompact, then $(X \times Y)_{(A\times Y)}$ is γ-paracompact. Furthermore, if $X_A \times Y$ satisfies condition $(*)$ in Theorem 1.1, then the converse holds.*

Proof: Assume $X_A \times Y$ is γ-paracompact. Let $\mathcal{U}$ be an open cover of $(X \times Y)_{(A\times Y)}$ of cardinality not greater than γ. Put

$$\mathcal{U}' = \big\{U \in \mathcal{U} \,\big|\, U \cap (A \times Y) \neq \emptyset\big\}.$$

Then $\bigcup\{\mathrm{Int}_{(X\times Y)} U \,\big|\, U \in \mathcal{U}'\} \supset A \times Y$. Hence $\{X_A \times Y \setminus A \times Y\} \cup \mathcal{U}'$ is an open cover of $X_A \times Y$ of cardinality not greater than γ. Since $X_A \times Y$ is γ-paracompact, there exists a locally finite open cover $\mathcal{V}$ of $X_A \times Y$ which refines $\mathcal{U}$. Put $\mathcal{V}' = \{V \in \mathcal{V} \,\big|\, V \cap (A \times Y) \neq \emptyset\}$. Then the collection

$$\mathcal{V}' \cup \big\{\{\langle x, y\rangle\} \,|\, \langle x, y\rangle \notin \bigcup \mathcal{V}'\big\}$$

is a locally finite open cover of $(X \times Y)_{(A\times Y)}$ and refines $\mathcal{U}$. Hence $(X \times Y)_{(A\times Y)}$ is γ-paracompact.

To prove the converse under $(*)$, assume that $(X \times Y)_{(A\times Y)}$ is γ-paracompact and $(*)$ holds. Let $\mathcal{U}$ be an open cover of $X_A \times Y$ of cardinality not greater than γ. Then $\mathcal{U}$ is an open cover of $(X \times Y)_{(A\times Y)}$ as well. By assumption there exists a locally finite open cover $\mathcal{V}$ of $(X \times Y)_{(A\times Y)}$ refining $\mathcal{U}$. Put

$$G = \big\{\langle x, y\rangle \in X \times Y \,\big|\, \mathcal{V} \text{ is locally finite at } \langle x, y\rangle \text{ in the product } X \times Y\big\}.$$

Then G is open in $X \times Y$ and $G \supset A \times Y$. Put $\mathcal{V}' = \big\{G \cap \mathrm{Int}_{(X\times Y)} V \,\big|\, V \in \mathcal{V}\big\}$. Then we have $\bigcup \mathcal{V}' \supset A \times Y$, and $\mathcal{V}'$ refines $\mathcal{U}$ and is locally finite at each $\langle x, y\rangle \in \bigcup \mathcal{V}'$ in $X \times Y$. By $(*)$ there exist open subsets O_1 and O_2 in $X_A \times Y$ such that

$$A \times Y \subset O_1 \subset \overline{O_1} \subset O_2 \subset \overline{O_2} \subset \bigcup \mathcal{V}'.$$

For every $x \in X \setminus A$, let $\mathcal{P}_x$ be a locally finite open cover of Y such that the collection $\big\{\{x\} \times P \,\big|\, P \in \mathcal{P}_x\big\}$ refines $\mathcal{U}$. Then the collection

$$\big\{(\{x\} \times P) \setminus \overline{O_1} \,\big|\, x \in X \setminus Y,\, P \in \mathcal{P}_x\big\} \cup \big\{V \cap O_2 \,\big|\, V \in \mathcal{V}'\big\}$$

is a locally finite open cover of $X_A \times Y$ which refines $\mathcal{U}$. Thus, $X_A \times Y$ is γ-paracompact. This completes the proof.

□

4. Proof of Theorem 1.2

First we prove

Theorem 4.1. *Let A be a subset of a space X and Y a space. Suppose that the product $A \times Y$ is γ-paracompact. If $X_A \times Y$ is normal, then $X_A \times Y$ is γ-paracompact.*

Proof: Assume that $X_A \times Y$ is normal. Then $(X \times Y)_{(A \times Y)}$ is normal by Theorem 1.1. Hence $A \times Y$ is normal and weakly C-embedded in $X \times Y$ by Theorem 2.1. Since $A \times Y$ is γ-paracompact, by [9, Lemma 4.6] $(X \times Y)_{(A \times Y)}$ is γ-paracompact. Since $X_A \times Y$ satisfies $(*)$, $X_A \times Y$ is γ-paracompact by Theorem 3.5. This completes the proof. □

Corollary 4.2. *Let A be a subset of a space X and Y a space. Suppose that the product $A \times Y$ is countably paracompact. If $X_A \times Y$ is normal, then $X_A \times Y$ is countably paracompact.*

Proof of Theorem 1.2: Let A be a subspace of a metric space X, and Y a normal and countably paracompact space. To prove the "only if" part, assume $X_A \times Y$ is normal. Since $A \times Y$ is closed in $X_A \times Y$, $A \times Y$ is also normal. Hence by Morita, Rudin-Starbird's theorem ([14], [16]), $A \times Y$ is countably paracompact. Hence $X_A \times Y$ is countably paracompact by Corollary 4.2.

To prove the converse, assume that $X_A \times Y$ is countably paracompact. Then similarly to above we have that $A \times Y$ is countably paracompact and normal. Then by [11] $A \times Y$ is z-embedded in $A \times \beta Y$, where βY is the Čech-Stone compactification of Y. Since $X_A \times \beta Y$ is paracompact, $A \times \beta Y$ is C-embedded in $X_A \times \beta Y$. It follows that $A \times Y$ is z-embedded in $X_A \times Y$, and hence it is weakly C-embedded in $X_A \times Y$. This easily implies that $A \times Y$ is weakly C-embedded in $X \times Y$. Hence $(X \times Y)_{(A \times Y)}$ is normal.

We next show that property $(*)$ in Theorem 1.1 is satisfied. Let $\{\mathcal{B}_n\}$ be a sequence of locally finite open covers of X such that $\{\, \mathrm{St}(x, \mathcal{B}_n) \mid n \in \mathbb{N}\}$ is a neighborhood base at each point x in X. Let $\mathcal{B}_n = \{B_{n\alpha} \mid \alpha \in \Omega_n\}$. Let us put

$$W(\alpha_1, \cdots, \alpha_n) = \bigcap \{B_{i\alpha_i} \mid i = 1, \ldots, n\}, \quad \text{for} \quad \alpha_i \in \Omega_i; \;\; i = 1, \ldots, n.$$

To prove $(*)$, let E be a closed subset of $X_A \times Y$ such that $E \cap (A \times Y) = \emptyset$. Put

$$G(\alpha_1, \cdots, \alpha_n) = \bigcup \{O \mid O \text{ is open in } Y,\ (W(\alpha_1, \cdots, \alpha_n) \times O) \cap E = \emptyset\}.$$

Then we have

$$G(\alpha_1, \cdots, \alpha_n) \subset G(\alpha_1, \cdots, \alpha_n, \alpha_{n+1})$$

for $\alpha_i \in \Omega_i,\ i = 1, \dots , n, n+1$, and

$$\big\{(W(\alpha_1,\cdots,\alpha_n)\cap A)\times G(\alpha_1,\cdots,\alpha_n)\,\big|\,\alpha_i\in\Omega_i,\,i=1,\dots,n;\,n\in\mathbb{N}\big\}$$

covers $A\times Y$. Since $A\times Y$ is normal and countably paracompact, by Morita [13] (see [8]) there exists a cozero-set $U(\alpha_1,\cdots,\alpha_n)$ of Y such that

$$U(\alpha_1,\cdots,\alpha_n)\subset G(\alpha_1,\cdots,\alpha_n)$$

and

$$\big\{(W(\alpha_1,\cdots,\alpha_n)\cap A)\times U(\alpha_1,\cdots,\alpha_n)\,\big|\,\alpha_i\in\Omega_i,\,i=1,\dots,n;\,n\in\mathbb{N}\big\}$$

covers $A\times Y$. Put

$$L=\bigcup\big\{W(\alpha_1,\cdots,\alpha_n)\times U(\alpha_1,\cdots,\alpha_n)\,\big|\,\alpha_i\in\Omega_i,\,i=1,\dots,n;\,n\in\mathbb{N}\big\}.$$

Then L is a cozero-set of $X\times Y$ and we have $L\supset A\times Y,\ L\cap E=\emptyset$. Since $X_A\times Y$ is countably paracompact, by [7] there exists an open subset H of $X\times Y$ such that $A\times Y\subset H\subset\overline{H}\subset L$. Hence $A\times Y$ and E are separated by open sets of $X_A\times Y$. This completes the proof of the theorem. □

The proof of the "if" part of Theorem 1.1 yields further the following result which seems of interest in itself.

Theorem 4.3. *Let A be a subset of a metric space X and Y a normal and γ-paracompact space. Then $(X\times Y)_{(A\times Y)}$ is γ-paracompact if and only if $A\times Y$ is normal.*

PROOF: To prove the "if" part, assume that $A\times Y$ is normal. Since Y is normal and γ-paracompact, so is $A\times Y$. Hence $(A\times Y)\times \mathrm{I}^\gamma$ is normal γ-paracompact, that is, $A\times \mathrm{I}^\gamma\times Y$ is normal, where $\mathrm{I}=[0,1]$. Hence, as is shown in the proof of Theorem 1.2, $(X\times(\mathrm{I}^\gamma\times Y))_{(A\times(\mathrm{I}^\gamma\times Y))}$ is normal. Since $(X\times(\mathrm{I}^\gamma\times Y))_{(A\times(\mathrm{I}^\gamma\times Y))}\cong((X\times Y)\times\mathrm{I}^\gamma)_{((A\times Y)\times\mathrm{I}^\gamma)}$, $((X\times Y)\times\mathrm{I}^\gamma)_{((A\times Y)\times\mathrm{I}^\gamma)}$ is normal. Thus, by Theorem 3.3 $(X\times Y)_{(A\times Y)}\times\mathrm{I}^\gamma$ is normal. Therefore, as is well-known, $(X\times Y)_{(A\times Y)}$ is γ-paracompact (see [6]). This completes the proof. □

Example 4.4. The condition "X is metric" cannot be excluded from Theorem 1.2. In fact, there exist compact spaces X and Y, and a subset A of X such that $A\times Y$ is normal and countably paracompact and $X_A\times Y$ is countably paracompact, but not normal. We use Bing's example G [3]. Let $\mathcal{P}(\omega_1)$ be the power set of ω_1 and

$$X=\big\{f\,\big|\,f:\mathcal{P}(\omega_1)\longrightarrow\{0,1\}\big\}.$$

For every $\alpha \in \omega_1$, let us define a function $f_\alpha : \mathcal{P} \longrightarrow \{0,1\}$ for $P \in \mathcal{P}(\omega_1)$ by

$$f_\alpha(P) = \begin{cases} 1 & \text{if } \alpha \in P, \\ 0 & \text{if } \alpha \notin P. \end{cases}$$

Put $A = \{f_\alpha \,|\, \alpha < \omega_1\}$. Then Bing's example G is precisely the space X_A. It is well-known that X_A is normal and countably paracompact, but it is not ω_1-collectionwise normal. Let Y be the one-point compactification of the discrete space of Card A. Since X_A is countably paracompact, $A \times Y$ is countably paracompact. Since A is $w(Y)$-paracompact, $A \times Y$ is normal. However, since X_A is not ω_1-collectionwise normal, by Alas [1] $X_A \times Y$ is not normal.

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(*Received* March 3, 2003)