Open maps do not preserve Whyburn property

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Abstract. We show that a (weakly) Whyburn space X may be mapped continuously via an open map f onto a non (weakly) Whyburn space Y. This fact may happen even between topological groups X and Y, f a homomorphism, X Whyburn and Y not even weakly Whyburn.

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1. Introduction

A subset $F \subset X$ of a topological space X is almost closed if $|\overline{F} \setminus F| = 1$. If F is almost closed and $\overline{F} \setminus F = \{x\}$ we shall write $F \to x$. A topological space X is weakly Whyburn if for any non-closed subset $A \subset X$ there exists a point $x \in \overline{A} \setminus A$ and an almost closed set $F \subset A$ such that $F \to x$. A topological space X is Whyburn if for any non-closed subset $A \subset X$ and for any point $x \in \overline{A} \setminus A$ there exists an almost closed set $F \subset A$ such that $F \to x$. Clearly any Whyburn space is weakly Whyburn.

A space X is pseudoradial if for any non-closed subset A of X there is a (possibly transfinite) sequence of points of A converging to a point $x \notin A$. If a sequence converging to x can be selected for any point $x \in \overline{A}$ the space is radial.

It is well known that radiality and pseudoradiality are preserved respectively by pseudo-open or closed maps and by quotient maps. It is a natural question to ask if the (weak) Whyburn property is preserved by some of these functions. As it is easily seen ([TY]), properties Whyburn and weakly Whyburn are preserved by closed maps. It is a natural question to ask if the (weak) Whyburn property is preserved by some of these functions. It has been remarked in [TY] that the quotient of a Whyburn space may fail to be Whyburn. Moreover in [O] it is shown that the quotient and even a pseudo-open image of a Whyburn space may fail to be even weakly Whyburn. Problem 4.6 in [TY] asks if the open image of a (weakly) Whyburn space is (weakly) Whyburn. During his talk at "Topology in Matsue 2002" conference (Japan), V.V. Tkachuk asked the same question, showing a special interest in getting an answer for topological groups and homomorphisms. The same question (for groups) appears as Problem 3.2 in [PTTW], where it is underlined that the answer is not known for general spaces but the group version seems to give more hope to a positive result. The aim of this note is to disprove such a conjecture by giving an example of a Whyburn topological group G, a non-weakly Whyburn topological group H and an open continuous homomorphism f on G onto H. Even if by far the third example is the most important, we think it might be of some interest to see two additional examples for topological spaces.

2. Examples

Example 2.1. *A continuous open map from a weakly Whyburn space onto a non-weakly Whyburn space.*

Let $\mathfrak{c} = \{\alpha : \alpha < \mathfrak{c}\}$ be the cardinal number $2^{\aleph_0}$ and let I be the unit compact interval of $\mathbb{R}$. Let $\varphi : \mathfrak{c} \to I$ be any bijection. Let Y be the disjoint union of $\mathfrak{c}$ and I. We consider the following topology on Y: any point $\alpha \in \mathfrak{c}$ is isolated in Y; a basic neighbourhood U of a point $x \in I$ is of the form

$$U = J \cup \{\alpha \in \mathfrak{c} : \ \alpha > \alpha_0, \ \varphi(\alpha) \in J\},$$

where J is an open neighbourhood of x in I and $\alpha_0 \in \mathfrak{c}$.

The space Y is not weakly Whyburn, since any non-closed (in Y) subset of $\mathfrak{c} \subset Y$ has $\mathfrak{c}$ accumulation points in I.

Let X be the disjoint union of the cartesian product $\mathfrak{c} \times \mathfrak{c}$ and I. We consider the following topology on X. Any point $\langle \alpha, \beta \rangle \in \mathfrak{c} \times \mathfrak{c}$ is isolated in X. A basic neighbourhood U of a point $x \in I$ is of the following form: let J be any neighbourhood of x in I; for any $y \in J$ let $\alpha_y \in \mathfrak{c}$ be any ordinal number and let J_y be any neighbourhood of y in J; then put

$$U = J \cup \bigcup_{y \in J} \{\langle \alpha, \varphi^{-1}(y) \rangle : \alpha > \alpha_y, \ \varphi(\alpha) \in J_y\}.$$

Finally let $f : X \to Y$ be the function defined by $f(\langle \alpha, \beta \rangle) = \alpha$ for any $\langle \alpha, \beta \rangle \in \mathfrak{c} \times \mathfrak{c}$; $f(x) = x$ for any $x \in I$.

We claim that X is a weakly Whyburn regular space, f is a continuous open map onto Y and Y is not weakly Whyburn.

Let $E \subset X$ be any non-closed subset. We may assume $E \cap I = \emptyset$ (if $E \cap I$ is closed and $x \in \overline{E} \setminus E$ pick an open neighbourhood U of x missing $E \cap I$ and replace E by $E \cap U$; if $E \cap I$ is not closed use sequentiality of I to pick a sequence in $E \cap I$ converging outside E).

Let us assume first that there is an element $y \in I$ such that $y \in \overline{E_y}$ where $E_y = \{\langle \alpha, \varphi^{-1}(y) \rangle : \alpha \in \mathfrak{c}\} \cap E$. Then clearly y is the only accumulation point of E_y and E_y is an almost closed subset of E converging outside E.

Assume now that no element $y \in I$ is an accumulation point of E_y. Then, for each y we can find a neighbourhood of the form

$$U_y = J_y \cup \bigcup_{u \in J_y} \{\langle \alpha, \varphi^{-1}(u) \rangle : \alpha > \alpha(y)_u, \ \varphi(\alpha) \in L(y)_u\};$$

where $L(y)_u$ is an open neighbourhood of u in J_y and both $L(y)_u$ and $\alpha(y)_u$ depend on both y and u, such that $U_y \cap E_y = \emptyset$.

In particular, if $\langle \alpha, \varphi^{-1}(y)\rangle$ is such that both $\alpha > \alpha(y)_y$ and $\varphi(\alpha) \in L(y)_y$, then $\langle \alpha, \varphi^{-1}(y)\rangle$ does not belong to E.

Let $x \in I$. Consider the open neighbourhood of x defined by

$$U = I \cup \bigcup_{y\in I} \{\langle \alpha, \varphi^{-1}(y)\rangle : \alpha > \alpha(y)_y,\ \varphi(\alpha) \in L(y)_y\}.$$

Then $U \cap E = \emptyset$. In fact let $\langle \alpha, \varphi^{-1}(y)\rangle \in U$, then both $\alpha > \alpha(y)_y$ and $\varphi(\alpha) \in L(y)_y$, hence $\langle \alpha, \varphi^{-1}(y)\rangle \notin E$. This contradicts the fact that E is not closed in X.

The function f is continuous. In fact let $U = J \cup \{\alpha \in \mathfrak{c} :\ \alpha > \alpha_0,\ \varphi(\alpha) \in J\}$ be any basic open neighbourhood of a point $x \in I \subset Y$. Then $f^{-1}(U) = J \cup \{\langle \alpha, \beta\rangle \in \mathfrak{c} \times \mathfrak{c} :\ \alpha > \alpha_0,\ \varphi(\alpha) \in J,\ \beta \in \mathfrak{c}\}$ is an open neighbourhood of $x \in X$.

The function f is also open. In fact let $U = J \cup \bigcup_{y\in J}\{\langle \alpha, \varphi^{-1}(y)\rangle : \alpha > \alpha_y,\ \varphi(\alpha) \in J_y\}$ be any basic open neighbourhood of a point $x \in I \subset X$. Let $y \in f(U)$. The set $W = J_y \cup \{\alpha \in \mathfrak{c} :\ \alpha > \alpha_y,\ \varphi(\alpha) \in J_y\}$ is an open neighbourhood of y contained in $f(U)$. Hence $f(U)$ is an open set in Y.

Example 2.2. *A continuous open map from a Whyburn space onto a non-Whyburn space.*

For any $\alpha < \omega_1$ let X_α be the ordered space $[0, \alpha[\, \subset \omega_1$ and consider the topological sum S of all spaces X_α for $\alpha < \omega_1$. We denote the point γ in the space X_α by $\langle \alpha, \gamma\rangle$. Let $X = S \cup \{\infty\}$ where $\infty \notin S$.

We consider the following topology on X: the topology on X_α is the order topology; for $\gamma < \rho < \omega_1$ a basic neighbourhood of ∞ is given by

$$U_{\rho,\gamma} = \{\infty\} \cup \bigcup_{\alpha>\rho} \{\langle \alpha, \beta\rangle : \beta \in\,]\gamma, \alpha[\,\}.$$

Let us show that the space X is Whyburn. Since for any α the space X_α is first countable we only need to check about the point ∞.

Assume $\infty \in \overline{E}$ for some $E \subseteq S$. Note that for any $\gamma < \omega_1$ there are uncountably many α's and points $\langle \alpha, \beta\rangle \in E$ such that $\beta > \gamma$. Let us construct by induction over ω_1 a "diagonal" in S of points of E as follows: let $\alpha_1 \in \omega_1$ be such that $E \cap X_{\alpha_1} \neq \emptyset$ and take any $\langle \alpha_1, \beta_1\rangle \in E \cap X_{\alpha_1}$. Assume we have defined $\langle \alpha_\delta, \beta_\delta\rangle \in E \cap X_{\alpha_\delta}$ for all $\delta < \overline{\delta}$ where both sequences $\{\alpha_\delta\}_{\delta<\overline{\delta}}$ and $\{\beta_\delta\}_{\delta<\overline{\delta}}$ are increasing. Let σ be an ordinal larger than both $\sup\{\beta_\delta < \overline{\delta}\}$ and $\sup\{\alpha_\delta < \overline{\delta}\}$. The set $U_{\sigma+1,\sigma}$ meets E, so we can pick a point $\langle \alpha_{\overline{\delta}}, \beta_{\overline{\delta}}\rangle \in E$ such that $\alpha_{\overline{\delta}} > \alpha_\delta$ and $\beta_{\overline{\delta}} > \beta_\delta$ for all $\delta < \overline{\delta}$ as required.

We claim that the set $E' = \{\langle \alpha_\delta, \beta_\delta\rangle : \delta < \omega_1\}$ is almost closed and $E' \to \infty$. In fact ∞ is clearly a limit point of E'; moreover for all α's the set E' meets X_α at most in one point, hence no points of S can be limit points of E'.

Now let us consider the map $f : X \to \omega_1 + 1$ defined by $f(\langle\alpha,\beta\rangle) = \beta$ for all $\langle\alpha,\beta\rangle \in S$ and $f(\infty) = \omega_1$. To check that the function f is continuous and open observe that $f^{-1}(\,]\gamma,\omega_1]) = U_{\gamma+1,\gamma}$ is an open set in X and $f(U_{\rho,\gamma}) = \,]\gamma,\omega_1]$ is an open set in $\omega_1 + 1$.

The space $\omega_1 + 1$ is hereditarily weakly Whyburn but not Whyburn ([PT], [TY]), hence we have obtained the required example.

Example 2.3. *A continuous open homomorphism from a Whyburn topological abelian group onto a non-weakly Whyburn topological abelian group.*

Let X be the abelian group of all real continuous functions on ω_1. If $x \in X$, the function x is eventually constant on ω_1; we denote by $k(x)$ this constant. The function $k : X \to \mathbb{R}$ is a group homomorphism.

We topologize X by considering the filter of neighbourhoods of $0 \in X$ defined by all sets U_γ for $\gamma < \omega_1$, where $U_\gamma := \{x \in X : x(\alpha) = 0 \text{ for all } \alpha \le \gamma\}$.

Since $U_\gamma + U_\gamma = -U_\gamma = U_\gamma$, this filter may be considered as the neighbourhood filter of a group topology on X.

Let Y be again the group of continuous real valued functions on ω_1. This time the filter of neighbourhoods of $0 \in Y$ will be defined by all possible sets $U_{\gamma,\varepsilon}$ for $\gamma < \omega_1$ and ε any positive real number, where $U_{\gamma,\varepsilon} := \{y \in Y : y(\alpha) = 0 \text{ for all } \alpha \le \gamma \text{ and } |k(y)| < \varepsilon\}$.

Since $-U_{\gamma,\varepsilon} = U_{\gamma,\varepsilon}$ and $U_{\gamma,\frac{\varepsilon}{2}} + U_{\gamma,\frac{\varepsilon}{2}} \subseteq U_{\gamma,\varepsilon}$ this filter may be considered as the neighbourhood filter of a group topology on Y.

Finally let us consider the map $f : X \times Y \to X \times \mathbb{R}$ defined by $f(\langle x,y\rangle) = \langle x,k(y)\rangle$.

We claim that the space $X \times Y$ is Whyburn, the space $X \times \mathbb{R}$ is not weakly Whyburn and the map f is an open homomorphism of $X \times Y$ onto $X \times \mathbb{R}$.

$X \times Y$ is Whyburn.

Let us note first that any countable set $E \subset X \times Y$ is closed, i.e. that $X \times Y$ is a weak P-space. In fact, assume $\langle a,b\rangle \notin E$; let us list all elements of E as $\{\langle u_n,v_n\rangle : n < \omega\}$. For any $n \in \omega$ there exists $\gamma(n) < \omega_1$ such that either $a(\gamma(n)) \ne u_n(\gamma(n))$ or $b(\gamma(n)) \ne v_n(\gamma(n))$. Pick $\gamma > \sup\{\gamma(n) : n \in \omega\}$. Then no element of E can belong to the open neighbourhood $\langle a,b\rangle + (U_\gamma \times U_{\gamma,1})$ of $\langle a,b\rangle$.

Assume now that $\langle 0,0\rangle \in \overline{E} \setminus E$. For any successor ordinal $\gamma = \eta + n$ (η limit, $n \in \omega$, $n \ne 0$) we can pick an element $\langle x_\gamma,y_\gamma\rangle \in E \cap (U_\gamma \times U_{\gamma,\frac{1}{n}})$. Let $F = \{\langle x_\gamma,y_\gamma\rangle : \gamma < \omega_1,\ \gamma \text{ successor ordinal}\}$. Then F is an almost closed set converging to $\langle 0,0\rangle$. In fact clearly $\langle 0,0\rangle \in \overline{F}$. Suppose $\langle a,b\rangle \in \overline{F} \setminus F$ and $\langle a,b\rangle \ne \langle 0,0\rangle$. Let $\xi \in \omega_1$ be such that either $a(\xi) \ne 0$ or $b(\xi) \ne 0$. Consider the neighbourhood Ω of $\langle a,b\rangle$ of the form $\Omega = \langle a,b\rangle + \big(U_\xi \times U_{\xi,1}\big)$. Then $\langle a,b\rangle \in \overline{(\Omega \cap F)}$. But $|\Omega \cap F|$ is countable because no element $\langle x_\gamma,y_\gamma\rangle$ with $\gamma > \xi$ can belong to Ω. This contradicts our initial remark. Therefore we must have $\langle a,b\rangle = \langle 0,0\rangle$. This shows that $X \times Y$ is Whyburn.

$X \times \mathbb{R}$ is not weakly Whyburn.

Let $\nu : \omega_1 \to \mathbb{R}$ be any injection such that $\nu(0) = 0$. For any $\gamma < \omega_1$ let us define $x_\gamma(\alpha) = 0$ for all $\alpha \leq \gamma$ and $x_\gamma(\alpha) = \nu(\gamma)$ for all $\alpha > \gamma$, and $E = \{\langle x_\gamma, \nu(\gamma)\rangle : 0 < \gamma < \omega_1\}$.

Let us observe that E is not closed. In fact the set $A := \{\nu(\gamma) : \ \gamma \in \omega_1\}$ has $\mathfrak{c}$ complete accumulation points in $\mathbb{R}$. Let p be one of these. Let $U = U_\eta \times]p-\varepsilon, p+\varepsilon[$ be a neighbourhood of $\langle 0, p\rangle$; there are ω_1 elements of A in $]p-\varepsilon, p+\varepsilon[$, hence there exists a $\gamma > \eta$ with $|\nu(\gamma) - p| < \varepsilon$. Then $\langle x_\gamma, \nu(\gamma)\rangle \in U$. This shows that $\langle 0, p\rangle \in \overline{E} \setminus E$.

We can also note that $\overline{E} \subseteq E \cup \{\langle 0, t\rangle : t \in \mathbb{R}\}$. In fact suppose that $x \neq 0$ and $\langle x, t\rangle \notin E$. If $x \neq x_\gamma$ for all $\gamma < \omega_1$, let $\xi \in \omega_1$ be such that $x(\xi) \neq 0$, and for all $\gamma < \xi$ pick an ordinal $\alpha(\gamma)$ such that $x(\alpha(\gamma)) \neq x_\gamma(\alpha(\gamma))$. Let $\eta > \sup(\{\alpha(\gamma) : \gamma \leq \xi\} \cup \{\xi\})$. Then no elements of E can belong to the neighbourhood Ω of $\langle x, t\rangle$ defined by $\Omega = \langle x, t\rangle + (U_\eta \times]t-1, t+1[\,)$. If $x = x_\xi$ for some $\xi \in \omega_1$, then $t \neq \nu(\xi)$; let $0 < \varepsilon < |t - \nu(\xi)|$, then no elements of E can belong to the neighbourhood Ω of $\langle x, t\rangle$ defined by $\Omega = \langle x, t\rangle + \big(U_{\xi+1} \times]-\varepsilon, \varepsilon[\,\big)$.

We claim that no subset $F \subset E$ can have a unique accumulation point outside E. In fact suppose $\overline{F} \setminus E \neq \emptyset$, then $|F| \geq \omega_1$. If not let $\langle 0, t\rangle \in \overline{F} \setminus E$ (since $\overline{E} \subseteq E \cup \{\langle 0, t\rangle : t \in \mathbb{R}\}$ the first coordinate must be 0) and $\eta > \sup\{\gamma : \langle x_\gamma, \nu(\gamma)\rangle \in F\}$. Consider the neighbourhood Ω of $\langle 0, t\rangle$ defined by $U_\eta \times \mathbb{R}$, then no elements of F can belong to Ω.

Since $|F| \geq \omega_1$ there are, in $\mathbb{R}$, $\mathfrak{c}$ complete accumulation points of the set $\{\nu(\gamma) : \langle x_\gamma, \nu(\gamma)\rangle \in F\}$. Let p be one of these, then $\langle 0, p\rangle \in \overline{F} \setminus E$.

The function $f : X \times Y \to X \times \mathbb{R}$ is continuous: let $\Omega = U_\gamma \times]-\varepsilon, \varepsilon[$ be an open neighbourhood of $\langle 0, 0\rangle$ in $X \times \mathbb{R}$. Then $U_\gamma \times U_{\gamma,\varepsilon} \subset f^{-1}(\Omega)$.

The function $f : X \times Y \to X \times \mathbb{R}$ is open: let $\Omega = U_\gamma \times U_{\gamma,\varepsilon}$ be an open neighbourhood of $\langle 0, 0\rangle$ in $X \times Y$. Then $U_\gamma \times]-\varepsilon, \varepsilon[\, \subset f(\Omega)$. In fact, if $\langle x, t\rangle \in U_\gamma \times]-\varepsilon, \varepsilon[$, define $y \in Y$ by $y(\alpha) = 0$ if $\alpha \leq \gamma$ and $y(\alpha) = t$ if $\alpha > \gamma$; then $\langle x, t\rangle = f(\langle x, y\rangle)$, and $\langle x, y\rangle \in \Omega$.

3. Final remarks

We have seen that open maps often do not preserve Whyburn properties, even in the class of topological groups. However these properties are certainly preserved in the class of compact spaces since closed maps do preserve Whyburn properties. We might ask if this happens for some other classes of spaces. For example open maps do preserve both Whyburn and weak Whyburn properties in the class of locally compact spaces, as it can be easily checked. We might also ask if there is a characterization of the functions that do preserve Whyburn properties. A study in this direction has been made for pseudoradiality and similar properties by G. Dimov, R. Isler and G. Tironi in [DIT]. I like to cite this paper also because in this work the notions of Whyburn and weakly Whyburn spaces have been introduced (with the name of gF-spaces and gs-spaces; Definitions 2.33) some years before [PT] and [S], the articles that recently raised all the interest on these spaces, but no mention of it has ever been made in the literature on this topic.

References

[DIT] Dimov G.D., Isler R., Tironi G., *On functions preserving almost radiality and their relations to radial and pseudoradial spaces*, Comment. Math. Univ. Carolinae **28.4** (1987), 357–360.

[O] Obersnel F., *Some notes on weakly Whyburn spaces*, Topology Appl., to appear.

[PTTW] Pelant J., Tkachenko M.G., Tkachuk V.V., Wilson R.G., *Pseudocompact Whyburn spaces need not be Fréchet*, to appear on PAMS.

[PT] Pultr A., Tozzi A., *Equationally closed subframes and representation of quotient spaces*, Cahiers de la Topologie et Géométrie Différentielle Categoriques **34** (1993), 167–183.

[S] Simon P., *On accumulation points*, Cahiers de la Topologie et Géométrie Différentielle Categoriques **35** (1994), 321–327.

[TY] Tkachuk V.V., Yashenko I.V., *Almost closed sets and topologies they determine*, Comment. Math. Univ. Carolinae **42.2** (2001), 395–405.

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