The fractional integral between weighted Orlicz and BMO_ϕ spaces on spaces of homogeneous type

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Abstract. In this work we give sufficient and necessary conditions for the boundedness of the fractional integral operator acting between weighted Orlicz spaces and suitable BMO_ϕ spaces, in the general setting of spaces of homogeneous type. This result generalizes those contained in [P1] and [P2] about the boundedness of the same operator acting between weighted L^p and Lipschitz integral spaces on $\mathbb{R}^n$. We also give some properties of the classes of pairs of weights appearing in connection with this boundedness.

Keywords: weights, Orlicz spaces, BMO, fractional integral

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1. Introduction and preliminaries

Let I_α, $0 < \alpha < n$, be the fractional integral operator on $\mathbb{R}^n$, that is

$$I_\alpha f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}\, dy, \quad x \in \mathbb{R}^n.$$

There are well known properties related to the boundedness of I_α acting on the Lebesgue spaces L^p for $1 < p \leq n/\alpha$, shortly: $I_\alpha : L^p \to L^q$, $1/q = 1/p - \alpha/n$ when $p < n/\alpha$ and $I_\alpha : L^{n/\alpha} \to BMO$ (in the last case an adequate extension is required by reasons of convergence). Moreover there are versions with weights of these results (see, for instance [MW] and [S]) and extensions of the weighted results for I_γ acting between Orlicz spaces ([KK]). Less known are results concerning the behaviour of I_α acting on L^p for $p > n/\alpha$. In this line we have [HSV1], where Harboure, the second author and Viviani prove one-weight boundedness inequalities for an appropriate extension of I_α between weak and strong L^p spaces and Lipschitz type integral spaces, characterizing the classes of weights involved. The same authors, in [HSV2], extend one of their results by considering Orlicz and BMO_ϕ spaces and, in addition, prove that the classes of weights considered coincide with the A_p Muckenhoupt's classes. In two later works, [P1] and [P2],

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the first author shows a two-weight version of some results of [HSV1] and, in particular, characterizes the pairs of weights (w, v) for which an extension of I_α acting between weighted versions of L^p and Lipschitz type integral spaces is bounded.

The purpose of this work is to try to give a unified view of the above results in the general setting of spaces of homogeneous type. In particular we characterize the classes of pairs of weights in connection with the boundedness of the fractional integral operator of order γ, with $0 < \gamma < 1$, between Orlicz and ϕ-Lipschitz integral spaces, but now working in the context of spaces of homogeneous type. Also, we study the properties of the classes of weights obtained and we give examples of them. We wish to remark that the unweighted case of these results in spaces of homogeneous type was studied in [GV] but for a slightly different version of the operator I_γ.

On the other hand, in [GGKK] (see Chapter 3, Section 3.6), the particular case $L^{n/\gamma} - BMO$ for two weights is studied for another version of I_γ (closer to ours than that one used in [GV]).

The paper is organized in the following way: the last part of this section contains basic definitions about spaces of homogeneous type and Orlicz spaces, Section 2 is dedicated to prove some properties of the classes of weights that we are going to use; in Section 3 we present our main results about I_γ, Section 4 contains the proofs of the main theorems; finally some examples of weights can be founded in Section 5.

Let X be a set. A function $d : X \times X \to \mathbb{R}_0^+$ is called a quasi-distance on X if the following conditions are satisfied:

(i) for every x and y in X, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,
(ii) for every x and y in X, $d(x, y) = d(y, x)$,
(iii) there exists a constant K such that $d(x, y) \leq K(d(x, z) + d(z, y))$ for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X generated by the d-balls $B(x, r) = \{y : d(x, y) < r\}$, with $x \in X$ and $r > 0$. We assume that μ satisfies the doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x, 2Kr)) \leq A\mu(B(x, r)) < \infty \tag{1.1}$$

holds for every ball $B \subset X$. The triple (X, d, μ), with d and μ as above, is called a space of homogeneous type.

We consider the function $\rho : X \times X \to \mathbb{R}_0^+$ defined by

$$\rho(x, y) = \begin{cases} \left(\mu(B(x, d(x, y))) + \mu(B(y, d(x, y)))\right)/2 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

It is easy to check that ρ is a quasi-distance on X. If there exists $\alpha \geq 1$ such that $\rho^{1/\alpha}$ is a distance, then we set $\eta = \rho^{1/\alpha}$. If that does not happen, we take for η a distance such that η^α is equivalent to ρ for some $\alpha \geq 1$ (from [MS], this choice it is always possible), that is, there exist two constants C_1 and C_2, such that

$$C_1\eta(x,y)^\alpha \leq \rho(x,y) \leq C_2\eta(x,y)^\alpha$$

holds for every x and y in X. With this choice of η and α, a version of the usual fractional integral operator of order γ, for $0 < \gamma < 1$, can be defined in (X, d, μ) as

$$I_\gamma f(x) = \int_X Q_\gamma(x,y) f(y) d\mu(y), \tag{1.2}$$

with

$$Q_\gamma(x,y) = \begin{cases} \eta(x,y)^{\alpha(\gamma-1)} & \text{if } x \neq y, \\ \mu(\{x\})^{\gamma-1} & \text{if } x = y. \end{cases}$$

Now we summarize a few facts about Orlicz spaces. We are going to deal with continuous functions ϕ defined and increasing on $[0,\infty)$ such that $\lim_{t\to 0^+} \phi(t) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. We also assume that the following conditions are satisfied:

(1.3) ϕ is of lower type p, $p > 1$, that is there exists a constant C such that

$$\phi(st) \leq Cs^p\phi(t)$$

for every $s \in [0,1]$ and every $t \geq 0$;

(1.4) ϕ is of upper type q, that is there exists a constant C such that

$$\phi(st) \leq Cs^q\phi(t)$$

for every $s \geq 1$ and every $t \geq 0$.

Given ϕ, the complementary function (with respect to ϕ) is defined by

$$\tilde{\phi}(s) = \sup_{t>0}(st - \phi(t))$$

for $s > 0$. It is well known (see for example [RR]) that $\tilde{\phi}$ satisfies similar properties as ϕ. In particular, if ϕ is of lower type p then $\tilde{\phi}$ is of upper type p' and if ϕ is of upper type q then $\tilde{\phi}$ is of lower type q', where $r' = r/(r-1)$. Moreover, it can be proved that there exist two constants C_1 and C_2 such that

$$C_1 t \leq \tilde{\phi}^{-1}(t)\phi^{-1}(t) \leq C_2 t \tag{1.5}$$

for every $t > 0$.

On the other hand, if ϕ is of lower type p then it is easy to check that $\tilde{\phi}^{-1}$ is of lower type $1/p'$ and if ϕ is of upper type q then $\tilde{\phi}^{-1}$ is of upper type $1/q'$.

Let ϕ be a function as above. For such ϕ, the Orlicz space L_ϕ is defined as the class of measurable functions $f : X \to \mathbb{R}$ such that

$$\int_X \phi(|f(x)|)\, d\mu(x) < \infty.$$

In this class we consider the following analog to the Luxemburg norm

$$\|f\|_\phi = \inf\left\{\lambda > 0 : \int_X \phi(|f(x)/\lambda|) \leq 1\right\}.$$

Note that $\|\cdot\|_\phi$ is not a norm, but in view of the properties of ϕ, it can be shown that it is equivalent to a norm. Moreover, the Hölder type inequality

$$\left|\int f(x)g(x)\, d\mu(x)\right| \leq C\|f\|_\phi \|g\|_{\tilde{\phi}} \tag{1.6}$$

holds for every $f \in L_\phi$ and every $g \in L_{\tilde{\phi}}$ where $L_{\tilde{\phi}}$ is the dual space of L_ϕ.

We define a weighted version of L_ϕ in the following way: given a weight w (that is, a non negative and locally integrable function defined in X), we say that $f \in L_{\phi,w}$ if $f/w \in L_\phi$.

2. Properties of the weights

Now, we give the definition of the classes of pairs of weights which we are going to use in connection with the boundedness of the fractional integral operator I_γ.

2.1 Definition. Given a function ϕ, $\beta \in \mathbb{R}$ and $0 < \gamma < 1$, we say that (w, v) belongs to $C_\gamma(\phi, \beta)$ if there exists a constant C such that the inequality

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+\gamma-1/\alpha-1}} \phi^{-1}(1/\mu(B)) \left\|\frac{v}{(\mu(B(x_B, d(x_B, .))) + \mu(B))^{1-\gamma+1/\alpha}}\right\|_\phi \leq C \tag{2.2}$$

holds for every ball $B \subset X$, where x_B denotes the centre of B.

2.3 Remark. It is clear that the above condition is verified if and only if the following pair of inequalities hold simultaneously for every $B \subset X$:

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^\beta} \phi^{-1}(1/\mu(B))\, ||v\chi_B||_\phi \leq C, \tag{2.4}$$

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+\gamma-1/\alpha-1}} \phi^{-1}(1/\mu(B)) \left\|\chi_{X-B} \frac{v}{\mu(B(x_B, d(x_B, \cdot)))^{1-\gamma+1/\alpha}}\right\|_\phi \leq C. \tag{2.5}$$

2.6 Remark. Replacing $\|(1/w)\chi_B\|_\infty$ by $\mu(B)/w(B)$ in (2.1) we obtain another class of weights involved in a different version of the boundedness of I_γ (see Section 3 below). We shall denote this class by $\tilde{C}_\gamma(\phi,\beta)$.

2.7 Lemma. *Let ϕ be an increasing function with lower type p, $\beta \in \mathbb{R}$ and $0<\gamma<1$. If $\mu(X)=\infty$ and $\beta < 1/p' - \gamma + 1/\alpha$ then $C_\gamma(\phi,\beta)$ is reduced to* (2.4). *If $\mu(X)<\infty$ then condition $C_\gamma(\phi,\beta)$ is reduced to* (2.4) *for every $\beta\in\mathbb{R}$.*

2.8 Remark. In the case $\mu(X)=\infty$, the result for $\beta = 1/p' - \gamma + 1/\alpha$ is not true. This fact was proved in [P1] for the case $\phi(t)=t^p$ in Euclidean spaces with the Lebesgue measure.

On the other hand, if $\mu(X)<\infty$, the proof is trivial and we omit it.

Proof of Lemma 2.7: Let $B=B(x_B,R)$ and $\tilde{B}=B(x_B,2KR)$. Note that $\mu(B(x_B,d(x_B,y)))\ge C_o\mu(B)$, with C_o independent of B, for every $y\in X-\tilde{B}$. We consider the sets $\Omega_j=\{y\in X : \mu(\bar{B}(x_B,d(x_B,y)))<2^jR_o\}$, $j=0,1,\dots$, with $R_o=C_o\mu(B)$ and $\bar{B}(x,r)=\{y : d(x,y)\le r\}$, then using Lemma 2.5 of [MT], we get $\Omega_{j+1}\subset \bar{B}(x_B,R_j)$ and

$$\mu(\bar{B}(x_B,R_j)) \le C2^{j+1}R_o, \tag{2.9}$$

where $R_j=\sup_{y\in X}\{d(x_B,y) : \mu(\bar{B}(x_B,d(x_B,y)))\le 2^{j+1}R_o\}$ and C only depends on the constants of the space. Since $X-\tilde{B}\subset X-\Omega_o$, we get

$$\begin{aligned}
&\int_{X-\tilde{B}} \phi\left(\frac{v(y)\phi^{-1}(1/\mu(B))}{\mu(B)^{\beta+\gamma-1/\alpha-1}\,\mu(B(x_B,d(x_B,y)))^{1-\gamma+1/\alpha}\inf_B w}\right)d\mu(y)\\
&\quad\le \int_{X-\Omega_o} \phi\left(\frac{v(y)\phi^{-1}(1/\mu(B))}{\mu(B)^{\beta+\gamma-1/\alpha-1}\,\mu(\bar{B}(x_B,d(x_B,y)))^{1-\gamma+1/\alpha}\inf_B w}\right)d\mu(y)\\
&\quad\le \sum_{j=0}^{\infty}\int_{\Omega_{j+1}-\Omega_j}\phi\left(C\frac{v(y)\phi^{-1}(1/\mu(B))}{\mu(B)^{\beta}\,2^{j(1-\gamma+1/\alpha)}\inf_B w}\right)d\mu(y),
\end{aligned}$$

where we have used that $\mu(\bar{B}(x_B,d(x_B,y)))\simeq 2^j\mu(B)$ if $y\in\Omega_{j+1}-\Omega_j$. Now, if $B_j^*=B(x_B,2R_j)$, from (2.9) we have that $\mu(B_j^*)\le C2^j\mu(B)$. Then, since ϕ^{-1} is of upper type $1/p$, we have that $\phi^{-1}(1/\mu(B_j^*))\ge \dfrac{C}{2^{j/p}}\phi^{-1}(1/\mu(B))$. So, from the fact that $\Omega_{j+1}-\Omega_j\subset B_j^*$ and $B\subset B_j^*$, we obtain that the last expression is bounded by

$$\begin{aligned}
&C\sum_{j=0}^{\infty}\int_{B_j^*}\phi\left(\frac{v(y)}{\mu(B_j^*)^{\beta}\,2^{j(1-\gamma+1/\alpha-\beta)}}\frac{\phi^{-1}(1/\mu(B))}{\inf_{B_j^*} w}\right)d\mu(y)\\
&\qquad\le C\sum_{j=0}^{\infty}\int_{B_j^*}\phi\left(C\frac{v(y)\phi^{-1}(1/\mu(B_j^*))}{\mu(B_j^*)^{\beta}2^{j(1/p'-\gamma+1/\alpha-\beta)}\inf_{B_j^*} w}\right)d\mu(y).
\end{aligned}$$

Finally, since $\beta < 1/p' - \gamma + 1/\alpha$ and ϕ is of lower type p, the last term is bounded by

$$C\sum_{j=0}^{\infty}\frac{1}{2^{j(1/p'-\gamma+1/\alpha-\beta)p}}\int_{B_j^*}\phi\left(\frac{v(y)\phi^{-1}(1/\mu(B_j^*))}{\mu(B_j^*)^{\beta}\inf_{B_j^*}w}\right)d\mu(y) \tag{2.10}$$

which is finite since (w,v) satisfies (2.4). So it is obvious that $(w,v)\in C_\gamma(\phi,\beta)$. From this fact and Remark 2.3 we conclude the proof. □

In connection with the previous lemma, it is important to note that the case $\beta > 1/p' - \gamma + 1/\alpha$ in many examples, lead to trivial weights. In fact, we have the following result.

2.11 Theorem. *Let $0<\gamma<1$ and let (X,d,μ) be a space of homogeneous type containing a sequence $\{B_i\}$ of balls in X with $\mu(B_i)\to 0$ and $B_{i+1}\subset B_i$. Then, if $\beta > 1/q' - \gamma + 1/\alpha$, condition $C_\gamma(\phi,\beta)$ is satisfied if and only if $v=0$ a.e. in X or if there exists i_o such that $v=0$ a.e. in B_i, $i\geq i_o$.*

Proof: Let $\{B_i\}$ be as in the assumption of the theorem. From condition (2.5) for such balls B_i we have

$$\int_{X-B_i}\phi\left(C\frac{v(y)\phi^{-1}(1/\mu(B_i))}{\mu(B_i)^{\beta+\gamma-1/\alpha-1}\,\mu(B(x_B,d(x_B,y)))^{1-\gamma+1/\alpha}\inf_{B_i}w}\right)d\mu(y)\leq 1. \tag{2.12}$$

From the fact that $\mu(B_i)\to 0$, there exists i_o such that $\mu(B_i)<1$ for $i>i_o$. Since ϕ^{-1} is of lower type $1/q$ we have that $\phi^{-1}(1/\mu(B_i))\geq C1/\mu(B_i)^{1/q}$ and then, using that ϕ has lower type p we get

$$\frac{1}{\mu(B_i)^{(\beta+\gamma-1/\alpha-1/q')p}}\int_{X-B_i}\phi\left(\frac{v(y)}{\mu(B(x_B,d(x_B,y)))^{1-\gamma+1/\alpha}\inf_{B_i}w}\right)d\mu(y)\leq 1. \tag{2.13}$$

Suppose that there exists $i_l > i_o$ such that $\inf_{B_{i_l}} w = l_o > 0$, otherwise, if $\inf_{B_i} w = 0$ for all $i > i_o$, from condition $C_\gamma(\phi,\beta)$ (see Remark 2.3), we obtain $v=0$ *a.e.* in B_i. Then, for every $i > i_l$, we have that

$$\int_{X-B_i}\phi\left(\frac{v(y)}{\mu(B(x_B,d(x_B,y)))^{1-\gamma+1/\alpha}}\right)d\mu(y)\leq C\mu(B_i)^{(\beta+\gamma-1/\alpha-1/q')p}. \tag{2.14}$$

The right hand side of (2.14) tends to 0 when $i\to\infty$, and then we get that $v=0$ *a.e.* in X. □

3. Statement of main results

First, we shall introduce the $\phi-$Lipschitz integral spaces that we are going to consider. The functions ϕ we are going to deal with, have the properties mentioned at the end of Section 1.

3.1 Definition. Let w be a weight, $0 < \gamma < 1$, $\beta \in \mathbb{R}$ and ϕ a function. We say that a locally integrable function f belongs to $\mathcal{L}_w^\phi(\beta, \gamma)$ if there exists a constant C such that

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+\gamma+1}\phi^{-1}(1/\mu(B))} \int_B |f(x) - m_B f| \, d\mu(x) \le C \tag{3.2}$$

for every ball $B \subset X$.

3.3 Definition. Let w be a weight, $0 < \gamma < 1$, $\beta \in \mathbb{R}$ and ϕ a function. We say that a locally integrable function f belongs to $\tilde{\mathcal{L}}_w^\phi(\beta, \gamma)$ if there exists a constant C such that

$$\frac{1}{w(B)\mu(B)^{\beta+\gamma}\phi^{-1}(1/\mu(B))} \int_B |f(x) - m_B f| \, d\mu(x) \le C \tag{3.4}$$

for every ball $B \subset X$.

3.5 Remark. For sake of simplicity we use two parameters β and γ in the definition of the spaces $\mathcal{L}_w^\phi(\beta, \gamma)$ and $\tilde{\mathcal{L}}_w^\phi(\beta, \gamma)$, that will be useful in the estimates.

It is easy to check that $\mathcal{L}_w^\phi(\beta, \gamma) \subseteq \tilde{\mathcal{L}}_w^\phi(\beta, \gamma)$. On the other hand, if w belongs to the A_1 Muckenhoupt's class with respect to X (that is $w(B)/\mu(B) \le C \inf_B w$ for every ball B), it is obvious that both spaces coincide. In the Euclidean case, if $\phi(t) = t^p$ and $\beta = (\delta - \gamma)/n + 1/p$ then the spaces $\mathcal{L}_w^\phi(\beta, \gamma)$ and $\tilde{\mathcal{L}}_w^\phi(\beta, \gamma)$ agree with the Lipschitz integral spaces $\mathcal{L}_w(\delta)$ and $\mathbb{L}_{\boldsymbol{w}}(\boldsymbol{\delta})$ defined in [P1] and [P2] respectively. The case with general ϕ and $w \in A_1$ was considered in [HSV2] for the one weight results, as we said before.

Now we are able to state the results about the boundedness of the fractional integral operator.

3.6 Theorem. *Let ϕ be an increasing function, $0 < \gamma < 1$, $\beta \in \mathbb{R}$ and $(w, v) \in C_\gamma(\tilde{\phi}, \beta)$. Then, the operator I_γ can be extended to a bounded linear operator $\tilde{I}_\gamma$ from $L_{\phi,v}$ into $\mathcal{L}_w^\phi(\beta, \gamma)$, i.e. there exists a constant C such that*

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+\gamma+1}\phi^{-1}(1/\mu(B))} \int_B \left| \tilde{I}_\gamma f(x) - m_B(\tilde{I}_\gamma f) \right| d\mu(x) \le C \|f/v\|_\phi \tag{3.7}$$

for every $B(x_B, R)$.

3.8 Remark. Even though we will restrict our attention in this paper to the boundedness of the fractional integral operator involving the spaces $\mathcal{L}_w^\phi(\beta,\gamma)$ and the corresponding classes $C_\gamma(\tilde{\phi},\beta)$, similar results can be obtained for the spaces $\tilde{\mathcal{L}}_w^\phi(\beta,\gamma)$ and the classes $\tilde{C}_\gamma(\tilde{\phi},\beta)$. The proof follows similar lines as Theorem 3.6 and we omit it.

For certain functions ϕ, a likewise reciprocal result of Theorem 3.6 holds if we restrict the balls $B = B(x_B, R)$ to those with R smaller than a fraction of $\mu(X)$. This is established in the following theorem.

3.9 Theorem. *Let ϕ be an increasing function with lower type p and upper type q, $0<\gamma<1$ and $\beta < 1/q - \gamma + 1/\alpha$. If (3.7) holds for every B then there exists a constant θ, $0<\theta<1$, only depending on the constants of the space X, such that $C_\gamma(\tilde{\phi},\beta)$ holds restricted to the balls with radius smaller than $\theta\mu(X)$ (that is (2.2) for $\tilde{\phi}$ and these balls).*

3.10 Corollary. *Let ϕ be an increasing function with upper type q, $0<\gamma<1$ and $\beta < 1/q - \gamma + 1/\alpha$. If $\mu(X)=\infty$ then (3.7) implies that $(w,v)\in C_\gamma(\tilde{\phi},\beta)$.*

3.11 Remark. If we suppose in the above theorem that $\|(1/w)\chi_X\|_\infty < \infty$ (that is $\inf_X w > 0$) and $v \in L_{\tilde{\phi}}(X)$ it can be proved that, when $\mu(X)<\infty$, condition $C_\gamma(\tilde{\phi},\beta)$ holds for every ball. Related to these conditions, note that if $\inf_B w = 0$ for any ball B then, from condition $C_\gamma(\tilde{\phi},\beta)$ we obtain that $v=0$ a.e. in B.

4. Proofs

Proof of Theorem 3.6: Let us first show that I_γ defined by (1.2) can be extended to an operator $\tilde{I}_\gamma$ such that $\tilde{I}_\gamma f(x)$ is finite in almost every $x\in X$ when $f\in L_{\phi,v}$. Take $x_o \in X$. With this point we consider the following definition for $\tilde{I}_\gamma$

$$\tilde{I}_\gamma f(x) = \int_X (Q_\gamma(x,y) - Q_\gamma(x_o,y)(1-\chi_{B(x_o,1)}(y)))\, f(y)\, d\mu(y). \tag{4.1}$$

For a ball $B = B(x_B,R)$, (4.1) can be formally written as $a_B + I(x)$, where

$$a_B = \int_X \Big((1-\chi_{\tilde{B}}(y))Q_\gamma(x_B,y) - (1-\chi_{B(x_o,1)}(y))Q_\gamma(x_o,y)\Big)\, f(y)\, d\mu(y), \tag{4.2}$$

$$I(x) = \int_X \big(Q_\gamma(x,y) - (1-\chi_{\tilde{B}}(y))Q_\gamma(x_B,y)\big)\ f(y)\, d\mu(y) \tag{4.3}$$

with $\tilde{B} = B(x_B, 2KR)$.

Let us first estimate a_B. With $B^* = B(x_o, K^2 d(x_o, x_B) + 2K^2 R + 1)$ we have $a_B = I + II$ where

$$I = \int_{\tilde{B^*}} \Big((1 - \chi_{\tilde{B}}(y))Q_\gamma(x_B, y) - (1 - \chi_{B(x_o,1)}(y))Q_\gamma(x_o, y)\Big)\, f(y)\, d\mu(y),$$

$$II = \int_{X - \tilde{B^*}} \Big((1 - \chi_{\tilde{B}}(y))Q_\gamma(x_B, y) - (1 - \chi_{B(x_o,1)}(y))Q_\gamma(x_o, y)\Big)\, f(y)\, d\mu(y).$$

If $y \in X - B^*$ then the expression enclosed by external parentheses in II is equal to

$$\begin{aligned} |Q_\gamma(x_B, y) - Q_\gamma(x_o, y)| &= |\eta(x_B, y)^{\alpha(\gamma-1)} - \eta(x_o, y)^{\alpha(\gamma-1)}| \\ &= \frac{\left|\eta(x_o, y)^{\alpha(1-\gamma)} - \eta(x_B, y)^{\alpha(1-\gamma)}\right|}{\eta(x_B, y)^{\alpha(1-\gamma)}\eta(x_o, y)^{\alpha(1-\gamma)}}. \end{aligned} \tag{4.4}$$

Since $\eta \simeq \rho^{1/\alpha}$, it is easy to see that $\eta(x_B, y) \simeq \mu(B(x_B, d(x_B, y)))^{1/\alpha}$. In fact, we have that $\mu(B(x_B, d(x_B, y))) \simeq \mu(B(y, d(x_B, y)))$ and, from the definition of ρ, the conclusion follows. On the other hand if $y \in X - B^*$ then $d(x_B, x_o) \leq K^2 d(x_o, x_B) + 2K^2 R + 1 \leq d(x_o, y)$, so $d(x_B, y) \leq K(d(x_B, x_o) + d(x_o, y)) \leq 2Kd(x_o, y)$. Then we have that $B(x_B, d(x_B, y)) \subset B(y, 4K^3 d(x_o, y))$. From the doubling property of μ we obtain that $\mu(B(x_B, d(x_B, y))) \leq C\mu(B(y, d(x_o, y)))$, and from the fact that

$$\eta(x_o, y)^\alpha \simeq \rho(x_o, y) \simeq \mu(B(y, d(x_o, y))),$$

we get

$$\eta(x_o, y)^\alpha \geq C\mu(B(x_B, d(x_B, y))).$$

Collecting these estimates and using the mean value theorem we have that the last expression in (4.4) is bounded by

$$\begin{aligned} C\frac{|\eta(x_o, y) - \eta(x_B, y)|}{\mu(B(x_B, d(x_B, y)))^{1-\gamma+1/\alpha}} &\leq C\frac{\eta(x_o, x_B)}{\mu(B(x_B, d(x_B, y)))^{1-\gamma+1/\alpha}} \\ &\leq C\frac{\mu(B^*)^{1/\alpha}}{\mu(B(x_B, d(x_B, y)))^{1-\gamma+1/\alpha}}. \end{aligned}$$

Then we have

$$II \leq C\mu(B^*)^{1/\alpha} \int_{X - \tilde{B^*}} \frac{|f(y)|}{\mu(B(x_B, d(x_B, y)))^{1-\gamma+1/\alpha}}\, d\mu(y).$$

Note that, if $y \in X - \tilde{B^*}$ then $\mu(B(x_B, d(x_B, y))) \simeq \mu(B(x_o, d(x_o, y)))$, so, the last inequality allows us to write

$$\begin{aligned}
II &\leq C\mu(B^*)^{1/\alpha} \int_{X-\tilde{B^*}} \frac{|f(y)|}{\mu(B(x_o, d(x_o, y)))^{1-\gamma+1/\alpha}}\, d\mu(y) \\
&\leq C\mu(B^*)^{1/\alpha} \, \|f/v\|_\phi \left\| \frac{v\chi_{X-\tilde{B^*}}}{\mu(B(x_o, d(x_o, .)))^{1-\gamma+1/\alpha}} \right\|_{\tilde{\phi}} \\
&\leq C\mu(B^*)^{\beta+\gamma} \phi^{-1}(1/\mu(B^*)) \inf_{B^*} w \, \|f/v\|_\phi < \infty,
\end{aligned}$$

where we have used the fact that $(w, v) \in C_\gamma(\tilde{\phi}, \beta)$.

To estimate I we first note that if $y \in \tilde{B^*}$ then

$$\begin{aligned}
&\Big((1 - \chi_{\tilde{B}}(y))Q_\gamma(x_B, y) - (1 - \chi_{B(x_o,1)}(y))Q_\gamma(x_o, y)\Big) \\
&\qquad\qquad\qquad \leq C\left(\mu(B)^{\gamma-1} + \mu(B(x_o, 1))^{\gamma-1}\right),
\end{aligned}$$

and thus we have

$$\begin{aligned}
I &\leq C\left(\mu(B)^{\gamma-1} + \mu(B(x_o, 1))^{\gamma-1}\right) \int_{B^*} |f(y)|\, d\mu(y) \\
&\leq C\left(\mu(B)^{\gamma-1} + \mu(B(x_o, 1))^{\gamma-1}\right) \|f/v\|_\phi \|v\chi_{\tilde{B^*}}\|_{\tilde{\phi}} \\
&\leq C\left(\mu(B)^{\gamma-1} + \mu(B(x_o, 1))^{\gamma-1}\right) \mu(\tilde{B^*})^{1+\beta} \phi^{-1}(1/\mu(B^*)) \inf_{\tilde{B^*}} w\|f/v\|_\phi < \infty.
\end{aligned}$$

Combining this inequality with the result for II we get that a_B is finite.

To see that $I(x)$ is finite, we first note that

$$\begin{aligned}
I(x) &= \int_{\tilde{B}} Q_\gamma(x, y)\, f(y)\, d\mu(y) + \int_{X-\tilde{B}} \left(Q_\gamma(x, y) - Q_\gamma(x_B, y)\right)\, f(y)\, d\mu(y) \\
&= I_1(x) + I_2(x).
\end{aligned}$$

From the definition of Q_γ it is clear that $Q_\gamma(x, y) \simeq \mu(B(y, d(x, y)))^{\gamma-1}$. On the other hand, if $y \in \tilde{B}$ and $x \in B$, then $\mu(B(y, d(x, y))) \leq C_o\mu(B)$. Now, let $y \in \tilde{B}$. Let us consider the same sets taken for the proof of Lemma 2.7, that is $\Omega_j = \{x \in B : \mu(B(y, d(x, y))) \leq 2^{-j}C_o\mu(B)\}$, for $j = 0, 1, \ldots$. So we have that $\Omega_j \subset \bar{B}(y, R_j)$ and $\mu(\bar{B}(y, R_j)) \leq C2^{-j}C_o\mu(B)$, with $R_j = \sup\{d(x, y) : \mu(B(y, d(x, y))) \leq 2^{-j}C_o\mu(B)\}$ where the sup is taken over all $x \in X$. Then we get

$$\begin{aligned}\int_B Q_\gamma(x,y)\,d\mu(x) &\le \sum_{j=0}^{\infty}\int_{\Omega_j-\Omega_{j+1}} Q_\gamma(x,y)\,d\mu(x)\\ &\le C\sum_{j=0}^{\infty}(2^{-j}\mu(B))^{\gamma-1}\mu(\Omega_j)\\ &\le C\sum_{j=0}^{\infty}2^{-j(\gamma-1)}\mu(B)^{\gamma-1}2^{-j}\mu(B)\\ &\le C\mu(B)^\gamma\sum_{j=0}^{\infty}2^{-j\gamma}\le C\mu(B)^\gamma.\end{aligned}$$

Using these facts we obtain

$$\begin{aligned}\int_B |I_1(x)|\,d\mu(x) &\le C\int_{\tilde B}|f(y)|\int_B Q_\gamma(x,y)\,d\mu(x)\,d\mu(y)\\ &\le C\mu(B)^\gamma\int_{\tilde B}|f(y)|\,d\mu(y).\end{aligned}\tag{4.5}$$

Since $(w,v)\in C_\gamma(\tilde\phi,\beta)$ the last expression is bounded by

$$C\mu(B)^{\gamma+\beta+1}\ \phi^{-1}(1/\mu(B))\inf_B w\|f/v\|_\phi.$$

On the other hand, if $x\in B$ and $y\in X-\tilde B$, proceeding as in (4.4) we have that

$$|Q_\gamma(x,y)-Q_\gamma(x_B,y)|\le C\frac{\mu(B)^{1/\alpha}}{\mu(B(x_B,d(x_B,y)))^{1-\gamma+1/\alpha}}\,.$$

Since $(w,v)\in C_\gamma(\tilde\phi,\beta)$ we get

$$\begin{aligned}|I_2(x)| &\le C\mu(B)^{1/\alpha}\int_{X-\tilde B}\frac{|f(y)|}{\mu(B(x_B,d(x_B,y)))^{1-\gamma+1/\alpha}}\,d\mu(y)\\ &\le C\mu(B)^{1/\alpha}\|f/v\|_\phi\left\|\frac{v\chi_{X-\tilde B}}{\mu(B(x_B,d(x_B,.)))^{1-\gamma+1/\alpha}}\right\|_{\tilde\phi}\\ &\le C\mu(B)^{\beta+\gamma}\phi^{-1}(1/\mu(B))\inf_B w\|f/v\|_\phi<\infty.\end{aligned}\tag{4.6}$$

So, from the estimates for $I(x)$ and a_B, we get $|\tilde I_\gamma f(x)|<\infty$ *a.e.* $x\in X$.

Now we are going to prove the boundedness result for $\tilde{I}_\gamma$. From the decomposition of $\tilde{I}_\gamma f(x)$ as $I(x) + a_B$ and the estimates (4.5) and (4.6), we can write

$$\begin{aligned}\int_B \left|\tilde{I}_\gamma f(x) - a_B\right| d\mu(x) &= \int_B |I(x)|\, d\mu(x)\\ &\le \int_B |I_1(x)|\, d\mu(x) + \int_B |I_2(x)|\, d\mu(x)\\ &\le C\mu(B)^{\beta+\gamma+1}\phi^{-1}(1/\mu(B)) \inf_B w\|f/v\|_\phi\end{aligned}$$

which proves (3.7) since

$$\begin{aligned}\int_B \left|\tilde{I}_\gamma f(x) - m_B(\tilde{I}_\gamma)\right| d\mu(x) &\le C\int_B \left|\tilde{I}_\gamma f(x) - a_B\right| d\mu(x)\\ &\le C\mu(B)^{\beta+\gamma+1}\phi^{-1}(1/\mu(B)) \inf_B w\|f/v\|_\phi.\end{aligned}$$

This completes the fact that $\tilde{I}_\gamma$ is a bounded linear operator $\tilde{I}_\gamma$ from $L_{\phi,v}$ into $\mathcal{L}_w^\phi(\beta,\gamma)$. □

In order to prove Theorem 3.9, we introduce two tools. First, let the function K_γ for $\gamma \in (0,1)$ be defined by

$$K_\gamma(x,z,y) = \begin{cases} \eta(x,y)^{\alpha(\gamma-1)} - \eta(z,y)^{\alpha(\gamma-1)} & \text{if } x \ne y \text{ and } z \ne y,\\ \mu(\{x\})^{\gamma-1} - \eta(z,y)^{\alpha(\gamma-1)} & \text{if } x = y \text{ and } z \ne y,\\ \eta(x,y)^{\alpha(\gamma-1)} - \mu(\{z\})^{\gamma-1} & \text{if } x \ne y \text{ and } z = y,\\ 0 & \text{if } x = y = z,\end{cases}$$

for x, y, and z in X.

It is easy to see that the left hand side of (3.7) is equivalent to the following expression involving K_γ

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+\gamma+2}\phi^{-1}(1/\mu(B))}\int_B\int_B\left|\int_X K_\gamma(x,z,y)f(y)\,d\mu(y)\right| d\mu(z)\,d\mu(x). \tag{4.7}$$

Now, we introduce a quasi-distance naturally associated to (X,d,μ). Let $\delta : X\times X \to \mathbb{R}$ be defined by

$$\delta(x,y) = \begin{cases}\mu(B(x,d(x,y))) & \text{if } x\ne y,\\ 0 & \text{if } x = y.\end{cases}$$

It is easy to see (see [MST]) that the function δ satisfies

(i) $\delta(x,y) \ge 0$ and $\delta(x,y) = 0$ if and only if $x = y$,
(ii) $\delta(x,y) \le A\delta(y,x)$ and
(iii) $\delta(x,y) \le A^2(\delta(x,z)+\delta(y,z))$ for every x, y, and z in X,

where A is the doubling constant of μ with the quasi-metric d.

We observe that $\delta(x,y)$ does not necessarily satisfy a symmetric condition as d. The function δ is called the non-necessarily symmetric quasi-distance associated to (X,d,μ). We denote by $B_\delta(x,r)$ the set $\{y : \delta(x,y) < r\}$. The above conditions on δ imply the existence of a constant D such that

$$0 < \mu(B_\delta(x, 2Kr)) \le D\,\mu(B_\delta(x,r)) < \infty. \tag{4.9}$$

In [MST] it is also proved that δ has the following properties:

(i) $B_\delta(x,r) = \{x\}$, if $0 < r < \mu(\{x\})$,
(ii) $\mu(B_\delta(x,r)) \le r$, if $\mu(\{x\}) \le r$,
(iii) $B_\delta(x,r) = X$, if $\mu(X) \le r$,
(iv) $A^{-2} r \le \mu(B_\delta(x,r))$, if $r < \mu(X)$.

In the proof of Theorem 3.9 we need the next lemma, which is a slight modification of Lemma 2.9 in [BS] for the case of a non-symmetric quasi-distance δ, so the proof is omitted here.

4.10 Lemma. *Let (X,δ,μ) be a space of homogeneous type with δ given by (4.8). For each $\gamma \in (0,1)$ there exist two constants, K_o and C, depending only on γ and the constants of the space such that, for every ball $B = B_\delta(x_B, R)$ satisfying $(2KA)^{-1}\mu(\{x_B\}) \le R \le K_o^{-1}\mu(X)$, the inequality*

$$\begin{aligned} \frac{1}{\mu(B^*)^2}\int_{B^*}\int_{B^*}\left|\int_B K_\gamma(x,z,y) f(y)\,d\mu(y)\right| d\mu(z)\,d\mu(x) \\ \ge C\frac{1}{\mu(B)^{1-\gamma}}\int_B f(x)\,d\mu(x) \end{aligned} \tag{4.11}$$

holds with $B^* = B_\delta(x_B, K_o R)$ for every non negative function f.

Proof of Theorem 3.9: Since $\tilde\phi$ is of lower type q' and $\beta < 1/q - \gamma + 1/\alpha$, by Lemma 2.7 we only need to estimate

$$\frac{\|(1/w)\chi_B\|}{\mu(B)^\beta}\tilde\phi^{-1}(1/\mu(B))\|v\chi_B\|_{\tilde\phi} \le C. \tag{4.12}$$

Let us first give an outline of the proof that is based on the following steps:

(a) The results holds if we have (X,δ,μ) in place of (X,d,μ).

(b) If condition (4.12) holds for δ-balls then it holds for d-balls.

(c) If (3.7) holds with d-balls then it holds with δ-balls.

To prove (a), suppose that (3.7) holds with δ-balls and let $B = B_\delta(x_B, R)$ be a ball such that $R \le K_o^{-1}\mu(X)$, where K_o is the same constant as in Lemma 4.10.

If $(2KA)^{-1}\mu(\{x_B\}) \le R$ from that lemma, (4.7) and the doubling property of μ we have

$$\begin{aligned}
&\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{1-\gamma}} \int_B |f(x)|\, d\mu(x) \\
&\le \frac{\|(1/w)\chi_{B^*}\|_\infty}{\mu(B^*)^2} \int_{B^*}\int_{B^*} \left|\int_B K_\gamma(x,z,y) f(y) d\mu(y)\right| d\mu(z)\, d\mu(x) \\
&\le C\mu(B)^{\beta+\gamma}\phi^{-1}(1/\mu(B))\|f/v\|_\phi
\end{aligned}$$

where $B^* = B_\delta(x_B, K_o R)$. Then, the operator T defined as

$$T(g) = \int_B g(x)v(x)\, d\mu(x)$$

satisfies

$$|T(f/v)| = \left|\int_B f(x)\, d\mu(x)\right| \le C\mu(B)^\beta \phi^{-1}(1/\mu(B)) \inf_B w \|f/v\|_\phi$$

for every f such that $f/v \in L_\phi$. So T belongs to $(L_\phi)^* = L_{\tilde\phi}$ and we can write

$$\|v\chi_B\|_{\tilde\phi} = \sup_{\|f/v\|_\phi \le 1} |T(f/v)| \le C\mu(B)^{\beta+1}\phi^{-1}(1/\mu(B)) \inf_B w$$

which proves the result for the case of δ-balls, with $(2KA)^{-1}\mu(\{x_B\}) \le R \le K_o^{-1}\mu(X)$.

Now, suppose $R \le (2KA)^{-1}\mu(\{x_B\}) \le K_o^{-1}\mu(X)$. If $K_o^{-1}\mu(X) < \mu(\{x_B\})$, by (i) of normality properties of δ (see after 4.9) we have that $B(x_B, R) = \{x_B\}$. Then we can choose $\tilde R$ such that $(2KA)^{-1}\mu(\{x_B\}) \le \tilde R \le K_o^{-1}\mu(X) < \mu(\{x_B\})$ and the conclusion follows from the above case since $B(x_B, \tilde R) = B(x_B, R) = \{x_B\}$.

On the other hand, if $\mu(\{x_B\}) \le K_o^{-1}\mu(X)$, we get the conclusion from the first case by taking $\tilde R$ such that $\mu(\{x_B\})(2KA)^{-1} < \tilde R < \mu(\{x_B\})$, since $B(x_B, R) = B(x_B, \tilde R) = \{x_B\}$.

Now we prove (b). Let S be any positive number and $\tilde R = \mu(B_d(x_B, S))$. Then

$$B_d(x_B, S) \subset \bar B_\delta(x_B, \tilde R) \subset B_\delta(x_B, 2\tilde R) \tag{4.13}$$

where $\bar B_\delta(x_B, \tilde R) = \{y : \delta(x_B, y) \le \tilde R\}$. Moreover, since (X, δ, μ) is a normal space of homogeneous type we have that

$$\mu(B_\delta(x_B, 2\tilde R)) \simeq \mu(B_\delta(x_B, \tilde R)) \simeq \tilde R = \mu(B_d(x_B, S)). \tag{4.14}$$

Since (4.12) holds for any δ-ball it holds for $B_\delta(x_B, 2\tilde{R})$. From (4.13) and (4.14) we obtain

$$\frac{\|(1/w)\chi_{B_d(x_B,S)}\|_\infty}{\mu(B_d(x_B,S))^\beta}\tilde{\phi}^{-1}(1/\mu(B_d(x_B,S))\|v\chi_{B_d(x_B,2KS))}\|_{\tilde{\phi}} \le C$$

for all $s > 0$, which says that $(w, v) \in C(\tilde{\phi}, \beta)$ with respect to d.

Finally we prove (c). If $S = \sup\{d(x_B, y) : y \in \bar{B}_\delta(x_B, R)\}$ then, from Lemma 2.5 of [MT] we have that

$$B_d(x_B, S) \subset \bar{B}_\delta(x_B, R) \subset \bar{B}_d(x_B, S). \tag{4.15}$$

Then we obtain that $B_\delta(x_B, R) \subset B_d(x_B, 2S)$ and

$$\mu(B_\delta(x_B, R)) \le C\mu(B_d(x_B, 2S)). \tag{4.16}$$

On the other hand, since μ satisfies the doubling condition, from (4.15), (1.1) and (4.9) we have that

$$\begin{aligned} \mu(B_d(x_B, 2S)) \le A\mu(B_d(x_B, S)) &\le A\mu(B_\delta(x_B, 2R)) \\ &\le AD\mu(B_\delta(x_B, R)). \end{aligned} \tag{4.17}$$

From (4.16) and (4.17) we obtain that

$$\mu(B_\delta(x_B, R)) \simeq C\mu(B_d(x_B, 2S)). \tag{4.18}$$

Now, since (3.7) holds for any d-ball, it holds for $B_d(x_B, 2S)$ and from (4.18) we have

$$\begin{aligned} &\frac{\|(1/w)\chi_{B_\delta(x_B,R)}\|_\infty}{\mu(B_\delta(x_B,R))^{\beta+\gamma+1}\phi^{-1}(1/\mu(B_\delta(x_B,R)))} \\ &\qquad \times \int_{B_\delta(x_B,R)} |\tilde{I}_\gamma f(x) - m_{B_\delta(x_B,R)}(\tilde{I}_\gamma f)|\, d\mu(x) \\ &\le C\frac{\|(1/w)\chi_{B_d(x_B,2S)}\|_\infty}{\mu(B_d(x_B,2S))^{\beta+\gamma+1}\phi^{-1}(1/\mu(B_d(x_B,2S)))} \\ &\qquad \times \int_{B_d(x_B,2S)} |\tilde{I}_\gamma f(x) - m_{B_d(x_B,2S)}(\tilde{I}_\gamma f)|\, d\mu(x) \\ &\le C\|f/v\|_\phi. \end{aligned}$$

Then we obtain (4.12) for δ-balls. We are done. □

4.19 Remark. From Theorem 3.9, if $\mu(X) < \infty$, we have that $C_\gamma(\tilde{\phi}, \beta)$ holds with R smaller than a fraction of $\mu(X)$. But, if $R > \mu(X)$ then, by (iii) of normality properties of δ, $B_\delta(x_B, R) = X$ and if $C\mu(X) \le R \le \mu(X)$, by (iv), $A^{-2}K_o^{-1}\mu(X) \le A^{-2}R \le \mu(B_\delta(x_B, R)) \le \mu(X)$ and thus $\mu(B_\delta(x_B, R)) \simeq \mu(X)$. So the first part of Remark 3.11 follows.

5. Examples

Now, we give some examples of pairs of weights belonging to $C_\gamma(\phi,\beta)$.

• Let us first consider $X=\mathbb{R}^n$, d the Euclidean metric, $\phi(t)=t^{p'}$ and μ the Lebesgue measure. Then, the condition $C(\phi,\beta)$ is the condition $\overline{\mathcal{H}}(p,\gamma n, n\beta - n/p+n\gamma)$ defined in [P] and, consequently, the pairs given there belong to $C_\gamma(\phi,\beta)$.

• Let us now consider $X=\left\{\frac{1}{2^i}\right\}_{i\in\mathbb{N}}$, d the Euclidean metric and μ such that $\mu(\{1/2^i\})=1/2^i$. It is easy to check that (X,d,μ) is a space of homogeneous type. We shall prove that the pair (w,v) defined by

$$w(x)=|x|^{\theta-\beta} \quad \textit{and} \quad v(x)=|x|^\theta,$$

belongs to $C_\gamma(\phi,\beta)$ for every γ in $(0,1)$, with $\phi(t)=t^\alpha$, $\alpha>1$ and $1/\alpha\le\theta\le\beta$. Since $\mu(X)<\infty$, from Remark 2.8 we only need to estimate (2.4) for every ball $B\subset X$. Let us first consider $B(\frac{1}{2^{i_0}},R)=\left\{\frac{1}{2^{i_0}}\right\}$, then

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+1/\alpha}}\left(\int_B v^\alpha\right)^{1/\alpha}=C\frac{1}{2^{i_0(\beta-\theta)}}2^{i_0(\beta+1/\alpha)}\left(\frac{1}{2^{i_0(\alpha\theta+1)}}\right)^{1/\alpha}\le C.$$

Let us now suppose that $B(\frac{1}{2^{i_0}},R)=\{x_1,\dots,x_n\}$, with $x_i\in X$. Then $R<\frac{1}{2^{i_0}}$ and

$$\mu(B)=\sum_{k=i_0}^{i_0+n}\frac{1}{2^k}=\frac{1}{2^{i_0}}\left(2-\frac{1}{2^n}\right).$$

Furthermore

$$\int_B v^\alpha=\sum_{k=i_0}^{i_0+n}\frac{1}{2^{k(\theta\alpha+1)}}=\frac{1}{2^{i_0(\theta\alpha+1)}}\frac{2^{(n+1)(\theta\alpha+1)}-1}{2^{\theta\alpha+1}-1}\frac{1}{2^{n(\theta\alpha+1)}}.$$

Then

$$\begin{aligned}&\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+1/\alpha}}\left(\int_B v^\alpha\right)^{1/\alpha}\\&=C\frac{2^{-i_o(\beta-\theta)}}{2^{-i_o(\beta+1/\alpha)}\,(2-(1/2^n))^{\beta+1/\alpha}}\frac{1}{2^{i_o(\theta+1/\alpha)}}\left(\frac{2^{\theta\alpha+1}}{2^{\theta\alpha+1}-1}\right)^{1/\alpha}\\&\le C\left(\frac{2^{\theta\alpha+1}}{2^{\theta\alpha+1}-1}\right)^{1/\alpha}\frac{1}{(2-(1/2^n))^{\beta+1/\alpha}}.\end{aligned}$$

Since $2 - 1/2^n \geq C > 0$ the last term in the above expression is bounded by a constant.

On the other hand, if $B(\frac{1}{2^{i_0}}, R)$ contains infinitely many $x_i \in X$, then $R \geq \frac{1}{2^{i_0}}$ and if $i_o \geq 3$, there exists $m \in \mathbb{N}$, $m \leq i_0 - 2$ such that $1/2^{m+1} - 1/2^{i_0} \leq R \leq 1/2^m - 1/2^{i_0}$. Then

$$\mu(B) = \sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^m} \quad \text{and} \quad \inf_B w = \frac{1}{2^{(m+1)(\theta-\beta)}}.$$

On the other hand

$$\int_B v^\alpha = \sum_{k=m+1}^{\infty} \frac{1}{2^{k(\theta\alpha+1)}} = \frac{1}{2^{m(\theta\alpha+1)}2^{\theta\alpha+1} - 1}.$$

Thus

$$\frac{\|(1/w)\chi_B\|_\infty}{\mu(B)^{\beta+1/\alpha}} \left(\int_B v^\alpha\right)^{1/\alpha} \leq C \frac{2^{(\theta-\beta)}}{\left(2^{\theta\alpha+1} - 1\right)^{1/\alpha}}. \tag{5.1}$$

If $i_o = 2$ or $i_o = 1$, then $B(1/2^{i_o}, R) = X$. So, we can take $m = 0$ and the same estimates hold. This proves that $(w, v) \in C_\gamma(\phi, \beta)$.

- With the same space (X, d, μ) from the previous example, let us now consider ϕ defined by

$$\phi(t) = \begin{cases} t^p & 0 \leq t \leq 1 \\ t^q & t > 1 \end{cases} \qquad \text{with} \quad 1 < p < q.$$

It is not too hard to see that ϕ is of lower type p and of upper type q. The pair (w, v) defined by

$$w(x) = |x|^\eta \quad \textit{and} \quad v(x) = |x|^\theta$$

with $-1/p < \eta < 0$, $\theta > 0$ and $-1/q < \beta < -1/p - \eta$ belongs to $C_\gamma(\phi, \beta)$ for every γ in $(0, 1)$. In fact, let us first suppose that $B(1/2^{i_0}, R) = \{x_1, \ldots, x_n\}$, with $x_i \in X$. In the previous example we obtain that $\mu(B) = \frac{1}{2^{i_0}}(2 - 1/2^n)$. From this estimate and the fact that ϕ is of lower type p and of upper type q, we get

$$\begin{aligned}
\int_{\tilde{B}} \phi\left(\frac{v(y)\phi^{-1}(1/\mu(B))}{\inf_B w\mu(B)^\beta}\right) &= \sum_{k=i_0}^{i_0+n} \phi\left(\frac{2^{-k\theta}\phi^{-1}(1/\mu(B))}{2^{-i_0\eta}(2 - 1/2^n)^\beta 2^{-i_0\beta}}\right) \mu(\{2^{-k}\}) \\
&\leq C \frac{2^{i_0((\beta+\eta)p+1)}}{(2 - 1/2^n)^{\beta q+1}} \sum_{k=i_0}^{i_0+n} 2^{-k(\theta p+1)} \\
&\leq C \frac{2^{i_0((\beta+\eta-\theta)p)}}{(2 - 1/2^n)^{(\beta q+1)}} \frac{1}{(1 - 2^{-(\theta p+1)})2^{\theta p+1}} \leq C.
\end{aligned}$$

If $B(1/2^{i_0}, R)$ contains infinitely many $x_i \in X$, proceeding as in the above example, we get that $\mu(B) = 1/2^m$ and $\inf_B w = 1/2^{(m+1)\eta}$. Then

$$\begin{aligned}\int_{\tilde{B}} \phi\left(\frac{v(y)\phi^{-1}(1/\mu(B))}{\inf_B w\mu(B)^\beta}\right) &= \sum_{k=m+1}^{\infty} \phi\left(\frac{2^{-k\theta}\phi^{-1}(1/\mu(B))}{2^{-\eta(m+1)}2^{-m\beta}}\right)\mu(2^{-k})\\ &\leq C2^{m+m(\beta+\eta)p+\eta p}\sum_{k=m+1}^{\infty} 2^{-k(\theta p+1)}\\ &\leq C2^{m(\beta+\eta-\theta)p+\eta p} \leq C.\end{aligned}$$

This completes the proof. □

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