The limit lemma in fragments of arithmetic

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Abstract. The recursion theoretic limit lemma, saying that each function with a Σ_{n+2} graph is a limit of certain function with a Δ_{n+1} graph, is provable in $\mathrm{B}\Sigma_{n+1}$.

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Let N be the set of all natural numbers and let a function $G : \mathrm{N}^{k+1} \to \mathrm{N}$ be such that for each $x_1, .., x_k$ the function $s \mapsto G(\underline{x}, s)$, where $\underline{x}$ is a shorthand for $x_1, .., x_k$, is eventually constant. Then we use $\lim_s G(\underline{x}, s)$ to denote the value the function $s \mapsto G(\underline{x}, s)$ assumes in each sufficiently large s. The *limit lemma* says that for each set $A \subseteq \mathrm{N}^k$ such that $A \in \Delta_2$ there exists a recursive function $G : \mathrm{N}^{k+1} \to \mathrm{N}$ such that $\lim_s G(\underline{x}, s) = 1$ whenever $[x_1, .., x_k] \in A$, and $\lim_s G(\underline{x}, s) = 0$ whenever $[x_1, .., x_k] \notin A$. For the definition of Σ_n, Π_n, and Δ_n, where $n \geq 1$, see e.g. [5], and recall that a set is Δ_1 if and only if it is recursive, and that $\Delta_n = \Sigma_n \cap \Pi_n$. The version of the limit lemma for functions says that for each function $F : \mathrm{N}^k \to \mathrm{N}$ whose graph is Σ_2 there exists a recursive $G : \mathrm{N}^{k+1} \to \mathrm{N}$ such that $F(\underline{x}) = \lim_s G(\underline{x}, s)$ for each k-tuple $[x_1, .., x_k]$. As can be seen e.g. from [4] and [2], the limit lemma is a useful tool in recursion theory.

Peano arithmetic PA is an axiomatic theory formulated in the arithmetical language $\{+, \cdot, 0, \mathrm{S}, \leq, <\}$; its axioms can be described as a finite set of base axioms plus the induction scheme. For details see e.g. [3]. *Bounded quantifiers* are quantifiers of the form $\forall v{\leq}x$, $\exists v{\leq}x$, $\forall v{<}x$, and $\exists v{<}x$. A *bounded formula*, or a Δ_0*-formula*, is a formula all quantifiers of which are bounded. A Σ_n*-formula* is a formula having the form $\exists v_1 \forall v_2 \exists \, .. \, v_n \varphi$, with n alternating quantifiers, where the first quantifier is existential and the matrix φ is a Δ_0-formula. A Π_n*-formula* is a formula of the form $\forall v_1 \exists v_2 \forall \, .. \, v_n \varphi$ where again $\varphi \in \Delta_0$. So $\Sigma_0 = \Pi_0 = \Delta_0$. The *theory* $\mathrm{I}\Gamma$, where Γ is Σ_n or Π_n, is PA with the induction scheme restricted to Γ-formulas. The *collection scheme* is the scheme

$$\forall \underline{y} \forall x (\forall v{\leq}x \, \exists z \varphi(v, z, \underline{y}) \to \exists t \forall v{\leq}x \, \exists z{\leq}t \, \varphi(v, z, \underline{y})).$$

The *theory* $\mathrm{B}\Gamma$, where again Γ is Σ_n or Π_n, is $\mathrm{I}\Delta_0$ extended by the collection scheme restricted to Γ-formulas. It is known that for each n the theories $\mathrm{I}\Sigma_n$

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and $\mathrm{I}\Pi_n$ are equivalent, and also $\mathrm{B}\Pi_n$ and $\mathrm{B}\Sigma_{n+1}$ are equivalent. $\mathrm{B}\Sigma_{n+1}$ is a theory stronger than $\mathrm{I}\Sigma_n$, but weaker than $\mathrm{I}\Sigma_{n+1}$. For details and proofs, see again e.g. [3]. A useful property of $\mathrm{I}\Sigma_n$ is that it proves induction for $\Sigma_0(\Sigma_n)$-formulas, i.e. for formulas built up from Σ_n-formulas using logical connectives and bounded quantification. Also the least number principle for $\Sigma_0(\Sigma_n)$-formulas is provable in $\mathrm{I}\Sigma_n$. A useful property of $\mathrm{B}\Sigma_{n+1}$ is that any formula obtained from Σ_{n+1}-formulas by bounded quantification is $\mathrm{B}\Sigma_{n+1}$-equivalent to a Σ_{n+1}-formula. This fact can be used to verify that each $\Sigma_0(\Sigma_n)$-formula is $\mathrm{B}\Sigma_{n+1}$-equivalent to a Σ_{n+1}-formula. We will also use the fact that $\Sigma_0(\Sigma_n)$-induction is provable in $\mathrm{B}\Sigma_{n+1}$.

P. Hájek and A. Kučera show in [2] that the limit lemma for sets is provable in $\mathrm{I}\Sigma_1$. P. Clote in an earlier paper [1] uses a version of the limit lemma for $\mathit{\Sigma}_{n+2}$ functions, saying that any function having a $\mathit{\Sigma}_{n+2}$ graph is a limit of a function having a $\mathit{\Delta}_{n+1}$ graph, and proves this version in $\mathrm{B}\Sigma_{n+2}$. I show that the results from [2] and [1] can be considerably improved: the limit lemma for $\mathit{\Sigma}_{n+2}$ functions is provable already in $\mathrm{B}\Sigma_{n+1}$.

Note that speaking about sets definable in a model, in the formulation of Lemma 1 and Theorem 1 below, is a way to overcome the difficulty that one cannot directly speak about sets and functions in the arithmetical language. In proofs of Lemma 1 and Theorem 1 we are less careful and ignore this difficulty. Recall that if $n \geq 1$ then a set is $\mathit{\Sigma}_n$ if and only if it is Σ_n-definable in the standard model of arithmetic. So a set simultaneously Σ_n- and Π_n-definable in a model corresponds to a set which, on metamathematical level, is $\mathit{\Delta}_n$.

Lemma 1. *Let* $\mathbf{M}$ *be a model of* $\mathrm{B}\Sigma_{n+1}$ *with domain* M *and let* $A \subseteq M^k$ *be simultaneously* Σ_{n+2}*- and* Π_{n+2}*-definable in* $\mathbf{M}$*. Then there exists a function* $G : M^{k+1} \to M$ *with a graph* $\Sigma_0(\Sigma_n)$*-definable in* $\mathbf{M}$ *such that* $\lim_s G(\underline{x}, s) = 1$ *whenever* $[x_1, \ldots, x_k] \in A$ *and* $\lim_s G(\underline{x}, s) = 0$ *whenever* $[x_1, \ldots, x_k] \notin A$.

PROOF: Let the set A be as specified and let φ and ψ be Σ_n-formulas such that $A = \{\, [x_1, \ldots, x_k]\,;\, \exists u \forall v \varphi(\underline{x}, u, v) \,\}$ and $\overline{A} = \{\, [x_1, \ldots, x_k]\,;\, \exists u \forall v \psi(\underline{x}, u, v) \,\}$, where $\overline{A}$ is the complement of A. Think of the k-tuple $\underline{x}$ as fixed and think of φ and ψ as two zero-one tables unbounded in two directions, with u running down and v running to the right. One and only one of the two tables contains rows consisting entirely of ones. Let the function H be defined as follows:

$$H(\underline{x}, s) = \begin{cases} 1 & \text{if } \forall u{\leq}s\,(\forall v{\leq}s\,\psi(\underline{x}, u, v) \to \exists u'{\leq}u \forall v{\leq}s\,\varphi(\underline{x}, u', v)) \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $[x_1, \ldots, x_k] \notin A$. Then $\exists u \forall v \psi(\underline{x}, u, v)$ and $\forall u \exists v \neg\varphi(\underline{x}, u, v)$. Let u_0 be *some* number satisfying $\forall v \psi(\underline{x}, u_0, v)$; note that the existence of least such number is not guaranteed in $\mathrm{B}\Sigma_{n+1}$. By $\mathrm{B}\Sigma_{n+1}$ there exists a number s_0 such that $\forall u{\leq}u_0\, \exists v{\leq}s_0\, \neg\varphi(\underline{x}, u, v)$. We can assume $s_0 \geq u_0$. If $s \geq s_0$ then there exists a number $u \leq s$, namely u_0, such that $\forall v{\leq}s\,\psi(\underline{x}, u, v)$ and simultaneously

$\forall u' \leq u \, \exists v \leq s \, \neg\varphi(\underline{x}, u', v)$. So $H(\underline{x}, s) = 0$ for all such s, i.e. $\lim_s H(\underline{x}, s) = 0$. The proof that $\lim_s H(\underline{x}, s) = 1$ whenever $[x_1, .., x_n] \in A$ is similar. The graph of H is $\Sigma_0(\Sigma_n)$. So the function H is as desired. □

Theorem 1. *Let* $\mathbf{M}$ *be a model of* $\mathrm{B}\Sigma_{n+1}$ *with domain* M *and let* $F : M^k \to M$ *have a graph* Σ_{n+2}*-definable in* $\mathbf{M}$*. Then there exists a function* $G : M^{k+1} \to M$ *with a graph* $\Sigma_0(\Sigma_n)$*-definable in* $\mathbf{M}$ *such that* $F(\underline{x}) = \lim_s G(\underline{x}, s)$ *for each* $\underline{x}$*.*

PROOF: Let $F \in \Sigma_{n+2}$ with k variables be given. It is clear that $F \in \Delta_{n+2}$ since for the complement of its graph we have $[\underline{x}, y] \notin F \Leftrightarrow \exists y'(y' \neq y \,\&\, [\underline{x}, y'] \in F)$. By Lemma 1 applied to the graph of F there exists a function $H \in \Sigma_0(\Sigma_n)$ such that $\lim_t H(\underline{x}, y, t) = 1$ whenever $F(\underline{x}) = y$ and $\lim_t H(\underline{x}, y, t) = 0$ whenever $F(\underline{x}) \neq y$. As in the proof of Lemma 1, let $\underline{x}$ be fixed and think of the function H as a table with t running down and y running to the right. Let the *score of a number* y *at stage* s be defined as the length of maximal contiguous segment of ones which lies in column y, the bottom end of which is in row s and the top end of which is in a row $t \geq y$.

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	...
1	0	0	0	0	0	0	0	0	...
2	0	0	0	0	0	1	0	0	...
3	0	0	0	0	0	1	0	0	...
4	0	0	1	1	0	1	0	0	...
5	0	0	1	1	0	1	0	0	...
6	0	0	0	1	0	1	0	0	...
7	0	0	1	1	0	1	0	0	...
8	0	0	1	1	0	1	0	0	...

Figure 1: Computing scores

If H is, for example, as in Figure 1 then the scores of numbers 2, 3, and 5 at stage 5 are 2, 2, and 1 respectively, and the score of any other number at stage 5 is zero. The scores of numbers 2, 3, and 5 at stage 8 are 2, 5, and 4. Let $G(\underline{x}, s)$ be defined as the least y having maximal possible score at stage s. So in our example from Figure 1 we have $G(\underline{x}, 5) = 2$ and $G(\underline{x}, 8) = 3$. It is evident that a score of a number $y \leq s$ at stage s is a number not exceeding $s + 1 - y \leq s + 1$ and that all y's greater than s have zero score at stage s. The formula

$$\exists u \leq s+1 \, (z + u = s + 1 \,\&\, y \leq u \,\&\, \forall t \leq s \, (u \leq t \to H(\underline{x}, y, t) = 1)),$$

i.e. the formula the score of y at stage s is at least z, is a $\Sigma_0(\Sigma_n)$-formula. So by $\Sigma_0(\Sigma_n)$-induction available in $\mathrm{B}\Sigma_{n+1}$, there exists a greatest z satisfying this formula, and the score of a number y at stage s is correctly defined. Also, the formulas the number z is the maximal score at stage s and the number y is the least number having the maximal score at stage s are $\Sigma_0(\Sigma_n)$-formulas. So again by $\Sigma_0(\Sigma_n)$-induction, the maximal score exists, and the function G is correctly defined. We have to verify that $\lim_s G(\underline{x}, s) = F(\underline{x})$. Let $y_0 = F(\underline{x})$. We know that $\lim_s H(\underline{x}, y_0, t) = 1$. So let the number t_0 be such that $t_0 \geq y_0$ and $\forall t(t \geq t_0 \rightarrow H(\underline{x}, y_0, t) = 1)$. We also know that $\lim_s H(\underline{x}, y, t) = 0$ for each $y \leq t_0$ such that $y \neq y_0$. Thus

$$\forall y{\leq}t_0\,(y \neq y_0 \rightarrow \exists t(t \geq t_0 \;\&\; H(\underline{x}, y, t) = 0)).$$

By Σ_{n+1}-collection (more precisely, by $\Sigma_0(\Sigma_n)$-collection available in $\mathrm{B}\Sigma_{n+1}$) there exists an s_0 such that

$$\forall y{\leq}t_0\,(y \neq y_0 \rightarrow \exists t{\leq}s_0\,(t \geq t_0 \;\&\; H(\underline{x}, y, t) = 0)).$$

This means that if $s \geq s_0$ then the score of all numbers $y \leq t_0$ such that $y \neq y_0$ at stage s is lower than the score of y_0. Since ones occurring in column y above the diagonal line do not count, the score of any $y > t_0$ at stage s is automatically lower than the score of y_0. So $G(\underline{x}, s) = y_0$ for each $s \geq s_0$, and thus $\lim_s G(\underline{x}, s) = y_0$. $\square$

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