The Poisson integral for a ball in spaces of constant curvature

Eleutherius Symeonidis

Abstract. We present explicit expressions of the Poisson kernels for geodesic balls in the higher dimensional spheres and real hyperbolic spaces. As a consequence, the Dirichlet problem for the projective space is explicitly solved. Comparison of different expressions for the same Poisson kernel lead to interesting identities concerning special functions.

Keywords: Poisson integral, Poisson kernel, Dirichlet problem, harmonic function, Riemannian manifold, hypergeometric function

Classification: Primary 31C12, 35J25; Secondary 33C90

1. Introduction

The question of determining a harmonic function on a ball by its values on the boundary sphere, the so-called Dirichlet problem, can be posed on any Riemannian manifold. Harmonicity refers to the Laplace-Beltrami operator and the ball is meant to be a geodesic one of a fixed radius. While theoretical arguments guarantee the existence of a unique harmonic function on such a ball extending continuously to the boundary for arbitrarily prescribed continuous boundary values [9], there is a lack of concrete formulas in other than euclidean situations. The only non-euclidean cases where an explicit integral representation of the harmonic function (a “Poisson integral”) is known seem to be the two-dimensional sphere and the hyperbolic plane ([13], [14]). In the present work we determine the Poisson kernel for a ball of arbitrary radius in the cases of the spheres and (real) hyperbolic spaces of any dimension by applying the method of separation of variables to Laplace’s equation (cf. [3, V. 9, VII.5] for the classical, euclidean situation). The method leads to a representation of the Poisson kernel in terms of an infinite series involving the hypergeometric function. This is done in Section 3. In Section 4 we search for equivalent but simpler expressions of the Poisson kernel. It turns out that in the case of a half sphere the kernel also appears as a product of two functions of one variable, one factor being hypergeometric (Theorem 3). The two different representations of one and the same kernel lead to a first identity in the context of special functions (Remark 1). Passing over from the half sphere to the projective space, the hypergeometric factor in the Poisson kernel degenerates and we are led to a remarkably simple and concise expression involving merely trigonometric functions (Section 5). In the last section we show that the Dirichlet problem for the entire hyperbolic space can be viewed as the case of a ball of “infinite” radius. In this way we establish the relation to a relevant result in the past [1] and arrive at a second identity concerning special functions.

2. Description of the problem

Let X be a complete Riemannian manifold, which here will be taken to be the n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} | \, |x| = 1\}$, the n-dimensional real projective space P^n (the Riemannian metric inherited from that of S^n), or the n-dimensional real hyperbolic space H^n. (For a concise introduction to the hyperbolic space the reader is referred to [6, pp. 151–153, 226, 227, 564]). We shall denote by $d(\cdot,\cdot)$ the distance function on X arising from the Riemannian metric. Let $B_R(o) \subseteq X$ be a geodesic ball of centre $o \in X$ and radius $R > 0$: $B_R(o) = \{z \in X | d(o,z) < R\}$ (for $X = S^n$ we assume $R < \pi$, for $X = P^n$ we take $R \leq \frac{\pi}{2}$). The boundary of $B_R(o)$ is the sphere $S_R(o) = \{x \in X | d(o,x) = R\}$. Each $x \in S_R(o)$ can be reached by a geodesic curve emanating from o and having length R.

A system of local coordinates is defined with the help of the exponential map $\exp_o$ of centre o (see, e.g., [18, Section 3.4]):

$$(0,R) \times S_o^{n-1} \ni (r,y) \mapsto \exp_o(ry) \in B_R(o),$$

S_o^{n-1} denoting the unit sphere in the tangent space T_oX. In these coordinates the line element ds is given by

$$\begin{aligned} ds^2 &= dr^2 + \sin^2 r \, d\sigma^2 \qquad \text{for} \quad X = S^n, \quad \text{and} \\ ds^2 &= dr^2 + \sinh^2 r \, d\sigma^2 \qquad \text{for} \quad X = H^n, \end{aligned} \tag{1}$$

where $d\sigma$ stands for the line element on S_o^{n-1}. In both cases the sphere $S_R(o)$ is a submanifold whose volume element dV is related to the Lebesgue measure $d\mu$ on S_o^{n-1} by (1):

$$\begin{aligned} dV &= \sin^{n-1} R \, d\mu \qquad \text{for} \quad X = S^n, \quad \text{and} \\ dV &= \sinh^{n-1} R \, d\mu \qquad \text{for} \quad X = H^n. \end{aligned} \tag{2}$$

We now consider a continuous function $f : S_R(o) \to \mathbb{R}$. The Dirichlet problem for the data $(B_R(o), f)$ consists in the determination of a harmonic function $H^f : B_R(o) \to \mathbb{R}$ (that is, H^f be annihilated by the Laplace-Beltrami operator Δ of X) which extends to a continuous function on the closure $\overline{B_R(o)}$ being equal to f on $S_R(o)$. Since the Laplace-Beltrami operator is elliptic with no zero order term and the boundary $S_R(o)$ is smooth, the Dirichlet problem is uniquely solvable [9, Theorem 21,I]. Moreover, there is an integral representation of the solution,

$$H^f(z) = \frac{1}{V(S_R(o))} \int_{S_R(o)} P^X_{B_R(o)}(z,x) f(x) \, dV(x), \qquad z \in B_R(o), \tag{3}$$

where the "Poisson kernel" $P^X_{B_R(o)}(z,x)$ is the negative of the outward normal derivative of the so-called Green function G [9, Theorem 21,VI]:

$$P^X_{B_R(o)}(z,x) = -\frac{\partial G(z,y)}{\partial y} \cdot \nu(y)|_{y=x}.$$

The aim of this work is to determine the kernel $P^X_{B_R(o)}(z,x)$ explicitly and study its properties. In the two-dimensional cases previous work ([13], [14]) supplies:

$$P^{S^2}_{B_R(o)}(z,x) = \frac{\sin^2\frac{R}{2} - \sin^2\frac{d(o,z)}{2}}{\sin^2\frac{d(z,x)}{2}} \quad \text{and} \quad P^{H^2}_{B_R(o)}(z,x) = \frac{\sinh^2\frac{R}{2} - \sinh^2\frac{d(o,z)}{2}}{\sinh^2\frac{d(z,x)}{2}}.$$

In the higher dimensional cases ($n \geq 3$) we have to face a far more difficult problem. Concerning the sphere S^n, stereographic projection no longer preserves harmonicity of functions, and on the different models for H^n euclidean and hyperbolic harmonicity is no longer the same. Moreover, the well-known reflection principle which leads to the Green function for a ball in $\mathbb{R}^n$ is not applicable in these spaces. We shall therefore try to solve the Dirichlet problem by returning to the Laplace equation $\Delta H^f = 0$ and applying the method of separation of variables to it. Such a treatment has been originally carried out in the case of a ball in $\mathbb{R}^3$ [3, V.9, VII.5].

3. Solving the Dirichlet problem

From now on we assume $n \geq 3$.

Harmonic functions remain harmonic after composition with isometries of the space X. Both S^n and H^n have the property that every rotation in T_oX is the differential of a unique isometry of X which leaves o fixed. We shall call such an isometry a *rotation* of X about o. The group of all rotations of X about o operates transitively on $S_R(o)$. The uniqueness of the Dirichlet solution implies a fundamental invariance property of the Poisson kernel:

$$P^X_{B_R(o)}(Az,Ax) = P^X_{B_R(o)}(z,x)$$

for every isometry $A : X \to X$ with $Ao = o$. To see this, we observe that for every continuous boundary function $f : S_R(o) \to \mathbb{R}$ it holds $H^{f\circ A} = H^f \circ A$ (the right hand side is harmonic and possesses the boundary values of $f \circ A$). We then use (3) and evaluate these functions at $z \in B_R(o)$ to obtain the equal expressions

$$\begin{aligned} H^{f\circ A}(z) &= \frac{1}{V(S_R(o))}\int_{S_R(o)} P^X_{B_R(o)}(z,x) f(Ax)\,dV(x) \quad \text{and} \\ H^f(Az) &= \frac{1}{V(S_R(o))}\int_{S_R(o)} P^X_{B_R(o)}(Az,x) f(x)\,dV(x) \\ &= \frac{1}{V(S_R(o))}\int_{S_R(o)} P^X_{B_R(o)}(Az,Ax) f(Ax)\,dV(x) \end{aligned}$$

(A an isometry), from which the invariance property of the Poisson kernel is deduced. This means that $P^X_{B_R(o)}$ is in fact a function of two real variables: the distance $r = d(o, z)$ and the angle θ formed by the geodesics oz and ox at o. (For $z = o$ the Poisson kernel is independent of x). In the sequel we shall therefore be writing $P^X_{B_R(o)}(r, \theta)$.

Because of this invariance property, the Poisson kernel will be computed by solving the Dirichlet problem for a continuous function $f : S_R(o) \to \mathbb{R}$ which is invariant with respect to rotations around a fixed (geodesic) axis ox_0, $x_0 \in S_R(o)$. The solution H^f keeps this property, so $H^f(z)$ only depends on $r = d(o, z)$ and the angle θ between the geodesics ox_0 and oz at o.

From now on we restrict ourselves to the case $X = S^n$. The hyperbolic case is treated similarly, and we shall give a brief account of it at the end of this section.

The Laplace-Beltrami operator Δ on a Riemannian manifold X with metric tensor g is expressed in local coordinates $x_1, \dots, x_n$ by

$$\Delta u = \frac{1}{\sqrt{\bar{g}}} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(\sum_{i=1}^{n} g^{ik} \sqrt{\bar{g}} \frac{\partial u}{\partial x_i} \right), \tag{4}$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_{ik}$, $\bar{g} = \det(g_{ij})$ (see, e.g., [18, Section 3.8]). Since the function H^f merely depends on r and θ, it follows by (1):

$$\Delta H^f = \frac{\partial^2 H^f}{\partial r^2} + (n-1)\cot r \cdot \frac{\partial H^f}{\partial r} + \frac{1}{\sin^2 r} \left[\frac{\partial^2 H^f}{\partial \theta^2} + (n-2)\cot\theta \cdot \frac{\partial H^f}{\partial \theta} \right] = 0.$$

The method of separation of the variables consists in the determination of all possible solutions of the equation $\Delta u = 0$ of the form $u(r, \theta) = U(r) \cdot \Phi(\theta)$. An appropriate (possibly infinite) linear combination of such functions u will eventually lead to H^f. It holds:

$$\begin{gathered} \Delta u = U'' \cdot \Phi + (n-1)\cot r \cdot U' \cdot \Phi + \frac{U}{\sin^2 r} \left[\Phi'' + (n-2)\cot\theta \cdot \Phi' \right] = 0 \\ \iff \sin^2 r \cdot \frac{U'' + (n-1)\cot r \cdot U'}{U} = -\frac{\Phi'' + (n-2)\cot\theta \cdot \Phi'}{\Phi}. \end{gathered}$$

Since the variables are separated here, both sides must be constant. Therefore there exists $\lambda \in \mathbb{R}$ such that

$$\sin^2 r \cdot [U'' + (n-1)\cot r \cdot U'] - \lambda U = 0 \quad \text{and} \tag{5}$$

$$\Phi'' + (n-2)\cot\theta \cdot \Phi' + \lambda\Phi = 0. \tag{6}$$

Equation (6) is in fact the eigenvalue problem for the Laplacian (of a radial function) on the $(n-1)$-dimensional sphere, and the different eigenvalues are of the form $-\lambda = -k(k+n-2)$, $k \in \mathbb{N} \cup \{0\}$ (cf. [6, Introduction, Section 3]). In any case the substitution $x = \frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$, $G(x) = \Phi(\theta)$ transforms (6) into

$$x(1-x)G'' + \left[\frac{n-1}{2} - (n-1)x\right] G' + \lambda G = 0. \tag{7}$$

The function Φ should be bounded, so the same condition is imposed on G for $0 \le x \le 1$. The point $x = 0$ is a so-called regular singular point of (7). The associated indicial equation $\rho(\rho-1) + \rho \cdot \frac{n-1}{2} = 0$ has the roots 0 and $\frac{3-n}{2}$, so up to a constant factor there is exactly one solution of (7) which is analytic at $x = 0$ (see [2, Chapter 9, Sections 6, 8]):

$$G(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_0 \neq 0. \tag{8}$$

For the coefficients a_k equation (7) implies that

$$(k+1)\left(k + \frac{n-1}{2}\right) a_{k+1} = [k(k+n-2) - \lambda] a_k\,. \tag{9}$$

The boundedness condition for G forces the power series in (8) to converge for $x = 1$. However, since

$$\frac{a_{k+1}}{a_k} = \frac{k^2 + (n-2)k - \lambda}{k^2 + \frac{n+1}{2}k + \frac{n-1}{2}},$$

by the convergence criterion of Gauss, $\sum_{k=0}^{\infty} a_k$ would diverge ($\frac{n+1}{2} - (n-2) = \frac{5-n}{2} \le 1$) (see, e.g., [4, §52]) unless $\lambda = k(k+n-2)$ for some $k \in \mathbb{N} \cup \{0\}$, in which case the right hand side of (8) is a finite sum and G a polynomial.

Equation (7) is then a special case of the general hypergeometric differential equation

$$x(1-x)G'' + [\gamma - (\alpha+\beta+1)x]G' - \alpha\beta G = 0, \tag{10}$$

namely for $\alpha = -k$, $\beta = k+n-2$ and $\gamma = \frac{n-1}{2}$. If we set $G = G_k(x) = \binom{k+n-3}{k} F\left(-k, k+n-2\,; \frac{n-1}{2}\,; x\right)$ (with the standard notation for the hypergeometric function[1]), then Φ turns out to be a Gegenbauer (also called ultraspherical) polynomial in $\cos\theta$ [15, V.7]:

$$\Phi = \Phi_k(\theta) = G_k\left(\frac{1-\cos\theta}{2}\right) = C_k^{\frac{n-2}{2}}(\cos\theta). \tag{11}$$

[1]For all the facts concerning the hypergeometric function, unless otherwise stated, the reader is referred to [8, §9.1–9.8].

Now equation (5) will be treated. The value of λ is $k(k+n-2)$ for a fixed $k \in \mathbb{N} \cup \{0\}$. In contrast to (6), equation (5) seems not to have been encountered before. In the first step it is transformed by $z = \sin^2 \frac{r}{2}$, $Q(z) = U(r)$ into

$$4z^2(1-z)^2 Q'' + 2nz(1-z)(1-2z)Q' - k(k+n-2)Q = 0. \tag{12}$$

Furthermore, the change of variables $w = \dfrac{1}{z}$, $R(w) = Q(z)$ leads to

$$(4w^2 - 8w + 4)R'' + \left(\frac{8-4n}{w} + 6n - 16 + 8w - 2nw\right) R' - k(k+n-2)R = 0.$$

It follows that $z = \infty$ is a regular singular point of (12), so (12) is a Fuchsian differential equation with precisely three singular points, namely 0, 1 and ∞. Such an equation can be transformed into a hypergeometric one by a change of the dependent variable [2, Chapter 9, Section 13]. Since $\frac{k}{2}$ is a root of the indicial polynomials at $z = 0$ and $z = 1$, we substitute

$$Q(z) = z^{\frac{k}{2}} (1-z)^{\frac{k}{2}} \tilde{Q}(z)$$

and obtain

$$z(1-z)\tilde{Q}'' + \left[\frac{n}{2} + k - (2k+n)z\right] \tilde{Q}' - k(k+n-1)\tilde{Q} = 0,$$

a hypergeometric equation (10) with $\alpha = k$, $\beta = k+n-1$, and $\gamma = \frac{n}{2} + k$. If we take

$$\tilde{Q} = \tilde{Q}_k(z) = 2^k \cdot F\left(k, k+n-1; \frac{n}{2} + k\,; z\right)$$

($z^{\frac{k}{2}}\tilde{Q}$ should be bounded in a neighbourhood of $z = 0$), it follows that

$$U = U_k(r) = \sin^k r \cdot F\left(k, k+n-1; \frac{n}{2} + k\,; \sin^2 \frac{r}{2}\right). \tag{13}$$

It is worth noticing that the relation

$$F(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{x}{x-1}\right), \quad x < 1, \tag{14}$$

for the hypergeometric function implies that

$$U_k(r) = 2^k \tan^k \frac{r}{2} \cdot F\left(k, 1 - \frac{n}{2}\,; \frac{n}{2} + k\,; -\tan^2 \frac{r}{2}\right),$$

which for $k \geq 1$ and even n is a polynomial of degree $k+n-2$ in $\tan \frac{r}{2}$.

Once equations (5) and (6) are solved, (11) and (13) can be put together to give harmonic functions

$$u = u_k(r,\theta) = \sin^k r \cdot F\left(k, k+n-1; \frac{n}{2}+k\,; \sin^2\frac{r}{2}\right) \cdot C_k^{\frac{n-2}{2}}(\cos\theta)\,, \quad k \in \mathbb{N}\cup\{0\}.$$

Our goal is to compute the Dirichlet solution H^f. To this end we proceed by setting

$$H^f = \sum_{k=0}^{\infty} a_k u_k \tag{15}$$

and trying to determine the coefficients $a_k \in \mathbb{R}$. The boundary condition suggests ($f(x)$ is henceforth taken as a function of θ, the angle between ox_0 and ox):

$$\begin{aligned} f(\theta) &= \sum_{k=0}^{\infty} a_k u_k(R,\theta) \\ &= \sum_{k=0}^{\infty} a_k \sin^k R \cdot F\left(k, k+n-1; \frac{n}{2}+k\,; \sin^2\frac{R}{2}\right) \cdot C_k^{\frac{n-2}{2}}(\cos\theta). \end{aligned} \tag{16}$$

The Gegenbauer polynomials $C_k^{\frac{n-2}{2}}(x)$ form an orthogonal system in the Hilbert space $L^2\left([-1,1]; (1-x^2)^{\frac{n-3}{2}}\right)$ (cf. [15, V.7]):

$$\int_{-1}^{1} (1-x^2)^{\frac{n-3}{2}} C_k^{\frac{n-2}{2}}(x) C_l^{\frac{n-2}{2}}(x)\,dx = \frac{2^{3-n}\pi\Gamma(k+n-2)}{k!(k+\frac{n-2}{2})\Gamma(\frac{n-2}{2})^2}\,\delta_{kl}\,.$$

Since the degree of $C_k^{\frac{n-2}{2}}$ is precisely k, it follows from the approximation theorem of Weierstrass that the system $\left(C_k^{\frac{n-2}{2}}\right)_k$ is complete. Hence (16) becomes a Fourier expansion if the coefficients are taken such that

$$a_k = \frac{1}{\sin^k R \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2\frac{R}{2})} \cdot \frac{k!(k+\frac{n-2}{2})\Gamma(\frac{n-2}{2})^2}{2^{3-n}\pi\Gamma(k+n-2)} \cdot \int_0^{\pi} f(\theta) C_k^{\frac{n-2}{2}}(\cos\theta) \sin^{n-2}\theta\,d\theta$$

for $k \in \mathbb{N}\cup\{0\}$.[2] Substitution into (15) and interchange of summation and inte-

[2] A Fourier expansion (16) converges uniformly on closed subintervals of $0 < \theta < \pi$ whenever the function f is piecewise continuously differentiable with a bounded derivative [10, §13]. However, we shall not need this fact.

gration provides:

$$H^f(r,\theta) = \frac{\Gamma(\frac{n-2}{2})^2}{2^{3-n}\pi} \cdot$$
$$\cdot \int_0^\pi \sum_{k=0}^\infty \frac{k!(k+\frac{n-2}{2})}{\Gamma(k+n-2)} \frac{\sin^k r \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{r}{2})}{\sin^k R \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{R}{2})} C_k^{\frac{n-2}{2}}(\cos\theta) C_k^{\frac{n-2}{2}}(\cos\theta') \cdot$$
$$\cdot f(\theta') \sin^{n-2}\theta' d\theta' .$$

On the other hand, in virtue of the invariance properties of the Poisson kernel, it follows from (2) and (3) that

$$H^f(r,0) = \frac{\Omega_{n-1}}{\Omega_n} \int_0^\pi P^{S^n}_{B_R(o)}(r,\theta') f(\theta') \sin^{n-2}\theta' d\theta' , \tag{17}$$

where Ω_k stands for the area of the unit sphere in $\mathbb{R}^k$: $\Omega_k = \frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}$. Comparison of the last two formulas gives:

$$P^{S^n}_{B_R(o)}(r,\theta) = \frac{\Omega_n}{\Omega_{n-1}} \cdot \frac{\Gamma(\frac{n-2}{2})^2}{2^{3-n}\pi} \cdot \tag{18}$$
$$\cdot \sum_{k=0}^\infty \binom{k+n-3}{k} \frac{k!(k+\frac{n-2}{2})}{\Gamma(k+n-2)} \cdot \frac{\sin^k r \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{r}{2})}{\sin^k R \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{R}{2})} C_k^{\frac{n-2}{2}}(\cos\theta)$$
$$= \sum_{k=0}^\infty \left(1 + \frac{2k}{n-2}\right) \cdot \frac{\sin^k r \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{r}{2})}{\sin^k R \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{R}{2})} C_k^{\frac{n-2}{2}}(\cos\theta)$$

(we have used the relation $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z)$). This remarkable result will be now proved correctly.

Theorem 1. *The Poisson kernel $P^{S^n}_{B_R(o)}(r,\theta)$ is given by* (18).

Proof: If the boundary function f is a polynomial in $\cos\theta$, then the Fourier series (16) is a finite sum, and so is (15), which presents the solution to the Dirichlet problem. So in this case it remains to justify the interchange of sum and integral in (17), after (18) is substituted for the Poisson kernel. To this end we shall show that (18) converges for every fixed $r < R$ uniformly in $0 \le \theta \le \pi$.

The hypergeometric functions involved in (18) can be estimated in the following two ways. With the standard notation $(\alpha)_0 := 1$, $(\alpha)_j := \alpha(\alpha+1)\ldots(\alpha+j-1)$

($j \in \mathbb{N}$) it holds for $x \geq 0$:

$$F\left(k, k+n-1; \frac{n}{2}+k\,; x\right) = \sum_{j=0}^{\infty} \frac{(k)_j (k+n-1)_j}{(\frac{n}{2}+k)_j} \cdot \frac{x^j}{j!}$$

$$\leq \sum_{j=0}^{\infty} \frac{(k+n-1)_j}{j!} x^j = (1-x)^{1-n-k},$$

$$F\left(k, k+n-1; \frac{n}{2}+k\,; x\right) = \sum_{j=0}^{\infty} \frac{(k)_j (k+n-1)_j}{(\frac{n}{2}+k)_j} \cdot \frac{x^j}{j!} \geq \sum_{j=0}^{\infty} \frac{(k)_j}{j!} x^j = (1-x)^{-k}.$$

By these estimates, we conclude that

$$\frac{\sin^k r \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{r}{2})}{\sin^k R \cdot F(k, k+n-1; \frac{n}{2}+k\,; \sin^2 \frac{R}{2})}$$

$$\leq \frac{\sin^k r \cdot \cos^{2-2n-2k} \frac{r}{2}}{\sin^k R \cdot \cos^{-2k} \frac{R}{2}} = \left(\frac{\tan \frac{r}{2}}{\tan \frac{R}{2}}\right)^k \cos^{2-2n} \frac{r}{2}\,.$$

On the other hand,

$$\left| C_k^{\frac{n-2}{2}}(\cos\theta) \right| \leq C_k^{\frac{n-2}{2}}(1) = \binom{k+n-3}{k} \tag{19}$$

for $0 \leq \theta \leq \pi$ (see [15, V.7]), and the absolute uniform convergence of (18) is herewith established.

The existence of the Poisson kernel was inferred by general arguments in the previous section. Its mere dependence on the variables r and θ was seen in the beginning of the present section. Its uniqueness is due to the uniqueness of the Dirichlet solution (for continuous boundary values). Since it acts like (18) on polynomials, the theorem follows by the approximation theorem of Weierstrass. □

The treatment of the equation $\Delta H^f = 0$ in the hyperbolic case is completely similar. Of course, since the local geometry of H^n is that of a sphere of "imaginary" radius i, it suffices to replace r and R by $\frac{r}{\mathrm{i}}$ and $\frac{R}{\mathrm{i}}$ in (18), and the hyperbolic Poisson kernel will be obtained. But for the sake of a rigourous proof we briefly describe the main steps in solving the equation $\Delta H^f = 0$.

The equation states (cf. (1)):

$$\Delta H^f = \frac{\partial^2 H^f}{\partial r^2} + (n-1)\coth r \cdot \frac{\partial H^f}{\partial r} + \frac{1}{\sinh^2 r}\left[\frac{\partial^2 H^f}{\partial \theta^2} + (n-2)\cot\theta \cdot \frac{\partial H^f}{\partial \theta}\right] = 0.$$

Separation of the variables in the form $u(r,\theta) = U(r)\cdot\Phi(\theta)$ leads to the same equation (6) for Φ, while for U we get:

$$\sinh^2 r\cdot U'' + (n-1)\sinh r\cosh r\cdot U' - \lambda U = 0.$$

Again, λ is of the form $k(k+n-2)$, $k\in\mathbb{N}\cup\{0\}$. The change of variables $z = -\sinh^2\frac{r}{2}$, $Q(z) = U(r)$ gives

$$4z^2(1-z)^2Q'' + 2nz(1-z)(1-2z)Q' - k(k+n-2)Q = 0,$$

from which by setting $Q(z) = (-z)^{\frac{k}{2}}(1-z)^{\frac{k}{2}}\tilde{Q}(z)$ we obtain

$$z(1-z)\tilde{Q}'' + \left[\frac{n}{2} + k - (2k+n)z\right]\tilde{Q}' - k(k+n-1)\tilde{Q} = 0,$$

the same hypergeometric equation as in the spherical case. Taking $\tilde{Q}(z) = 2^k\cdot F\left(k, k+n-1; \frac{n}{2}+k\,; z\right)$ we conclude that

$$\begin{aligned} U = U_k(r) &= \sinh^k r\cdot F\left(k, k+n-1; \frac{n}{2}+k\,; -\sinh^2\frac{r}{2}\right)\\ &= 2^k\tanh^k\frac{r}{2}\cdot F\left(k, 1-\frac{n}{2}\,; \frac{n}{2}+k\,; \tanh^2\frac{r}{2}\right), \end{aligned}$$

for $k\geq 1$ and even n a polynomial of degree $k+n-2$ in $\tanh\frac{r}{2}$.

Assuming for H^f a representation of the form

$$H^f(r,\theta) = \sum_{k=0}^{\infty} a_k U_k(r)\Phi_k(\theta)\,,\quad \Phi_k(\theta) = C_k^{\frac{n-2}{2}}(\cos\theta),$$

such that for $r=R$ the right hand side is the Fourier series of f with respect to the system $\left(C_k^{\frac{n-2}{2}}\right)_k$, the same procedure leads to the Poisson kernel:

$$\text{(20)}\quad P_{B_R(o)}^{H^n}(r,\theta) = \\ = \sum_{k=0}^{\infty}\left(1+\frac{2k}{n-2}\right)\cdot\frac{\sinh^k r\cdot F(k,k+n-1;\frac{n}{2}+k\,;-\sinh^2\frac{r}{2})}{\sinh^k R\cdot F(k,k+n-1;\frac{n}{2}+k\,;-\sinh^2\frac{R}{2})}C_k^{\frac{n-2}{2}}(\cos\theta).$$

Theorem 2. *The Poisson kernel* $P_{B_R(o)}^{H^n}(r,\theta)$ *is given by* (20).

PROOF: Since the hypergeometric function $F(\alpha,\beta;\gamma;x)$ is symmetric in α and β, it follows from (14):

$$\begin{aligned} &\sinh^k r\cdot F\left(k, k+n-1; \frac{n}{2}+k\,; -\sinh^2\frac{r}{2}\right)\\ &\qquad = 2^k\tanh^k\frac{r}{2}\cdot F\left(\frac{n}{2}, k+n-1; \frac{n}{2}+k\,; \tanh^2\frac{r}{2}\right)\cdot\cosh^{2-2n}\frac{r}{2}, \end{aligned}$$

whence

$$\frac{\sinh^k r\cdot F\left(k,k+n-1;\frac{n}{2}+k\,;-\sinh^2\frac{r}{2}\right)}{\sinh^k R\cdot F\left(k,k+n-1;\frac{n}{2}+k\,;-\sinh^2\frac{R}{2}\right)}$$
$$=\left(\frac{\tanh\frac{r}{2}}{\tanh\frac{R}{2}}\right)^k\cdot\frac{F\left(\frac{n}{2},k+n-1;\frac{n}{2}+k\,;\tanh^2\frac{r}{2}\right)}{F\left(\frac{n}{2},k+n-1;\frac{n}{2}+k\,;\tanh^2\frac{R}{2}\right)}\cdot\left(\frac{\cosh\frac{R}{2}}{\cosh\frac{r}{2}}\right)^{2n-2}$$
$$\le\left(\frac{\tanh\frac{r}{2}}{\tanh\frac{R}{2}}\right)^k\cdot\left(\frac{\cosh\frac{R}{2}}{\cosh\frac{r}{2}}\right)^{2n-2}.$$

The convergence properties of (20) and the rest of the proof now follow as in the case of Theorem 1. □

The method of separation of variables in the equation $\Delta H^f=0$ applies, of course, in the euclidean situation too (cf. [3] for dimension $n=3$). Equation (6) is there the same whereas (5) has to be replaced by

$$r^2\cdot U''+(n-1)r\cdot U'-k(k+n-2)U=0$$

with solution r^k (bounded in a neighbourhood of $r=0$). We conclude for the Poisson kernel for $B_R(o)$:

$$P^{\mathbb{R}^n}_{B_R(o)}(r,\theta)=$$
$$=\sum_{k=0}^{\infty}\left(1+\frac{2k}{n-2}\right)\cdot\left(\frac{r}{R}\right)^k C_k^{\frac{n-2}{2}}(\cos\theta)=\frac{R^2-r^2}{(R^2+r^2-2Rr\cos\theta)^{\frac{n}{2}}}R^{n-2},$$

having made use of the generating function for the Gegenbauer polynomials: $\sum_{k=0}^{\infty}C_k^{\frac{n-2}{2}}(x)z^k=(1-2xz+z^2)^{\frac{2-n}{2}}$ for $|z|<1$ (see [15, V.7]).

The last expression makes it natural to ask whether (18) and (20) can be simplified as well. An answer (in a certain sense) is given in the next section.

4. In quest of a simpler expression for the Poisson kernel

It has just been mentioned that if the background space is euclidean, the Poisson kernel for the ball $B_R(o)$ takes the simpler form

$$P^{\mathbb{R}^n}_{B_R(o)}(z,x)=$$
$$=\frac{R^2-|z-o|^2}{[R^2+|z-o|^2-2R|z-o|\cos\angle(zox)]^{\frac{n}{2}}}\cdot R^{n-2}=\frac{R^2-|z-o|^2}{|x-z|^n}\cdot R^{n-2}.$$

Hence $P^{\mathbb{R}^n}_{B_R(o)}(z,x)$ is a function of $|z-o|$ and $|x-z|$ in which these variables are *separated*. On the other hand, by virtue of their invariance properties as exhibited in the beginning of the previous section, both kernels $P^{S^n}_{B_R(o)}(z,x)$ and $P^{H^n}_{B_R(o)}(z,x)$ only depend on $r := d(o,z)$ and $t := d(z,x)$ since the latter variable is related to $\theta := \angle(zox)$ according to the laws of cosine (see, e.g., [11, Section 58], [5, VI.3]):

$$\begin{aligned} \cos t &= \cos R \cos r + \sin R \sin r \cos\theta \qquad &\text{for } \ X = S^n, \ \text{ and} \\ \cosh t &= \cosh R \cosh r - \sinh R \sinh r \cos\theta \qquad &\text{for } \ X = H^n\,. \end{aligned} \tag{21}$$

Thus it is natural to ask whether in the non-euclidean Poisson kernels the variables r and t remain separated. The answer is given in the next theorem.

Theorem 3. *The Poisson kernel* $P^X_{B_R(o)}(z,x)$ ($z \in B_R(o)$, $x \in S_R(o)$, $X \in \{S^n, H^n\}$, $n \geq 3$) *cannot be expressed as a product of a function only depending on* $r = d(o,z)$ *with a function only depending on* $t = d(z,x)$ *unless* $X = S^n$ *and* $R = \frac{\pi}{2}$ (*the half sphere case*), *where it holds:*

$$\begin{aligned} P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,x) &= \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)\sqrt{\pi}} \cos r \cdot F\left(n, 1; \frac{n}{2}+1; \cos^2\frac{t}{2}\right) \\ &= \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)\sqrt{\pi}} \cdot \frac{\cos r}{\sin^n \frac{t}{2}} \cdot F\left(1-\frac{n}{2}, \frac{n}{2}; \frac{n}{2}+1; \cos^2\frac{t}{2}\right)\,. \end{aligned}$$

(*The formulae hold in the two-dimensional case too, cf. Section* 2).

The proof will not be completed before the end of this section. The following lemma will be needed.

Lemma 1. *For every* $\theta \in (0,\pi]$ *we have* $\lim_{\substack{r\to R\\ r<R}} P^X_{B_R(o)}(r,\theta) = 0$ *whereas* $\lim_{\substack{r\to R\\ r<R}} P^X_{B_R(o)}(r,0) = +\infty$.

Proof: At first the spherical situation will be considered. From (18) we obtain:

$$\begin{aligned} &P^{S^n}_{B_R(o)}(r,\theta) = \\ &= \sum_{k=0}^{\infty} \left(1+\frac{2k}{n-2}\right) \frac{\sin^k r \cdot F(k,k+n-1;\frac{n}{2}+k\,;\sin^2\frac{r}{2})}{\sin^k R \cdot F(k,k+n-1;\frac{n}{2}+k\,;\sin^2\frac{R}{2})} C_k^{\frac{n-2}{2}}(\cos\theta) \\ &= \sum_{k=0}^{\infty} \left(1+\frac{2k}{n-2}\right) \frac{\sin^k r}{\sin^k R} C_k^{\frac{n-2}{2}}(\cos\theta) \\ &- \sum_{k=0}^{\infty} \left(1+\frac{2k}{n-2}\right) \frac{\sin^k r}{\sin^k R} \left[1 - \frac{F(k,k+n-1;\frac{n}{2}+k\,;\sin^2\frac{r}{2})}{F(k,k+n-1;\frac{n}{2}+k\,;\sin^2\frac{R}{2})}\right] C_k^{\frac{n-2}{2}}(\cos\theta)\,. \end{aligned}$$

Recalling the expression of the euclidean Poisson kernel from the end of last section we observe that the first term tends to zero for $r \to R$. From previous considerations it follows that the second term is an absolutely convergent series. Since

$$0 \leq \frac{\sin^k r}{\sin^k R}\left[1 - \frac{F(k, k+n-1; \frac{n}{2}+k; \sin^2 \frac{r}{2})}{F(k, k+n-1; \frac{n}{2}+k; \sin^2 \frac{R}{2})}\right] \underset{\substack{r \to R\\ r<R}}{\searrow} 0$$

(the derivative[3] with respect to r is negative in a region $R - \delta < r < R$), it converges to zero by Beppo Levi's theorem. The second statement follows easily by means of (19).

We next consider the hyperbolic case. The integral representation

$$F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}\, dt \tag{22}$$

for $0 < \alpha < \gamma$ and $x < 1$ implies that the function

$$r \mapsto F\left(k, k+n-1; \frac{n}{2}+k\,; -\sinh^2 \frac{r}{2}\right)$$

is positive and decreasing. Thus

$$0 \leq \frac{\sinh^k r}{\sinh^k R}\left[\frac{F(k, k+n-1; \frac{n}{2}+k\,; -\sinh^2 \frac{r}{2})}{F(k, k+n-1; \frac{n}{2}+k\,; -\sinh^2 \frac{R}{2})} - 1\right] \underset{\substack{r \to R\\ r<R}}{\searrow} 0\,,$$

again by testing the derivative. The convergence of (20) to zero now follows as above, the second statement too. $\square$

The Poisson kernels (18) and (20) are harmonic functions of the interior point $z \in B_R(o)$. This is a consequence of the harmonicity of the Green function (cf. Section 2) but can also be deduced directly: Differentiation under the sum sign is justified by arguments as those in the proof of Theorem 1 and 2 under the use of the relation

$$\frac{d}{dx}F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1; \gamma+1; x)$$

and its consequence

$$C_k^{\frac{n-2}{2}}(x)' = (n-2)C_{k-1}^{\frac{n}{2}}(x)\,.$$

(In the hyperbolic case we also use the relation

$$F\left(\frac{n}{2}, k+n; \frac{n}{2}+k+1; x\right) \leq F\left(\frac{n}{2}, k+n-1; \frac{n}{2}+k\,; x\right)$$

[3] In fact, $\frac{d}{dx}F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1; \gamma+1; x)$.

for $x \geq 0$).

Assuming now that the Poisson kernel $P^X_{B_R(o)}$ is of the form $u(r,t) = U(r)\cdot V(t)$ we proceed by testing whether such a function can be harmonic. By virtue of Lemma 1, U and V are assumed to be non-constant. For the current study it is necessary to express Δu in the new coordinates r and t. At first the spherical situation will be considered. The relation between the coordinate systems r, θ and r, t is contained in (21). Since the variable r appears in both coordinate systems, $\frac{\partial u}{\partial r}$ has two different meanings. To avoid any confusion, we shall denote by $\frac{{}^\theta\partial u}{\partial r}$ respectively $\frac{{}^t\partial u}{\partial r}$ that partial derivative, for which θ respectively t is being held constant. We compute:

$$\frac{{}^\theta\partial u}{\partial r} = \frac{{}^t\partial u}{\partial r} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial r} = \frac{{}^t\partial u}{\partial r} + \frac{\cos R \sin r - \sin R \cos r \cos\theta}{\sin t}\frac{\partial u}{\partial t}\,. \tag{23}$$

Differentiation of (21) also gives

$$\frac{\partial\theta}{\partial r} = \frac{\sin R \cos r \cos\theta - \cos R \sin r}{\sin R \sin r \sin\theta}, \qquad \frac{\partial\theta}{\partial t} = \frac{\sin t}{\sin R \sin r \sin\theta}\,,$$

by use of which it follows from (23):

$$\begin{aligned} \frac{{}^\theta\partial^2 u}{\partial r^2} &= \frac{{}^t\partial^2 u}{\partial r^2} + \frac{\sin^2 R \cos t \sin^2\theta}{\sin^3 t}\frac{\partial u}{\partial t} \\ &\quad + 2\,\frac{\cos R \sin r - \sin R \cos r \cos\theta}{\sin t}\frac{\partial^2 u}{\partial r \partial t} \\ &\quad + \frac{(\cos R \sin r - \sin R \cos r \cos\theta)^2}{\sin^2 t}\frac{\partial^2 u}{\partial t^2}\,. \end{aligned} \tag{24}$$

Moreover, by the use of $\dfrac{\partial t}{\partial\theta} = \left(\dfrac{\partial\theta}{\partial t}\right)^{-1} = \dfrac{\sin R \sin r \sin\theta}{\sin t}$, we obtain

$$\frac{\partial u}{\partial\theta} = \frac{\sin R \sin r \sin\theta}{\sin t}\frac{\partial u}{\partial t}\,, \tag{25}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial\theta^2} &= \frac{\sin^2 R \sin^2 r \sin^2\theta}{\sin^2 t}\frac{\partial^2 u}{\partial t^2} \\ &\quad + \sin R \sin r \cdot \frac{\cos\theta - (\cos R \cos r \cos\theta + \sin R \sin r)\cos t}{\sin^3 t}\frac{\partial u}{\partial t}\,. \end{aligned} \tag{26}$$

Substitution of (23)–(26) into

$$\Delta u = \frac{{}^\theta\partial^2 u}{\partial r^2} + (n-1)\cot r \cdot \frac{{}^\theta\partial u}{\partial r} + \frac{1}{\sin^2 r}\left[\frac{\partial^2 u}{\partial\theta^2} + (n-2)\cot\theta \cdot \frac{\partial u}{\partial\theta}\right]$$

gives at last

$$\Delta u = \\ = \frac{{}^t\partial^2 u}{\partial r^2} + (n-1)\cot r \cdot \frac{{}^t\partial u}{\partial r} + 2\,\frac{\cos R - \cos r \cos t}{\sin r \sin t}\,\frac{\partial^2 u}{\partial r \partial t} + \frac{\partial^2 u}{\partial t^2} + (n-1)\cot t \cdot \frac{\partial u}{\partial t}\,,$$

a symmetric expression in r and t.[4]

Now if $u(r,t) = U(r)\cdot V(t)$ is harmonic, we have

$$\Delta u = [U'' + (n-1)\cot r \cdot U']\cdot V + 2\,\frac{\cos R - \cos r \cos t}{\sin r \sin t} U' \cdot V' \\ + U\cdot [V'' + (n-1)\cot t \cdot V'] = 0,$$

which implies that the expression

$$\frac{U''}{U} + (n-1)\cot r \cdot \frac{U'}{U} + 2\,\frac{\cos R - \cos r \cos t}{\sin r \sin t} \cdot \frac{U'}{U} \cdot \frac{V'}{V} \tag{27}$$

does not depend on r. Hence

$$\frac{d}{dr}\left[\frac{U''}{U} + (n-1)\cot r \cdot \frac{U'}{U}\right] + 2\,\frac{\cos t - \cos R \cos r}{\sin^2 r \sin t} \cdot \frac{U'}{U} \cdot \frac{V'}{V} \\ +\ 2\,\frac{\cos R - \cos r \cos t}{\sin r \sin t} \cdot \frac{U''U - (U')^2}{U^2} \cdot \frac{V'}{V} = 0$$

which means that

$$\frac{V'}{V \sin t}\left[\frac{\cos R \cos r - \cos t}{\sin r} U' + (\cos r \cos t - \cos R)\frac{U''U - (U')^2}{U}\right] \tag{28}$$

does not depend on t. This implies that in the expression within the square brackets, denoted in the sequel by $A(r,t)$, r and t are actually separated. If $A \equiv 0$, the coefficient of $\cos t$ should vanish identically. This would imply that $U(r) = a|\cos r|^b$ $(a, b \in \mathbb{R})$, which would contradict Lemma 1 in the case $P^{S^n}_{B_R(o)} = U\cdot V$ that we always have in mind unless $R = \frac{\pi}{2}$. For the time being we assume $R \neq \frac{\pi}{2}$. If $\frac{\partial A}{\partial t} \equiv 0$, we have from (28): $\frac{d}{dt}\frac{V'}{V \sin t} = 0$. However, the general solution $V(t) = a e^{b \cos t}$ $(a, b \in \mathbb{R})$ would again contradict the lemma. Thus

$$\frac{A \sin t}{\frac{\partial A}{\partial t}} = -\cos t + \cos R \cdot \frac{[U''U - (U')^2]\sin r - U'U \cos r}{[U''U - (U')^2]\sin r \cos r - U'U}\,,$$

[4] By the law of cosine (21), the coefficient of the mixed derivative is equal to $2\cos\angle(ozx)$. This is a common fact for the spherical, hyperbolic, and euclidean case.

and it only depends on t.[5] If $R \neq \frac{\pi}{2}$, the coefficient of $\cos R$ has to be constant. This can only happen if U is of the form $a|1 - c\cos r|^b$ or $ae^{c\cos r}$ $(a, b, c \in \mathbb{R})$. To satisfy Lemma 1 we impose the condition $\lim_{\substack{r\to R\\ r<R}} U(r) = 0$, so $U(r) = a(\cos r - \cos R)^\alpha$ with $\alpha > 0$. If this is substituted into (27), henceforth denoted by $G(r,t)$, we obtain

$$G(r,t) = \frac{\alpha}{(\cos r - \cos R)^2} \cdot \\ \cdot \left[(\alpha - 1)\sin^2 r - (\cos r - \cos R)\left(n\cos r + 2\,\frac{\cos R - \cos r\cos t}{\sin t} \cdot \frac{V'}{V}\right)\right],$$

which should not depend on r. Letting $r \to R$ we infer $\alpha = 1$. Then

$$\begin{aligned} G(r,t) &= \frac{1}{\cos r - \cos R}\left(-n\cos r - 2\,\frac{\cos R - \cos r\cos t}{\sin t} \cdot \frac{V'}{V}\right) \\ &= -n - \frac{\cos R}{\cos r - \cos R}\left(n + 2\,\frac{1 - \cos t}{\sin t} \cdot \frac{V'}{V}\right) + 2\cot t \cdot \frac{V'}{V}, \end{aligned}$$

so the expression in the parentheses should vanish, which implies that V is proportional to $(1 - \cos t)^{-\frac{n}{2}} = 2^{-\frac{n}{2}} \sin^{-n} \frac{t}{2}$. On the other hand, $G(r,t)$ is a part of Laplace's equation, so it must hold

$$-n + 2\cot t \cdot \frac{V'}{V} = G(r,t) = -\frac{V''}{V} - (n-1)\cot t \cdot \frac{V'}{V},$$

which is equivalent to $n = 2$, a contradiction $(n \geq 3)$ (justifying, anyway, the expression for the two-dimensional Poisson kernel as it was given at the end of Section 2). Thus $R = \frac{\pi}{2}$ remains as the only possibility for the Poisson kernel to have its variables r and t separated. That this is indeed so, will now be demonstrated.

Let $R = \frac{\pi}{2}$. If $P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,x) = U(r) \cdot V(t)$, we infer from (28) that $\frac{V'\cos t}{V\sin t}$ is constant unless $A(r,t) \equiv 0$. The first relation would imply that $V(t) = a|\cos t|^b$ $(a, b \in \mathbb{R})$, which would contradict Lemma 1 for $t \to 0$. So $A(r,t) \equiv 0$, that is,

$$\left[U''U - (U')^2\right]\cos r \sin r - U'U = 0.$$

The general (non-constant) solution is $U(r) = a\cos^\alpha r$. Because of the lemma it must hold: $\alpha > 0$. As above, we finally conclude that $\alpha = 1$. The function $V(t)$ will now be determined from Laplace's equation directly:

$$\begin{aligned} 0 = \Delta\left[\cos r \cdot V(t)\right] &= \left[V'' + (n+1)\cot t \cdot V' - nV\right] \cdot \cos r \\ \iff V'' + (n+1)\cot t \cdot V' - nV &= 0. \end{aligned} \tag{29}$$

[5]Here and in the sequel we have tacitly assumed that U and V are analytic. This is a consequence of the analyticity of every harmonic function (cf. [9, Theorem 19,I]).

The change of variables $x = \cos^2 \frac{t}{2}$, $G(x) = V(t)$ transforms (29) to

$$x(1-x)G'' + \left[\frac{n}{2} + 1 - (n+2)x\right] G' - nG = 0, \tag{30}$$

a hypergeometric differential equation (10) with $\alpha = n$, $\beta = 1$, $\gamma = \frac{n}{2} + 1$. We take

$$V(t) = G(\cos^2 \frac{t}{2}) = F\left(n, 1; \frac{n}{2} + 1; \cos^2 \frac{t}{2}\right) = \sin^{-n} \frac{t}{2} \cdot F\left(1 - \frac{n}{2}, \frac{n}{2}; \frac{n}{2} + 1; \cos^2 \frac{t}{2}\right),$$

having used the relation $F(\alpha, \beta; \gamma; x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta; \gamma; x)$ for $x < 1$. This solution of (30) indeed satisfies the unboundedness condition for $t \to 0$. We therefore should have

$$P^{S^n}_{B_{\frac{\pi}{2}}(o)}(r,t) = c_n \cdot \cos r \cdot F\left(n, 1; \frac{n}{2} + 1; \cos^2 \frac{t}{2}\right), \quad c_n \in \mathbb{R}. \tag{31}$$

The coefficient c_n is calculated by setting $r = 0$ ($\Rightarrow t = \frac{\pi}{2}$):

$$c_n = F\left(n, 1; \frac{n}{2} + 1; \frac{1}{2}\right)^{-1} = F\left(\frac{n}{2}, \frac{1}{2}; \frac{n}{2} + 1; 1\right)^{-1} = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2} + 1)\sqrt{\pi}},$$

where we have made use of the identity

$$F\left(\alpha, \beta; \alpha + \beta + \frac{1}{2}; x\right) = F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; \frac{1 - \sqrt{1-x}}{2}\right) \tag{32}$$

for $x < 1$, $\alpha + \beta + \frac{1}{2} \notin \mathbb{Z}_-$ as well as the property

$$\lim_{\substack{x \to 1 \\ x < 1}} F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \quad \text{in the case } \gamma - \alpha - \beta > 0. \tag{33}$$

We now proceed to give a rigourous proof of the result we have obtained somewhat heuristically. The arguments rely on the following lemma and are completely analogous to those in the euclidean case (cf. [7, Chapter 4, Section 3]).

Lemma 2. *Let $Q(z,x)$ be the right hand side of* (31) ($r = d(o,z)$, $t = d(z,x)$). *It holds:*

(a) *Q is analytic on $B_{\frac{\pi}{2}}(o) \times S_{\frac{\pi}{2}}(o)$ and harmonic in the first variable;*
(b) *Q is positive on $B_{\frac{\pi}{2}}(o) \times S_{\frac{\pi}{2}}(o)$;*
(c) $\frac{1}{V(S_{\frac{\pi}{2}}(o))} \int_{S_{\frac{\pi}{2}}(o)} Q(z,x)\, dV(x) = 1$ *for every $z \in B_{\frac{\pi}{2}}(o)$;*
(d) *for every $x_0 \in S_{\frac{\pi}{2}}(o)$, $\lim_{z \to x_0} Q(z,x) = 0$ uniformly in x for $d(x_0,x) \geq \delta > 0$.*

Proof of Lemma 2: Properties (a), (b) and (d) are obvious.

As it was computed for (17), in the by now standard notation we have:

$$\frac{1}{V(S_{\frac{\pi}{2}}(o))}\int_{S_{\frac{\pi}{2}}(o)} Q(z,x)\,dV(x) = \frac{\Omega_{n-1}}{\Omega_n}\int_0^\pi Q(z,x)\sin^{n-2}\theta\,d\theta$$

$$= \frac{c_n\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\pi}}\cos r\cdot\int_0^\pi F\left(n,1;\frac{n}{2}+1;\frac{1+\sin r\cos\theta}{2}\right)\sin^{n-2}\theta\,d\theta \qquad \text{by (21)}$$

$$= \frac{2\Gamma(\frac{n+1}{2})}{n\pi\Gamma(\frac{n-1}{2})}\cos r\cdot\sum_{j=0}^\infty\frac{(n)_j}{(\frac{n}{2}+1)_j}\int_0^\pi\left(\frac{1+\sin r\cos\theta}{2}\right)^j\sin^{n-2}\theta\,d\theta$$

(the hypergeometric series converges uniformly, since $\frac{1+\sin r\cos\theta}{2}$ is bounded away from 1)

$$= \frac{2\Gamma(\frac{n+1}{2})}{n\pi\Gamma(\frac{n-1}{2})}\cos r\cdot\sum_{j=0}^\infty\frac{(n)_j}{2^j(\frac{n}{2}+1)_j}\sum_{l=0}^j\binom{j}{l}\sin^l r\int_0^\pi\cos^l\theta\sin^{n-2}\theta\,d\theta =: M.$$

But

$$\int_0^\pi\cos^l\theta\sin^{n-2}\theta\,d\theta = [1+(-1)^l]\int_0^{\frac{\pi}{2}}\cos^l\theta\sin^{n-2}\theta\,d\theta$$

$$= \frac{1+(-1)^l}{2}\int_0^1(1-s)^{\frac{l-1}{2}}s^{\frac{n-3}{2}}\,ds \quad (s=\sin^2\theta)$$

$$= \frac{1+(-1)^l}{2}B\left(\frac{l+1}{2},\frac{n-1}{2}\right) = \frac{1+(-1)^l}{2}\cdot\frac{\Gamma(\frac{l+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{l+n}{2})}$$

(see, e.g., [8, §1.5]), so

$$M = \frac{2\Gamma(\frac{n+1}{2})}{n\pi}\cos r\cdot\sum_{j=0}^\infty\frac{(n)_j}{2^j(\frac{n}{2}+1)_j}\sum_{l=0}^j\binom{j}{l}\frac{1+(-1)^l}{2}\cdot\frac{\Gamma(\frac{l+1}{2})}{\Gamma(\frac{l+n}{2})}\sin^l r$$

$$= \frac{2\Gamma(\frac{n+1}{2})}{n\pi}\cos r\cdot\sum_{l=0}^\infty\frac{1+(-1)^l}{2}\cdot\frac{\Gamma(\frac{l+1}{2})}{\Gamma(\frac{l+n}{2})}\cdot\frac{(n)_l}{2^l(\frac{n}{2}+1)_l}\sin^l r\cdot$$

$$\cdot\sum_{j=l}^\infty\frac{(n+l)_{j-l}(1+l)_{j-l}}{(\frac{n}{2}+1+l)_{j-l}}\cdot\frac{1}{2^{j-l}(j-l)!}$$

$$= \frac{2\Gamma(\frac{n+1}{2})}{n\pi}\cos r\cdot$$

$$\cdot\sum_{l=0}^\infty\frac{1+(-1)^l}{2}\cdot\frac{\Gamma(\frac{l+1}{2})}{\Gamma(\frac{l+n}{2})}\cdot\frac{\Gamma(n+l)\Gamma(\frac{n}{2}+1)}{2^l\Gamma(n)\Gamma(\frac{n}{2}+1+l)}\sin^l r\cdot F\left(n+l,1+l;\frac{n}{2}+1+l;\frac{1}{2}\right)$$

$$= \frac{2\Gamma(\frac{n+1}{2})}{n\pi} \cos r \cdot$$

$$\cdot \sum_{l=0}^{\infty} \frac{1+(-1)^l}{2} \cdot \frac{\Gamma(\frac{l+1}{2})}{\Gamma(\frac{l+n}{2})} \cdot \frac{\Gamma(n+l)\Gamma(\frac{n}{2}+1)}{2^l\Gamma(n)\Gamma(\frac{n}{2}+1+l)} \sin^l r \cdot \underbrace{F\left(\frac{n+l}{2}, \frac{1+l}{2}; \frac{n}{2}+1+l; 1\right)}_{= \frac{\Gamma(\frac{n}{2}+1+l)\sqrt{\pi}}{\Gamma(\frac{l}{2}+1)\Gamma(\frac{n+l+1}{2})}}$$

$$= \frac{2^n\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+1)}{n\Gamma(n)\pi} \cos r \cdot \sum_{l=0}^{\infty} \frac{1+(-1)^l}{2} \frac{\Gamma(\frac{l+1}{2})}{\Gamma(\frac{l}{2}+1)} \sin^l r$$

(we have used the relation $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z)$)

$$= \frac{\cos r}{\sqrt{\pi}} \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)} \sin^{2\nu} r = \frac{\cos r}{\sqrt{\pi}} \sum_{\nu=0}^{\infty} \frac{\sqrt{\pi}\Gamma(2\nu+1)\cdot 2^{-2\nu}}{\Gamma(\nu+1)^2} \sin^{2\nu} r$$

$$= \cos r \cdot \sum_{\nu=0}^{\infty} \frac{(2\nu)!}{(\nu!)^2} \left(\frac{\sin^2 r}{4}\right)^{\nu} = \cos r \cdot \sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu} (-\sin^2 r)^{\nu}$$

$$= \cos r \cdot (1-\sin^2 r)^{-\frac{1}{2}} = 1.$$

This proves (c). □

Now let f be an arbitrary continuous function on $S_{\frac{\pi}{2}}(o)$. To prove (31) it remains to show that

$$\lim_{z\to x_0} \frac{1}{V(S_{\frac{\pi}{2}}(o))} \int_{S_{\frac{\pi}{2}}(o)} Q(z,x) f(x)\, dV(x) = f(x_0)$$

for every $x_0 \in S_{\frac{\pi}{2}}(o)$. This follows by a standard argument (see [7, Chapter 4, Section 3]) from statement (d) of Lemma 2 and the continuity of f.

By now we have completed the proof of Theorem 3 in the case $X = S^n$. We briefly discuss the case $X = H^n$, since it is treated analogously.

For $u = u(r,t)$ Laplace's equation states:

$$\Delta u = \frac{{}^t\partial^2 u}{\partial r^2} + (n-1)\coth r \cdot \frac{{}^t\partial u}{\partial r} + 2\frac{\cosh r \cosh t - \cosh R}{\sinh r \sinh t} \frac{\partial^2 u}{\partial r \partial t} + \frac{\partial^2 u}{\partial t^2}$$
$$+ (n-1)\coth t \cdot \frac{\partial u}{\partial t} = 0.$$

From this we infer that $\tilde{u}(\rho,\tau) := u(\mathrm{i}\rho, \mathrm{i}\tau)$ satisfies the spherical equation of Laplace:

$$\frac{{}^\tau\partial^2 \tilde{u}}{\partial \rho^2} + (n-1)\cot\rho \cdot \frac{{}^\tau\partial \tilde{u}}{\partial \rho} + 2\frac{\cos\tilde{R} - \cos\rho\cos\tau}{\sin\rho\sin\tau} \frac{\partial^2 \tilde{u}}{\partial\rho\partial\tau} + \frac{\partial^2 \tilde{u}}{\partial\tau^2} + (n-1)\cot\tau \cdot \frac{\partial\tilde{u}}{\partial\tau} = 0,$$

where we have put $\tilde{R} = \dfrac{R}{\mathrm{i}}$. Since $\tilde{u}$ cannot have its variables separated, the same must hold for u.

At this point, Theorem 3 is established.

Remark 1. Theorems 1 and 3 provide two different expressions for the Poisson kernel $P^{S^n}_{B_{\frac{\pi}{2}}(o)}$. By comparison and under the use of (32) and (33) we obtain the following interesting identity in the context of special functions:

$$\sum_{k=0}^{\infty} \frac{\Gamma(\frac{n+k}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n}{2}+k-1)} \sin^k r \cdot F\left(k, k+n-1; \frac{n}{2}+k; \sin^2\frac{r}{2}\right) C_k^{\frac{n-2}{2}}(\cos\theta)$$
$$\equiv \frac{\Gamma(\frac{n+1}{2})(n-2)}{2\Gamma(\frac{n}{2}+1)} \cos r \cdot F\left(n, 1; \frac{n}{2}+1; \frac{1+\sin r\cos\theta}{2}\right).$$

5. The Dirichlet problem for the real projective space P^n

In this section we solve the Dirichlet problem for the projective space by adapting to it the Poisson integral for a half sphere, which was computed in the last section. We are thus led to a surprisingly simple expression for the projective Poisson kernel.

Let $n \geq 2$, P^n be the n-dimensional real projective space, $p : S^n \to P^n$ the canonical projection identifying the endpoints of the diameters. We introduce a Riemannian metric on P^n such that p becomes an isometry. The diameter of P^n is then equal to $\frac{\pi}{2}$ and the space can be identified with a closed half sphere $\overline{B_{\frac{\pi}{2}}(o)} = B_{\frac{\pi}{2}}(o) \cup S_{\frac{\pi}{2}}(o) \subseteq S^n$ where opposite points on the boundary $S_{\frac{\pi}{2}}(o)$ are being identified. Given a continuous and even function $f : S_{\frac{\pi}{2}}(o) \to \mathbb{R}$, we now ask for a continuous extension $H^f_{B_{\frac{\pi}{2}}(o)} : \overline{B_{\frac{\pi}{2}}(o)} \to \mathbb{R}$, harmonic on $B_{\frac{\pi}{2}}(o)$. Obviously, it is given by the Poisson integral of the previous section. It holds:

$$\begin{aligned}
H^f_{B_{\frac{\pi}{2}}(o)}(z) &= \frac{1}{V(S_{\frac{\pi}{2}}(o))} \int_{S_{\frac{\pi}{2}}(o)} P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,x) f(x)\, dV(x) \\
&= \frac{1}{V(S_{\frac{\pi}{2}}(o))} \int_{S_{\frac{\pi}{2}}(o)} P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,-x) \underbrace{f(-x)}_{=f(x)}\, dV(x) \\
&= \frac{1}{V(S_{\frac{\pi}{2}}(o))} \int_{S_{\frac{\pi}{2}}(o)} \frac{P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,x) + P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,-x)}{2} f(x)\, dV(x) \\
&= \frac{1}{V(\frac{1}{2}S_{\frac{\pi}{2}}(o))} \int_{\frac{1}{2}S_{\frac{\pi}{2}}(o)} \frac{P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,x) + P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,-x)}{2} f(x)\, dV(x),
\end{aligned}$$

where the half sphere "$\frac{1}{2}S_{\frac{\pi}{2}}(o)$" is taken such that $d(z,x) \le \frac{\pi}{2}$ for all $x \in \frac{1}{2}S_{\frac{\pi}{2}}(o)$. We proceed to compute the kernel. Setting $t = d(z,x)$, $t' = d(z,-x)$ we have $t + t' = \pi$, so $\cos\frac{t'}{2} = \sin\frac{t}{2}$ and

$$F\left(n,1;\frac{n}{2}+1;\cos^2\frac{t'}{2}\right) = F\left(n,1;\frac{n}{2}+1;1-\cos^2\frac{t}{2}\right). \tag{34}$$

According to the relation

$$\begin{aligned}F(\alpha,\beta;\gamma;x) &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;1+\alpha+\beta-\gamma;1-x)\\ &\quad + (1-x)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta;1-\alpha-\beta+\gamma;1-x)\end{aligned}$$

for $0 < x < 1$ if $\alpha+\beta-\gamma \notin \mathbb{Z}$, the right hand side of (34) equals

$$\begin{aligned}&\frac{\Gamma(\frac{n}{2}+1)\Gamma(-\frac{n}{2})}{\Gamma(1-\frac{n}{2})\Gamma(\frac{n}{2})}F\left(n,1;\frac{n}{2}+1;\cos^2\frac{t}{2}\right)\\ &\quad + \cos^{-n}\frac{t}{2}\cdot\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2})}{\Gamma(n)}\cdot F\left(1-\frac{n}{2},\frac{n}{2};1-\frac{n}{2};\cos^2\frac{t}{2}\right)\\ &= -F\left(n,1;\frac{n}{2}+1;\cos^2\frac{t}{2}\right) + \frac{2^n\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1)}{\Gamma(n)}\sin^{-n} t\end{aligned}$$

($F(\alpha,\beta;\alpha;x) = (1-x)^{-\beta}$). By Theorem 3, we conclude:

$$\begin{aligned}&\frac{P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,x) + P^{S^n}_{B_{\frac{\pi}{2}}(o)}(z,-x)}{2}\\ &\quad = \frac{\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{n}{2}+1)\sqrt{\pi}}\cos r\cdot\left[F\left(n,1;\frac{n}{2}+1;\cos^2\frac{t}{2}\right) + F\left(n,1;\frac{n}{2}+1;\cos^2\frac{t'}{2}\right)\right]\\ &\quad = \frac{2^{n-1}\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})}{\Gamma(n)\sqrt{\pi}}\cos r\cdot\sin^{-n} t = \cos r\cdot\sin^{-n} t \quad (!).\end{aligned}$$

We have thus proved

Theorem 4. *Let $o \in P^n$. The solution to the Dirichlet problem for the data $(B_{\frac{\pi}{2}}(o), f)$ is given by a Poisson integral* (3) *with the Poisson kernel*

$$P^{P^n}_{B_{\frac{\pi}{2}}(o)}(z,x) = \frac{\cos r}{\sin^n t} \tag{35}$$

where $r = d(o,z)$, $t = d(z,x)$.

6. The Dirichlet problem for the real hyperbolic space H^n

The Dirichlet problem for the entire hyperbolic space H^n has been extensively studied in the past. A Poisson integral for the solution can be found in [1]. In this section we show that, as one expects, the Poisson kernel $P^{H^n}_{B_R(o)}$ of Section 3 converges for $R \to \infty$ and so the limit must be equal to the kernel in [1]. We are thus led to another remarkable identity in the context of special functions.

Let $o \in H^n$ $(n \geq 2)$, S_o^{n-1} be the unit sphere in the tangent space T_oH^n and $f : S_o^{n-1} \to \mathbb{R}$ a continuous function. The Dirichlet problem lies in the determination of a harmonic function $h : H^n \to \mathbb{R}$ with the property

$$\lim_{\substack{s\to+\infty \\ S_o^{n-1} \ni v \to v_0}} h(\exp_o(sv)) = f(v_0) \tag{36}$$

for every $v_0 \in S_o^{n-1}$. In contrast to the euclidean situation, in this case it is always a uniquely solvable problem.

If we formally let $R \to \infty$ in (20), we obtain by means of (14) and (33):

$$\begin{aligned} &\lim_{R\to\infty} P^{H^n}_{B_R(o)}(r,\theta) = \\ &\qquad = \sum_{k=0}^{\infty} \frac{\binom{k+n-2}{k}}{\binom{k+\frac{n-4}{2}}{k}} \tanh^k \frac{r}{2} \cdot F\left(k, 1-\frac{n}{2}; \frac{n}{2}+k; \tanh^2\frac{r}{2}\right) C_k^{\frac{n-2}{2}}(\cos\theta) \end{aligned} \tag{37}$$

for $n \geq 3$. From the integral representation (22) it is inferred that the function $x \mapsto F\left(k, 1-\frac{n}{2}; \frac{n}{2}+k; x\right)$ is positive and decreasing. This implies that

$$F\left(k, 1-\frac{n}{2}; \frac{n}{2}+k; \tanh^2\frac{r}{2}\right) \leq 1,$$

and because of (19) the right hand side of (37) is majorized by the convergent series

$$\sum_{k=0}^{\infty} \frac{\binom{k+n-2}{k}\binom{k+n-3}{k}}{\binom{k+\frac{n-4}{2}}{k}} \tanh^k \frac{r}{2}\,.$$

Therefore, (37) is indeed convergent. In the two-dimensional case we have by Section 2:

$$\begin{aligned} \lim_{R\to\infty} P^{H^2}_{B_R(o)}(r,\theta) &= \lim_{R\to\infty} \frac{\sinh^2\frac{R}{2} - \sinh^2\frac{r}{2}}{\sinh^2\frac{t}{2}} = \lim_{R\to\infty} \frac{\cosh R - \cosh r}{\cosh t - 1} \\ &= \lim_{R\to\infty} \frac{\cosh R - \cosh r}{\cosh R\cosh r - \sinh R \sinh r\cos\theta - 1} = \frac{1}{\cosh r - \sinh r\cos\theta}\,. \end{aligned}$$

On the other hand, in the conformal ball model $B_1(0) \subseteq \mathbb{R}^n$ for the hyperbolic space the Poisson kernel states (see [1])

$$P^{H^n}(z,v) = \left(\frac{1-|z|^2}{|v-z|^2}\right)^{n-1}$$

for $z \in B_1(0)$, $v \in S_1(0) = S^{n-1}$. In hyperbolic terms, this is equal to

$$(\cosh r - \sinh r \cos\theta)^{1-n} \tag{38}$$

($0 \in \mathbb{R}^n$ is taken for $o \in H^n$, $r = d(o,z) = 2\operatorname{artanh}|z|$, $\theta = \angle(oz, ov)$). We conclude:

Theorem 5. *The solution to the Dirichlet problem for the data* $(H^n = B_\infty(o), f)$ $(n \geq 2)$ *is given by a Poisson integral of the form*

$$\frac{1}{\Omega_n}\int_{S_o^{n-1}} P^{H^n}(z,v) f(v)\, d\mu(v)$$

with the Poisson kernel

$$P^{H^n}(z,v) = \lim_{R\to\infty} P^{H^n}_{B_R(o)}(r,\theta) = (\cosh r - \sinh r\cos\theta)^{1-n}$$

in the usual notation.

Remark 2. Comparison of (37) and (38) leads to the identity

$$\sum_{k=0}^{\infty} \frac{\binom{k+n-2}{k}}{\binom{k+\frac{n-4}{2}}{k}} t^k \cdot F\left(k, 1-\frac{n}{2}; \frac{n}{2}+k; t^2\right) C_k^{\frac{n-2}{2}}(x) = \left(\frac{1-t^2}{1-2tx+t^2}\right)^{n-1}.$$

References

[1] Ahlfors L.V., *Möbius Transformations in Several Dimensions*, Ordway Professorship Lectures in Mathematics, University of Minnesota, Minneapolis, 1981.

[2] Birkhoff G., Rota G.-C., *Ordinary Differential Equations*, Third Edition, Wiley, New York, 1978.

[3] Courant R., Hilbert D., *Methods of Mathematical Physics*, Volume I, First English Edition, Interscience, New York, 1953.

[4] Dörrie H., *Unendliche Reihen*, R. Oldenbourg, Munich, 1951, also Sändig GmbH, Wiesbaden, 1989.

[5] Fenchel W., *Elementary Geometry in Hyperbolic Space*, de Gruyter, Berlin, 1989.

[6] Helgason S., *Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press, Orlando, 1984.

[7] John F., *Partial Differential Equations*, Third Edition, Springer, New York, 1978.

[8] Lebedew N.N., *Spezielle Funktionen und ihre Anwendung*, B. I.-Wissenschaftsverlag, Bibliographisches Institut, Zurich, 1973.

[9] Miranda C., *Partial Differential Equations of Elliptic Type*, Second Revised Edition, Springer, New York, 1970.

[10] Nikiforov A., Ouvarov V., *Eléments de la théorie des fonctions spéciales*, Traduction française Editions Mir, Moscow, 1976.

[11] Sigl R., *Ebene und sphärische Trigonometrie*, Akademische Verlagsgesellschaft, Frankfurt, 1969.

[12] Symeonidis E., *The Poisson Integral as an appropriate mean value*, Analysis **19** (1999), 13–18.

[13] Symeonidis E., *The Poisson Integral for a disk in the hyperbolic plane*, Exposition. Math. **17** (1999), 239–244.

[14] Symeonidis E., *The Dirichlet problem for a disk on the sphere* S^2, Exposition. Math. **17** (1999), 365–370.

[15] Tricomi F.G., *Vorlesungen über Orthogonalreihen*, Second Revised Edition, Springer, Berlin, 1970.

[16] Vignati M, *A geometric property of functions harmonic in a disk*, El. Math. **47** (1992), 33–38.

[17] Vignati M., *On the factorization of Poisson integrals on spheres*, preprint, Università degli studi di Milano, 1990.

[18] Willmore T.J., *Riemannian Geometry*, Oxford Science Publications, Oxford University Press, New York, 1993.

Mathematisch-Geographische Fakultät, Katholische Universität Eichstätt-Ingolstadt, Ostenstrasse 26, 85072 Eichstätt, Germany

E-mail: e.symeonidis@ku-eichstaett.de

(*Received* March 28, 2003)