Mappings on the dyadic solenoid

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Abstract. Answering an open problem in [3] we show that for an even number k, there exist no k to 1 mappings on the dyadic solenoid.

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Suppose that $P = (p_1, p_2, \dots)$ is a sequence of prime numbers. The P-adic solenoid S_P is the inverse limit sequence (S, f_n) where $S \approx \mathbb{R}/\mathbb{Z}$, the circle, and the bonding maps are homomorphisms $f_n(z) = p_n \cdot z \mod 1$. The P-adic solenoid is a compact abelian group. In case P is a constant sequence of 2's, the inverse limit is called the *dyadic solenoid*, denoted by S_2. We shall prove the following result.

Theorem 1. *Suppose that $f\colon S_2 \to S_2$ is a k to 1 map of the dyadic solenoid. Then k is odd.*

This answers a question in [3] and shows that the result in [7] is correct. The main ingredient in our proof is Scheffer's theorem [6] (see [5] for a recent application of Scheffer's theorem).

Theorem 2. *Suppose that G, H are compact and connected groups and that H is abelian. Suppose that $f\colon (G, e) \to (H, e)$ is a continuous map that preserves the unit element. Then f is homotopic to a unique homomorphism. The homotopy preserves the unit element.*

Solenoids have a local product structure of a Cantor set and an arc, [2]. The arc component Γ_e of the unit element e is a dense subgroup that is a $1-1$ homomorphic image of $\mathbb{R}$. The other arc components are translates of Γ.

Proposition 3. *Suppose that $f\colon S_2 \to S_2$ is a non-trivial homomorphism. Then f bijectively maps arc-components onto arc-components.*

Proof: Since f is a homomorphism, it suffices to verify that the restriction to the unit component Γ_e is a bijection. Now Γ_e is an image of $\mathbb{R}$. A non-trivial homomorphism on $\mathbb{R}$ is of the form $x \to rx$ for $r \neq 0$. In particular, it is a bijection. $\square$

Proposition 4. *Suppose that $f: S_2 \to S_2$ is not homotopic to a constant function. Then f maps arc-components onto arc-components.*

PROOF: By composing f with a translation, if necessary, we may assume that f preserves the unit element. By Scheffer's theorem, f is homotopic to a non-trivial homomorphism h. The difference map $h - f: S_2 \to S_2$ has a compact image that is contained in Γ_e. So $f(x) = h(x) + t(x)$ for some $t(x)$ in a compact subset of Γ_e. Since h maps arc-components onto arc-components so does f. □

Under Pontryagin duality, the category of compact abelian groups is contravariantly equivalent to the category of discrete groups. The Pontryagin dual of the dyadic solenoid S_2 is isomorphic to the additive group $Q_2 = \{\frac{k}{2^n}: k \in \mathbb{Z}, n \geq 0\}$, see [4]. Each non-zero element of Q_2 has a unique representation $\frac{k}{2^n}$ for an odd number k and a non-negative integer n.

Lemma 5. *Suppose that $f: S_2 \to S_2$ is not homotopic to a constant map and that Γ is an arc-component. Then $f^{-1}(\Gamma)$ consists of an odd number of arc-components.*

PROOF: As S_2 is homogeneous, we may assume that Γ is the component of e. By the corollary, $f^{-1}(\Gamma)$ is a collection of arc-components that is necessarily the same for all mappings in the homotopy class of f. By Scheffer's theorem, we may assume that f is a homomorphism and we see that the number of components in $f^{-1}(\Gamma)$ is the same for every possible choice of Γ. Consider the dual homomorphism $\hat{f}: Q_2 \to Q_2$. It is determined by the value $\hat{f}(1) = \frac{k}{2^n}$, for some odd number k. The image of $\hat{f}$ is a subgroup of odd index k. By the contravariance of Pontryagin duality, the kernel of f is a subgroup of odd order k. The number of elements in the kernel is equal to the number of arc components by Proposition 3. □

Recall that $f: R \to R$ has a *proper* local maximum in c if there is an open interval I such that $c \in I$ and $f(x) < f(c)$ for all $c \neq x \in I$. A proper local minimum is defined likewise. A proper local extreme is either a maximum or a minimum. The value $f(c)$ is called a proper extreme value.

Proposition 6. *Suppose $f: \mathbb{R} \to \mathbb{R}$ is a continuous map with finite fibers. Then the set of proper extreme values of f is countable.*

PROOF: It suffices to show that the set of proper local maxima is countable. As the fibers of f are finite, f has a proper local maximum in x whenever it has a local maximum in x. For each proper local maximum x, select an interval $I(x)$ with rational endpoints as in the definition of proper local maximum. Note that $I(x) \neq I(y)$ whenever $x \neq y$. The claim now follows as there are only countably many intervals with rational end points. □

Lemma 7. *Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous surjection with finite fibers. Then the parity of $f^{-1}(z)$ is odd for each z that is not a proper extreme value.*

Proof: Suppose z is not a proper extreme value of f. The graph of $y = f(x)$ intersects the horizontal line $y = z$ transversally, in finitely many points. As f is a surjection, we have $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, or the other way around. □

Proof of Theorem 1: Suppose that $f: S_2 \to S_2$ is a continuous k to 1 map. In particular, f is a surjection so it is not homotopic to a constant map. Without loss of generality we may assume that $f(e) = e$. By Lemma 5, $f^{-1}(\Gamma_e)$ consists of an odd number of arc-components. Each of these components is an image of the real line and is mapped surjectively onto Γ_e. As each of these maps can be lifted to $\mathbb{R}$, see [1], this results in an odd number of maps from the real line onto itself. By Lemma 7, outside a countable set of extreme values, each of these maps has fibers of odd parity. Now the sum of an odd number of odd numbers is odd, so k has to be odd. □

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