Perfect mappings in topological groups, cross-complementary subsets and quotients

A.V. Arhangel'skii

Abstract. The following general question is considered. Suppose that G is a topological group, and F, M are subspaces of G such that $G = MF$. Under these general assumptions, how are the properties of F and M related to the properties of G? For example, it is observed that if M is closed metrizable and F is compact, then G is a paracompact p-space. Furthermore, if M is closed and first countable, F is a first countable compactum, and $FM = G$, then G is also metrizable. Several other results of this kind are obtained. An extensive use is made of the following old theorem of N. Bourbaki [5]: if F is a compact subset of a topological group G, then the natural mapping of the product space $G \times F$ onto G, given by the product operation in G, is perfect (that is, closed continuous and the fibers are compact). This fact provides a basis for applications of the theory of perfect mappings to topological groups. Bourbaki's result is also generalized to the case of Lindelöf subspaces of topological groups; with this purpose the notion of a G_δ-closed mapping is introduced. This leads to new results on topological groups which are P-spaces.

Keywords: topological group, quotient group, locally compact subgroup, quotient mapping, perfect mapping, paracompact p-space, metrizable group, countable tightness

Classification: 22A05, 54H11, 54D35, 54D60

In a topological group the product operation generates a variety of continuous mappings. Some of them are well known and play a fundamental role: the translations, for example. However, it is natural to try to treat the subject in a more systematic way. We are going to consider the following general questions. Suppose that G is a topological group, A and B are subsets of G, and f is the mapping of the product space $A \times B$ onto the subspace AB of G given by the product operation in G. Under certain natural restrictions on A and B, what can be said about the properties of the mapping f? How the properties of the subspace AB are related to the properties of the subspaces A and B? We establish some results in this direction, in particular, we show that the theory of perfect mappings, a well developed chapter of General Topology, can be effectively applied to answer some of these questions. Under this approach, the quotient groups also naturally enter the picture. In particular, we observe that a quotient mapping with respect to a locally compact subgroup is always compact-covering.

We follow notation and terminology in [4], [6], and [12]. We assume T_1-separation axiom.

Let G be a topological group, and A and B subsets of G. Let us say that A and B are *cross-complementary* (in G) if $G = AB = \{ab : a, b \in G\}$. Suppose that $\mathcal{P}$ is a topological or algebraic-topological property, and A is a subset of a topological group G. We will say that A *has a $\mathcal{P}$-grasp* on G if there exists a subset B of G such that B is cross-complementary to A and has the property $\mathcal{P}$. In particular, $A \subset G$ has a *compact grasp* on G if there exists a compact subspace B of G such that $AB = G$. Similarly, A has a *metrizable grasp* on G if $G = AB$, for some metrizable subspace B of G.

Proposition 1. *Suppose that G is a topological group, and H a metrizable invariant subgroup of G such that H has a countably tight compact grasp on G. Then G is metrizable.*

Proof: The closure of H in G is also a metrizable invariant subgroup of G (observe that the closure of H in G is first countable, since the space G is regular). Thus, we may assume that H is closed in G. There exists a compact subspace B of G such that $HB = G$ and the tightness of B is countable.

Let us consider now the quotient group G/H and the quotient mapping p. Since $HB = G$, we have $p(B) = G/H$. Therefore, G/H is a compactum of countable tightness. Since G/H is also a topological group, and every compact group of countable tightness is metrizable [4], we conclude that G/H is metrizable. Since H is metrizable as well, it follows from a theorem of M.I. Graev [8] that G is also metrizable. □

The above result will be strengthened below. However, we presented it here and in this form, since the above argument, after an obvious modification, can be turned into a proof of the following statement.

Theorem 2. *Suppose that G is a topological group, and H a closed invariant subgroup of G such that H has a countably tight compact grasp on G and the pseudocharacter of H is countable. Then the pseudocharacter of G is also countable.*

Theorem 3. *Suppose that G is a topological group, and H a closed subgroup of G such that the pseudocharacter of H is countable and H has a countable networkweight grasp B on G. Then the pseudocharacter of G is also countable.*

Proof: Since $HB = G$, we have $p(B) = G/H$. Therefore, the quotient G/H has a countable network, too. Hence, the pseudocharacter of G/H is countable, and therefore, the pseudocharacter of G is countable. □

The next statement is proved in [5, Chapter 3, Section 4, Proposition 1]:

Proposition 4. *Suppose that G is a topological group, and F a compact subspace of G. Then the restriction f of the product mapping $G \times G \to G$ to the subspace $G \times F$ is a perfect mapping of $G \times F$ onto G.*

The mapping f in Proposition 4 is also open, since the translations in G are homeomorphisms.

Corollary 5. *Suppose that G is a topological group, F a compact subspace of G, and M a closed subspace of G. Then the restriction f of the product mapping $G \times G \to G$ to the subspace $M \times F$ is a perfect mapping of $M \times F$ onto a closed subspace MF of G.*

PROOF: To derive this corollary from Proposition 4, we only have to observe that the restriction of a perfect mapping to a closed subspace is again a perfect mapping. □

Note that the perfect mapping of $M \times F$ onto MF in Corollary 5 need not be open. Observe also that the assumption that M is closed cannot be dropped in Corollary 5.

Example 6. Take any compact metrizable group G with a proper dense subgroup M. Then the natural mapping of the space $G \times M$ onto G, generated by the product operation, is not closed, since the closed subset $\{e\} \times M$ of $G \times M$ maps onto the non-closed subset M of G.

Corollary 5 is one of the main technical tools in this article. Of course, it works in combination with other results of the theory of perfect mappings, some of which are quite deep. Here is a typical application of Corollary 5.

Theorem 7. *Suppose that G is a topological group, F a compact subspace of G, and M a closed metrizable subspace of G. Then FM and MF are paracompact p-spaces. In particular, if $G = FM$ or $G = MF$, then G is a paracompact p-space.*

PROOF: Indeed, $M \times F$ is a paracompact p-space (see [1]), and MF is an image of $M \times F$ under a perfect mapping, by Corollary 5. Now it follows from a remarkable theorem of V.V. Filippov [7] that MF is a paracompact p-space as well. □

Theorem 8. *Suppose that G is a topological group, F a compact subspace of G, M a closed subspace of G, and both M and F have countable tightness. Then the tightness of MF is also countable. In particular, if $G = FM$, then the tightness of G is countable.*

PROOF: The tightness of $M \times F$ is countable (see [4]). Perfect mappings do not increase the tightness [1], [4]. Therefore, it follows from Corollary 5 that the tightness of MF is also countable. □

Theorem 9. *Suppose that G is a topological group such that $G = FM$, where F is a compactum of countable tightness and M a metrizable closed subspace of G. Then G is metrizable.*

PROOF: Indeed, by Theorem 7, G is a paracompact p-space. From Theorem 8 it follows that the tightness of G is countable. However, every topological group of countable tightness, which is a paracompact p-space, is metrizable. Indeed, every such group G must contain a compact subgroup H with a countable base of neighbourhoods [12]. Then H is metrizable, since every compact group is a dyadic compactum (see [10], [13]) and every dyadic compactum of countable tightness is metrizable [4], [6]. It follows that G is first countable and therefore, metrizable [12]. □

Corollary 10. *Suppose that G is a topological group such that $G = FM$, where F is a metrizable compactum and M is a metrizable closed subspace of G. Then the space G is metrizable.*

The last statement also follows from Corollary 5 and the well known theorem that perfect mappings preserve metrizability (in the direction of the image) [6], [4].

Corollary 10 can be improved as follows. The class of bisequential spaces contains the class of first countable spaces and is a subclass of the class of sequential spaces. It also includes all images of first countable spaces under perfect mappings. For the definition, see [11].

Theorem 11. *Suppose that G is a topological group such that $G = FM$, where F is a bisequential compactum and M a bisequential closed subspace of G. Then G is metrizable.*

Proof: The space G is an image of the space $M \times F$ under a perfect mapping. The space $M \times F$ is also bisequential [11]. Since every perfect mapping is biquotient, the space G is bisequential as well, by a theorem of E. Michael [11]. Every bisequential topological group is metrizable [3]. Therefore, G is metrizable. □

Corollary 12. *Suppose that G is a topological group such that $G = FM$, where F is a first countable compactum and M a first countable closed subspace of G. Then G is metrizable.*

It is not clear, whether the conclusion will remain true if, in Corollary 12, we replace the assumption that F is a first countable compactum with the weaker assumption that F is a compactum of countable tightness.

The above techniques can also be applied to the study of topological structure of a topological group under certain weaker assumptions than in Theorems 9 and 11. For example, we have:

Theorem 13. *Suppose that G is an Abelian topological group and H is a subgroup of G such that H is algebraically generated by a compact metrizable subspace F, and $G = H + M$, where M is a closed metrizable subspace of G. Then G is the union of a countable family of closed metrizable subspaces.*

Proof: Clearly, we can assume that F is symmetric: $F = F^{-1}$. Now we inductively define a sequence of subspaces M_n of G by the rule: $M_0 = M$, and $M_{n+1} = F + M_n$, for each $n \in \omega$. Since F generates H and $G = H + M$, we have:

$\bigcup\{M_n : n \in \omega\} = G$. Corollaries 5 and 10 guarantee that each M_n is a closed metrizable subspace of G. □

Theorem 14. *Suppose that G is a topological group, F a compact subspace of G, and M a paracompact, closed subspace of G. Then FM is also a paracompact space. In particular, if $G = FM$, then G is paracompact.*

Proof: Since the product of a paracompact space with a compact space is paracompact, this follows from Corollary 5 and the well known theorem that paracompactness is preserved by perfect mappings (see [6], [4]). □

The next result is somewhat unexpected: it shows that certain closed subsets of a topological group with a compact grasp on this group must be paracompact.

Theorem 15. *Suppose that G is a topological group such that $G = FM$, where F is compact and M is a Čech-complete closed subspace of G. Then G is also Čech-complete, and both M and G are paracompact.*

Proof: The product of a Čech-complete space and a compact space is Čech-complete. Therefore, the space $M \times F$ is Čech-complete. Since Čech-completeness is preserved by perfect mappings [9], it follows from Corollary 5 that G is Čech-complete. However, every Čech-complete topological group is paracompact [12]. Hence, G is paracompact. Since M is a closed subspace of G, the space M is also paracompact. □

Now we are going to generalize Proposition 4. We start with a definition.

A subset A of a topological space X is said to be G_δ-*closed* if, for each $x \in X \setminus A$, there exists a G_δ-subset P of X such that $x \in P \subset X \setminus A$.

Of course, in a space of countable pseudocharacter every subset is G_δ-closed. On the other hand, a Lindelöf subspace is G_δ-closed in every larger Tychonoff space, this is well known (see [6]).

A mapping f of a topological space X into a topological space Y will be called G_δ-*closed* if, for every closed subset P of X, the image $f(P)$ is G_δ-closed in Y. Clearly, every closed mapping is G_δ-closed. We also have:

Proposition 16. *Every continuous mapping f of a Lindelöf space X into a Tychonoff space Y is G_δ-closed.*

Proof: This follows from the already mentioned fact that every Lindelöf subspace of a Tychonoff space is G_δ-closed in that space. □

Theorem 17. *Suppose that G is a topological group, and F a Lindelöf subspace of G. Then the restriction f of the product mapping $G \times G \to G$ to the subspace $F \times G$ is a G_δ-closed mapping of $F \times G$ onto G.*

Proof: Consider a mapping $s : F \times G \to F \times G$ defined as follows: $s(x, y) = (x, xy)$, for each $(x, y) \in F \times G$. Obviously, s is continuous, one-to-one, and $s(F \times G) = F \times G$. Clearly, the inverse mapping s^{-1} is described by the formula

$s^{-1}(x, y) = (x, x^{-1}y)$. Therefore, s^{-1} is also continuous. Hence, s is a homeomorphism. Since $xy = p(x, xy) = ps(x, y)$, where $p : F \times G \to G$ is the natural projection mapping given by $p(x, z) = z$, for each $(x, z) \in F \times G$, we conclude that f is the composition of s and p, that is, $f = ps$. However, since F is Lindelöf, the mapping p is G_δ-closed (the proof of this practically coincides with the proof of Theorem 3.1.16 in [6]). Therefore, the mapping f, as a composition of a homeomorphism with a G_δ-closed mapping, is itself G_δ-closed. $\square$

Here is just one application of Theorem 17. Recall that a space X is *a P-space* if every G_δ-subset of X is open.

Theorem 18. *Suppose that G is a topological group and a P-space, F a Lindelöf subspace of G, and M a paracompact closed subspace of G. Then FM is a paracompact closed subspace of G. In particular, if $G = FM$, then G is paracompact.*

Proof: Since F is Lindelöf, M is paracompact, and both F and M are P-spaces, the product space $F \times M$ is paracompact. The mapping f of $F \times G$ onto G, given by the product operation, is continuous and G_δ-closed, by Theorem 17. Since G is a P-space, it follows that f is closed. Since $F \times M$ is closed in $F \times G$, it follows that the restriction of f to $F \times M$ is a closed mapping of $F \times M$ into G. Hence, by a well known theorem of E. Michael (see [4] and [6]), FM is paracompact and closed in G. $\square$

Let us discuss one more modification of Proposition 4 which can be proved by almost the same argument as Proposition 4 and Theorem 17, we just have to refer to Theorem 3.10.7 in [6].

Theorem 19. *Suppose that G is a sequential topological group, and F a countably compact subspace of G. Then the restriction f of the product mapping $G \times G \to G$ to the subspace $G \times F$ is a closed mapping of $G \times F$ onto G.*

We say that a mapping $f : X \to Y$ is *locally perfect* if, for each $x \in X$, there exists an open neighbourhood U of x such that the restriction of f to the closure of U is a perfect mapping of $\overline{U}$ into Y. The following generalization of Proposition 4 is sometimes useful.

Theorem 20. *Suppose that G is a topological group, and Y a locally compact subspace of G. Then the restriction f of the product mapping $G \times G \to G$ to the subspace $G \times Y$ is an open locally perfect mapping of $G \times Y$ onto G.*

Proof: This obviously follows from Proposition 4 and the observation following it. $\square$

A natural general question to consider is the following one. Suppose that H is a closed subgroup of a topological group G. When does H have a compact grasp on G? Clearly, a necessary condition for this is compactness of the quotient space G/H. Below we refer to some known examples of topological groups and their quotients showing that this condition is not sufficient. We also give a complete answer to the above question when the subgroup H is locally compact.

Recall that, for any topological group G and any compact subgroup H of G, the natural quotient mapping π of G onto the quotient space G/H is perfect (in addition to being open) [12]. This statement was recently generalized as follows [2]:

Theorem 21. *Suppose that G is a topological group, and H a locally compact subgroup of G. Then the quotient mapping $\pi : G \to G/H$ is locally perfect.*

We will now present a few corollaries of this theorem. Recall that a mapping $f : X \to Y$ is said to be *compact-covering*, or *k-covering* [4], if, for each compact subspace F of Y, there exists a compact subspace Φ of X such that $f(\Phi) = F$. The relevance of this notion to our considerations is revealed by the following statement, the proof of which is obvious.

Proposition 22. *Suppose that G is a topological group and H a closed subgroup of G. Then H has a compact grasp on G if and only if the quotient space G/H is compact and the quotient mapping $\pi : G \to G/H$ is compact-covering.*

Theorem 23. *Suppose that G is a topological group and H a locally compact subgroup of G. Then the quotient mapping $\pi : G \to G/H$ is compact-covering.*

Proof: Taking into account Theorem 21 and the fact that the quotient mapping π is open, Theorem 23 follows from the next statement: $\square$

Proposition 24. *Every open locally perfect mapping f of a topological space X onto a topological space Y is compact-covering.*

Proof: Let F be any compact subspace of Y. For each $x \in X$ fix an open neighbourhood V_x of x such that the restriction of f to the closure of V_x is a perfect mapping of $\overline{V_x}$ into Y.

Since f is open and $f(X) = Y$, the family $\gamma = \{f(V_x) : x \in X\}$ is an open covering of Y. Since F is compact and $F \subset \bigcup \gamma$, there exists a finite subset K of X such that $F \subset \bigcup\{f(V_x) : x \in K\}$. Put $F_x = F \cap f(\overline{V_x})$ for $x \in K$. Observe, that $f(\overline{V_x})$ is closed in Y, since the restriction f_x of f to $\overline{V_x}$ is a perfect mapping. Hence, F_x is compact, for each $x \in K$. From the fact that f_x is perfect it follows also that $P_x = \overline{V_x} \cap f^{-1}(F_x) = f_x^{-1}(F_x)$ is a compact subset of X. Obviously, $f(P_x) = F_x$. Put $P = \bigcup\{P_x : x \in K\}$. Then P is compact, and $f(P) = F$. $\square$

Let us call a mapping $f : X \to Y$ *Lindelöf-covering*, or *l-covering* if, for each Lindelöf subspace M of Y, there exists a Lindelöf subspace L of X such that $f(L) = M$. Introducing obvious changes in the proofs of the last two statements, we obtain the following results.

Proposition 25. *Every open locally perfect mapping f of a topological space X onto a topological space Y is Lindelöf-covering.*

Theorem 26. *Suppose that G is a topological group and H a locally compact subgroup of G. Then the quotient mapping $\pi : G \to G/H$ is Lindelöf-covering.*

Theorem 23 allows to clarify completely when a locally compact subgroup H of a topological group G has a compact grasp on G:

Theorem 27. *A locally compact subgroup H of a topological group G has a compact grasp on G if and only if the quotient space G/H is compact.*

Proof: Assume that $G = FH$, where F is compact. Then $G/H = \pi(G) = \pi(F)$, where π is the quotient mapping. Therefore, by continuity of π, the space G/H is compact. The inverse statement follows from Theorem 23. □

Similarly, we have:

Theorem 28. *A locally compact subgroup H of a topological group G has a Lindelöf grasp on G if and only if the quotient space G/H is Lindelöf.*

Proof: The proof is the same as that of Theorem 27; we just refer to Theorem 26. □

In connection with Theorem 23, we have to mention that the quotient mapping of a topological group G onto a quotient group G/H need not be compact-covering even under very strong restrictions on G/H and H. Indeed, V.G. Pestov constructed an Abelian topological group G which is not a paracompact p-space but has a closed metrizable subgroup H such that the quotient group G/H is compact. In this case the natural quotient mapping $\pi : G \to G/H$ is not compact-covering. A consistent example in the same direction with somewhat stronger properties was described in [2].

References

[1] Arhangelskii A.V., *On a class of spaces containing all metric and all locally bicompact spaces*, Mat. Sb. (N.S.) **67** (109) (1965), 55–88; English translation: Amer. Math. Soc. Transl. **92** (1970), 1–39.

[2] Arhangel'skii A.V., *Quotients with respect to locally compact subgroups*, to appear in Houston J. Math.

[3] Arhangel'skii A.V., *Bisequential spaces, tightness of products, and metrizability conditions in topological groups*, Trans. Moscow Math. Soc. **55** (1994), 207–219.

[4] Arhangel'skii A.V., Ponomarev V.I., *Fundamentals of General Topology: Problems and Exercises*, Reidel, 1984.

[5] Bourbaki N., *Elements de Mathématique, Premiere Partie, Livre 3, Ch. 3*, 3-me ed., Hermann, Paris, 1949.

[6] Engelking, *General Topology*, Warszawa, 1977.

[7] Filippov V.V., *On perfect images of paracompact p-spaces*, Soviet Math. Dokl. **176** (1967), 533–536.

[8] Graev M.I., *Theory of topological groups, 1*, Uspekhi Mat. Nauk **5** (1950), 3–56.

[9] Henriksen M., Isbell J.R., *Some properties of compactifications*, Duke Math. J. **25** (1958), 83–106.
[10] Ivanovskij L.N., *On a hypothesis of P.S. Alexandroff*, Dokl. Akad. Nauk SSSR **123** (1958), 785–786.
[11] Michael E., *A quintuple quotient quest*, General Topology Appl. **2** (1972), 91–138.
[12] Roelke W., Dierolf S., *Uniform Structures on Topological Groups and Their Quotients*, McGraw-Hill, New York, 1981.
[13] Uspenskij V.V., *Topological groups and Dugundji spaces*, Mat. Sb. **180:8** (1989), 1092–1118.

Department of Mathematics, Ohio University, Athens, Ohio 45701, USA
E-mail: arhangel@math.ohiou.edu

(*Received* March 31, 2003, *revised* July 13, 2003)