Characterizing polyhedrons and manifolds

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Abstract. In [5], W. Taylor shows that each particular compact polyhedron can be characterized in the class of all metrizable spaces containing an arc by means of first order properties of its clone of continuous operations. We will show that such a characterization is possible in the class of compact spaces and in the class of Hausdorff spaces containing an arc. Moreover, our characterization uses only the first order properties of the monoid of self-maps. Also, the possibility of characterizing the closed unit interval of the real line and some related objects in the category of partially ordered sets and monotonous maps will be illustrated.

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1. Introduction

This paper is a result of research motivated by W. Taylor's monograph [5]. In the named paper, among other results, each particular compact polyhedron is characterized in the class of all metrizable spaces containing an arc by means of the first order properties of its clone of continuous operations. In the present paper we will see that such a characterization is possible in the class of all compact spaces and in the class of spaces containing a homeomorphic copy of the unit interval. Also, we will show that n-dimensional topological manifolds can be characterized in the named classes. Moreover, our characterizations are monoid-theoretical.

Our starting point is the following result.

Proposition 1.1. *There exists a monoid-theoretic formula* Int_1 *valid in the monoid* $M(I)$ *of the closed unit interval* I *such that the following holds:*

if the monoid of continuous self-maps $M(X)$ *of a space* X *satisfies*

$$M(X) \models \mathrm{Int}_1,$$

then

(1) *up to a bijection,* X *is a topology on* I *finer than the natural topology of the interval;*
(2) X *is homogeneous in the sense that it is homeomorphic to any its subinterval* (*with the topology induced by* X).

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Proposition 1.1 was proved by the author in [1].

As immediate corollaries we obtain the following results:

Corollary 1.2. *The monoid of continuous self-maps $M(X)$ of a compact space X satisfies*

$$M(X) \models \mathrm{Int}_1$$

if and only if X is homeomorphic to the closed unit interval I of the real line.

By a compact space we mean a space with the finite subcover property, hence Hausdorffness is not needed.

Corollary 1.3. *The monoid of continuous self-maps $M(X)$ of a space X containing an arc satisfies*

$$M(X) \models \mathrm{Int}_1$$

if and only if X is homeomorphic to I.

Note that a monoid-theoretical characterization of the interval was already obtained by Magill and Subbiah in [3] for Tychonoff spaces containing an arc. Actually, they prove the following proposition.

Proposition 1.4. *A Tychonoff space X containing an arc is homeomorphic to I if and only if $M(X)$ has exactly two regular $\mathfrak{D}$-classes.*

This probably needs more detailed explanation. Given a semigroup S, one can define three equivalences on it known as Green's relations. Say $a, b \in S$ are $\mathfrak{L}$-related, if they generate the same left ideal, and $\mathfrak{R}$-related, if they generate the same right ideal. The relations $\mathfrak{L}$ and $\mathfrak{R}$ are known to commute, hence their composition

$$\mathfrak{D} = \mathfrak{L} \circ \mathfrak{R}$$

is again an equivalence. If a $\mathfrak{D}$-equivalence class contains a regular element, then all its members are regular; such classes are named *regular $\mathfrak{D}$-classes*.

It is easy to see that Proposition 1.4 actually gives a first-order condition for $M(X)$: having two regular $\mathfrak{D}$-classes is equivalent to having two nonequivalent regular elements to some of which any other regular element is $\mathfrak{D}$-related. The latter is clearly a first order condition as both regularity and $\mathfrak{D}$-equivalence are given in first order terms.

Remark 1.5. Note that the formula Int_1 actually characterizes the unit interval in the union of the class of compact spaces and the class of spaces containing an arc.

Before going any further let us make a few conventions. The language of the theory of monoids will be denoted $\mathcal{L}_m$. It consists of one binary functional symbol $\circ$, which will be often omitted in our formulas. Also parentheses will be omitted in view of the associativity of monoids.

The formula

$$\mathrm{Const}(x) \equiv (\forall \zeta)(x\zeta = x)$$

characterizes constant self-maps of a space. Constants play quite a significant role in our formulas. We will use abbreviations like $\mathrm{Const}(x, y)$ instead of $\mathrm{Const}(x) \,\&\, \mathrm{Const}(y)$.

The first thing to do is to characterize the boundary of the closed unit interval. Actually this is done in [1]: take

$$\begin{aligned}\mathrm{Bd}(x, y) \equiv \mathrm{Const}(x, y) \,\&\, (\forall f)[(\exists u, v)(fu = x \,\&\, fv = y) \to \\ \to (\forall z)(\mathrm{Const}(z) \to (\exists w) fw = z)].\end{aligned}$$

Thus, Bd says that whenever x and y are in the range of f, then f is onto. Clearly $\mathrm{Bd}(x, y)$ holds in $M(I)$ if and only if x and y are constants pointing the boundary of I, i.e. 0 and 1. Hence the formula

(1) $$\mathrm{Int}(x, y) \equiv \mathrm{Int}_1 \;\&\; \mathrm{Bd}(x, y)$$

characterizes the closed unit interval of the real line together with its boundary in the class of compact spaces and spaces containing an arc. This will be used in Section 3, where we show the existence of monoid-theoretical characterizations of compact polyhedrons and topological manifolds.

Section 4 illustrates the usefulness of our methods in a different category, namely in the category of partially ordered sets and their monotonous mappings.

The main tool is developed in Section 2. We show how the first-order properties of the monoid of a retract can be described by the first-order properties of the monoid of the space itself. Actually, the notions and results in Section 2 are put in a more general setting, which allows us to use them in Section 4. The following definitions will be useful in Sections 2 and 4.

Let X be an object in an arbitrary category $\mathcal{K}$. Suppose all the finite powers of X exist in $\mathcal{K}$. Following [5], we define the *clone* $\mathrm{Cl}(X)$ *of* X as the ω-sorted algebra

$$\langle C_n; e_i^{(n)}; S_m^n \rangle,$$

where

- $C_n = \mathcal{K}(X^n, X)$ for $n \in \omega$,
- $e_i^{(n)} : X^n \to X$ for $i < n < \omega$ are the product projections and
- $S_m^n : C_n \times (C_m)^n \to C_m$ for $n, m \in \omega$ are heterogeneous operations defined as follows:

$$S_m^n(f; g_1, \ldots, g_n) = f \circ (g_1 \triangle \cdots \triangle g_n).$$

($\triangle$ denotes the diagonal product of morphisms.)

Thus, the clone $\mathrm{Cl}(X)$ of an object X is an extension of the monoid $M(X)$ of endomorphisms. Likewise, the language $\mathcal{L}_c$ of the theory of clones is an extension of that of monoids.

So, $\mathcal{L}_c$ contains ω sorts of variables $f_i^{(n)}$, $n, i \in \omega$, each $f_i^{(n)}$ being of the n-th sort; constant symbols $e_i^{(n)}$ for $i < n < \omega$ (again $e_i^{(n)}$ being of the n-th sort); and operational symbols S_m^n of type $n \times m^n \to m$.

The n-th sort terms are constructed by the following rules:

- each $f_i^{(n)}$ is an n-th sort term;
- each $e_i^{(n)}$ is an n-th sort term;
- whenever t is an n-th sort term and $t_1, \ldots, t_n$ are m-th sort terms, then also $S_m^n(t; t_1, \ldots, t_n)$ is an m-th sort term.

Atomic formulas have the form $t_1 = t_2$ for terms t_1 and t_2 of the same sort. Formulas are then constructed in the usual way:

- each atomic formula is a formula;
- whenever φ and ψ are formulas, so are also $\neg\varphi$, $\varphi \mathbin{\&} \psi$, $(\forall f_i^{(n)})\varphi$.

For topological spaces, the language of the theory of clones is actually stronger than that of the theory of monoids. This was first proved in [6], where V. Trnková showed that there exist topological spaces X and Y such that the monoids $M(X)$ and $M(Y)$ are isomorphic but the clones $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$ are not elementarily equivalent. This result was further strengthened in [4].

In the present paper, however, all the topological characterizations will be done in the language $\mathcal{L}_m$ of the theory of monoids.

2. Retracts

For the purposes of Section 3, the current section could deal with only monoids and topological spaces. However, the concepts and results presented here will also be used in Section 4 for partially ordered sets, which is why we present this section in full generality. (Thanks go to J. Velebil for noting that the results presented in this section work in arbitrary categories.)

Recall that $\mathcal{L}_c$ denotes the language of the theory of clones. Let $\mathcal{L}_c' = \mathcal{L}_c \cup \{h\}$ be the same language enriched with a new 1-st sort variable h. (One would like it better to have a new constant but we will later want to quantify it, hence a new variable is better suited. This new variable will be later interpreted as an idempotent, but at this point that is not important.)

For each formula φ in the language $\mathcal{L}_c$ define its *restriction* φ^h *on* h according to the following scheme:

- if φ is the atomic formula $t_1 = t_2$, where t_1 and t_2 are terms of sort n, take

$$\varphi^h \equiv S_n^n(t_1; S_n^1(h; e_0^{(n)}), \ldots, S_n^1(h; e_{n-1}^{(n)})) = S_n^n(t_2; S_n^1(h; e_0^{(n)}), \ldots, S_n^1(h; e_{n-1}^{(n)}));$$

- if $\varphi \equiv \neg\psi$ then $\varphi^h \equiv \neg(\psi^h)$, if $\varphi \equiv \psi \,\&\, \theta$ then $\varphi^h \equiv (\psi^h) \,\&\, (\theta^h)$;
- if $\varphi \equiv (\forall f_i^{(n)})\psi$, then

$$\varphi^h \equiv (\forall f_i^{(n)})[S_n^1(h; S_n^n(f_i^{(n)}; S_n^1(h; e_0^{(n)}), \dots, S_n^1(h; e_{n-1}^{(n)}))) = \\ = S_n^n(f_i^{(n)}; S_n^1(h; e_0^{(n)}), \dots, S_n^1(h; e_{n-1}^{(n)})) \to \psi^h].$$

Let now $\mathcal{K}$ be a category and $X, Y \in \mathrm{obj}(\mathcal{K})$. Let $r : X \to Y$ be a retraction and $c : Y \to X$ its corresponding coretraction (so $r \circ c = 1_Y$). Put $h = c \circ r : X \to X$. The morphism $h : X \to X$ will be the interpretation of the variable h of the enriched language $\mathcal{L}_c'$. Suppose that all the finite powers of X and Y exist in $\mathcal{K}$, so we can speak of the clone of X, which is the algebraic structure

$$\mathrm{Cl}(X) = \langle X^n; S_m^n; \pi_i^{(n)} \rangle.$$

For any morphism $F : X^n \to X$ let $F^h : Y^n \to Y$ denote its *restriction* $r \circ F \circ c^n$ *on* Y. The clone of Y will be constructed as the restriction of $\mathrm{Cl}(X)$. In other words, we will choose

$$p_i^{(n)} = [\pi_i^{(n)}]^h$$

as the respective projections of the clone

$$\mathrm{Cl}(Y) = \langle Y^n; S_m^n; p_i^{(n)} \rangle.$$

A morphism $F : X^k \to X$ will be said to be *h-invariant* if

$$h \circ F \circ h^k = F \circ h^k.$$

Note that each projection is h-invariant.

The following simple remarks will be useful in proving the Proposition 2.3.

Remark 2.1. For any m-th sort term t in the language $\mathcal{L}_c$ with free variables among $f_1, \dots, f_n$, where f_i is an abbreviation for $f_{j_i}^{(k_i)}$, and any collection $F_1, \dots, F_n$ of h-invariant $\mathcal{K}$-morphisms, $F_i : X^{k_i} \to X$, the morphism

$$t^X(F_1 \dots, F_n)$$

is also h-invariant. In other words, any composition of h-invariant maps is h-invariant.

Remark 2.2. For any m-th sort term t in the language $\mathcal{L}_c$ with free variables among $f_1, \ldots, f_n$, where f_i is an abbreviation for $f_{j_i}^{(k_i)}$, and any collection $F_1, \ldots, F_n$ of h-invariant $\mathcal{K}$-morphisms, $F_i : X^{k_i} \to X$, the following holds:

$$[t^X(F_1, \ldots, F_n)]^h = t^Y(F_1^h, \ldots, F_n^h).$$

Again, in human language, this says that the restriction of a composition is the composition of restrictions.

Proposition 2.3. *Let φ be a formula in the language $\mathcal{L}_c$ with free variables among $f_1, \ldots, f_n$, where f_i is an abbreviation for $f_{j_i}^{(k_i)}$. Let $F_1, \ldots, F_n$ be h-invariant $\mathcal{K}$-morphisms, $F_i : X^{k_i} \to X$. Then*

$$\mathrm{Cl}(X) \models \varphi^h[F_1, \ldots, F_n] \quad \textit{iff} \quad \mathrm{Cl}(Y) \models \varphi[F_1^h, \ldots, F_n^h].$$

PROOF: The proof goes by induction on the complexity of φ.

Suppose φ is the atomic formula $t_1 = t_2$, the terms t_1 and t_2 being of sort m. Then, by Remarks 2.1 and 2.2,

$$\begin{aligned} &t_i^X(F_1, \ldots, F_n) \circ h^m = h \circ t_i^X(F_1, \ldots, F_n) \circ h^m \\ &= c \circ r \circ t_i^X(F_1, \ldots, F_n) \circ c^m \circ r^m \\ &= c \circ t_i^Y(F_1^h, \ldots, F_n^h) \circ r^m. \end{aligned}$$

Thus

$$\begin{aligned} &\mathrm{Cl}(X) \models \varphi^h[F_1, \ldots, F_n] \quad \text{iff} \\ &t_1^X(F_1, \ldots, F_n) \circ h^m = t_2^X(F_1, \ldots, F_n) \circ h^m \quad \text{iff} \\ &c \circ t_1^Y(F_1^h, \ldots, F_n^h) \circ r^m = c \circ t_2^Y(F_1^h, \ldots, F_n^h) \circ r^m \quad \text{iff} \\ &t_1^Y(F_1^h, \ldots, F_n^h) = t_2^Y(F_1^h, \ldots, F_n^h) \quad \text{iff} \\ &\mathrm{Cl}(Y) \models \varphi[F_1^h, \ldots, F_n^h]. \end{aligned}$$

(The third iff is because of c being a mono and r being an epi.)

If $\varphi \equiv \neg\psi$, then obviously

$$\begin{aligned} &\mathrm{Cl}(X) \models \varphi^h[F_1, \ldots, F_n] \quad \text{iff} \\ &\mathrm{Cl}(X) \not\models \psi^h[F_1, \ldots, F_n] \quad \text{iff} \\ &\mathrm{Cl}(Y) \not\models \psi[F_1^h, \ldots, F_n^h] \quad \text{iff} \\ &\mathrm{Cl}(Y) \models \varphi[F_1^h, \ldots, F_n^h]. \end{aligned}$$

Similarly, if $\varphi \equiv \psi \,\&\, \theta$, then

$$\begin{aligned}
&\mathrm{Cl}(X) \models \varphi^h[F_1, \ldots, F_n] \quad \text{iff} \\
&\mathrm{Cl}(X) \models \psi^h[F_1, \ldots, F_n] \quad \text{and} \quad \mathrm{Cl}(X) \models \theta^h[F_1, \ldots, F_n] \quad \text{iff} \\
&\mathrm{Cl}(Y) \models \psi[F_1^h, \ldots, F_n^h] \quad \text{and} \quad \mathrm{Cl}(Y) \models \theta[F_1^h, \ldots, F_n^h] \quad \text{iff} \\
&\mathrm{Cl}(Y) \models \varphi[F_1^h, \ldots, F_n^h].
\end{aligned}$$

Now let $\varphi \equiv (\forall f_i^{(k)})\psi(f_i^{(k)}, f_1, \ldots, f_n)$. Suppose

$$\mathrm{Cl}(X) \models \varphi^h[F_1, \ldots, F_n]. \tag{2}$$

Let $H : Y^k \to Y$ be arbitrary; set $F = c \circ H \circ r^k$. Then

$$F^h = r \circ F \circ c^k = r \circ c \circ H \circ r^k \circ c^k = H.$$

Also, F is h-invariant as $h \circ F \circ h^k = h \circ c \circ H \circ r^k \circ h^k = c \circ H \circ r^k \circ h^k = F \circ h^k$. Thus, as $\mathrm{Cl}(X) \models \psi^h[F, F_1, \ldots, F_n]$ surely holds, we have the induction hypothesis that

$$\mathrm{Cl}(Y) \models \psi[H, F_1^h, \ldots, F_n^h].$$

This proves the validity of

$$\mathrm{Cl}(Y) \models \varphi[F_1^h, \ldots, F_n^h]. \tag{3}$$

Conversely, suppose (3) holds. Let $F : X^k \to F$ be any h-invariant morphism. Then

$$\mathrm{Cl}(Y) \models \psi[F^h, F_1^h, \ldots, F_n^h].$$

But by the induction hypothesis this exactly means that

$$\mathrm{Cl}(X) \models \psi^h[F, F_1, \ldots, F_n].$$

As F was arbitrary h-invariant this proves that (2) holds. The proof is complete. □

3. Manifolds and polyhedrons

Throughout this section we work with first order formulas in the language $\mathcal{L}_m$.

Suppose X is a compact space and $h : X \to X$ is a retraction on an arc $Y \subseteq X$. Then $M(Y) \models \mathrm{Int}_1$. Hence, by Proposition 2.3,

$$M(X) \models \mathrm{Int}_1^h.$$

On the other hand, if $M(X) \models \mathrm{Int}_1^h$, then

$$M(Y) \models \mathrm{Int}_1 .$$

As the image of a compact space is compact, we get Y is an arc.

Now notice that retractions can be characterized by the formula

$$\mathrm{Ret}(h) \equiv (h \circ h = h).$$

Hence the formula

$$(\exists h)\, \mathrm{Ret}(h) \,\&\, \mathrm{Int}_1^h$$

characterizes in the class of compact spaces those having the closed interval as their retract.

Note that replacing Int_1^h by $\mathrm{Int}^h(x,y) \,\&\, \mathrm{Const}(x,y) \,\&\, hx = x \,\&\, hy = y$ in the above formula we get characterizations of retractions on intervals with endpoints x and y. Let us denote the formula thus obtained as $I\,\mathrm{Ret}(h,x,y)$.

Now we are ready to characterize more complicated objects in the class of compact spaces. Take

$$\begin{aligned}
&\mathrm{Cell}_n(f_1,\dots,f_n,g_1,\dots,g_n) \equiv \\
&\Big\{ \bigwedge_{i=1}^{n} (\mathrm{Ret}(f_i) \,\&\, \mathrm{Ret}(g_i)) \,\&\, \\
&(\exists h_1)\dots(\exists h_n)(\exists u_1,v_1)\dots(\exists u_n,v_n)\Big[\bigwedge_{i=1}^{n} I\,\mathrm{Ret}(h_i,u_i,v_i) \,\&\, \\
&(\forall x_1)\dots(\forall x_n)\Big(\bigwedge_{i=1}^{n} (\mathrm{Const}(x_i) \,\&\, h_i x_i = x_i) \to (\exists_1 x)\Big(\bigwedge_{i=1}^{n} h_i x = x_i\Big)\Big) \,\&\, \\
&\bigwedge_{i=1}^{n} (\forall x)(\mathrm{Const}(x) \to (f_i \circ x = x \leftrightarrow h_i \circ x = u_i) \,\&\, (g_i \circ x = x \leftrightarrow h_i \circ x = v_i))\Big]\Big\},
\end{aligned}$$

where $(\exists_1 x)\dots$ means "there exists exactly one x such that $\dots$".

Proposition 3.1. *For any compact space X and its continuous self-maps*

$$F_1,\dots,F_n,G_1,\dots,G_n \in M(X)$$

the following statements are equivalent:

(i) $M(X) \models \mathrm{Cell}_n(F_1,\dots,F_n,G_1,\dots,G_n)$,

(ii) *X is homeomorphic to the n-cell I^n and the functions F_i and G_i are projections on the respective edges.*

Proof: Clearly the formula holds for the n-cell and its projections on edges. So the opposite direction should only be proved.

Suppose X is a compact space, $F_i, G_i : X \to X$ are continuous maps for $i = 1, \dots, n$ and suppose

$$M(X) \models \mathrm{Cell}_n(F_1, \dots, F_n, G_1, \dots, G_n)$$

holds. The first raw in the above definition of Cell_n says that all the F_i's and G_i's are retractions. The second raw acquires the existence of n retractions $H_i : X \to X$ and $2n$ constants $u_i, v_i \in X$ such that u_i and v_i lie in the range $Y_i = H_i(X)$ of H_i and, moreover, each Y_i is homeomorphic to the closed unit interval I, whereas u_i and v_i are its boundary points.

At this point we can form the diagonal product of the maps H_i: let

$$H = H_1 \triangle \dots \triangle H_n : X \to Y = Y_1 \times \dots \times Y_n.$$

The third raw in the definition simply says that thus defined H is a bijection. Hence we have a continuous bijection of the compact space X onto a Hausdorff space Y. Clearly this bijection should be a homeomorphism. But each Y_i is in turn homeomorphic to I, consequently X is homeomorphic to the n-cell I^n.

Now it suffices to notice that the last raw in the definition of Cell_n only says that the range of F_i (G_i) is the preimage of u_i (v_i, respectively) under H_i. Under the homeomorphism H it transforms to the preimage of the boundary point u_i (v_i) under the i-th projection

$$\pi_i : Y_1 \times \dots \times Y_n \to Y_i.$$

So the homeomorphism H transforms X to the n-cell Y and transforms each F_i and G_i to the projection onto the corresponding edge. This is what should have been proved. $\square$

Remark 3.2. For $n = 0$ one can take

$$\mathrm{Cell}_0 \equiv (\forall x)(\forall y) x = y.$$

This is clearly a characterization of the 0-cell — a singleton.

It is clear how one can build a formula

$$\mathrm{Cell\,Ret}_n(h; f_1 \dots, f_n, g_1 \dots, g_n)$$

from the above Cell_n which characterizes all the retractions of the space onto an n-cell, with $f_1, \dots, f_n$ and $g_1, \dots, g_n$ being the corresponding retractions of the whole space onto the edges of the cell. To be more precise, we want the equivalence of the following statements:

- $M(X) \models \mathrm{Cell\,Ret}_n(H; F_1 \dots, F_n, G_1 \dots, G_n)$,
- $H : X \to X$ is a retraction with the range $Y = H(X)$ being homeomorphic to the n-cell I^n and the maps F_i and G_i are retractions on the corresponding edges of Y.

Note that an n-cell can also be looked at as an n-simplex. To make also the boundary "simplicial" we can glue up the second n-tuple of edges into one. Formally, let

$$\begin{aligned}\mathrm{Simplex}_n(f_1,\ldots,f_n,f_{n+1}) \equiv\ & f_{n+1}\circ f_{n+1}=f_{n+1}\,\&\\ & (\exists g_1)\ldots(\exists g_n)\,\mathrm{Cell}_n(f_1,\ldots,f_n,g_1,\ldots,g_n)\,\&\\ & (\forall x)(\mathrm{Const}(x)\to(f_{n+1}\circ x=x\leftrightarrow\bigvee_{i=1}^{n} g_i\circ x=x)).\end{aligned}$$

The formula $\mathrm{Simplex}_n(F_1,\ldots,F_{n+1})$ holds for a space X if and only if X is an n-simplex and $F_1,\ldots,F_{n+1}$ are retractions onto all of its edges. Again we can construct a new formula $\mathrm{Sim\,Ret}_n(h;f_1,\ldots,f_{n+1})$ which characterizes the retractions onto an n-simplex and retractions on its edges.

Now it is easy to characterize *every particular* compact polyhedron by a first-order formula in the language of monoids in the class of all compact spaces.

Really, let a compact polyhedron K be given. Recall that any such K can be triangulated by a finite simplicial complex. The latter is given by a finite set V of vertices and a set S of non-empty subsets of V with the properties that

(i) $\bigcup S=V$,
(ii) if $s\in S$ and $\emptyset\neq r\subseteq s$, then $r\in S$.

For such a polyhedron K define

$$\begin{aligned}\mathrm{Complex}_K \equiv\ & (\exists h^{(s)})_{s\in S}(\exists f_v^{(s)})_{v\in s\in S}\{\bigwedge_{s\in S}\mathrm{Sim\,Ret}_{|s|-1}(h^{(s)};f_v^{(s)})_{v\in s}\,\&\\ & \bigwedge_{\substack{s\subseteq r\\ s,r\in S}}(\forall x)(\mathrm{Const}(x)\to(h^{(s)}\circ x=x\leftrightarrow\bigwedge_{v\in r\setminus s} f_v^{(r)}\circ x=x))\,\&\\ & \bigwedge_{\substack{s\cap r=\emptyset\\ s,r\in S}}(\forall x)(\mathrm{Const}(x)\to(h^{(s)}\circ x\neq x\vee h^{(r)}\circ x\neq x))\,\&\\ & \bigwedge_{\substack{s\cap r=t\\ s,r,t\in S}}(\forall x)(\mathrm{Const}(x)\to(h^{(t)}\circ x=x\leftrightarrow(h^{(s)}\circ x=x\,\&\,h^{(r)}\circ x=x)))\,\&\\ & (\forall x)(\mathrm{Const}(x)\to\bigvee_{s\in S}h^{(s)}\circ x=x)\}.\end{aligned}$$

The formula needs some explanation. First, the existence of $|S|$ retractions is required (so another way to write the first quantifier is

$$(\exists h^{(s_1)})(\exists h^{(s_2)})\ldots(\exists h^{(s_{|S|})})\ldots).$$

Then, for each $s \in S$ we need the existence of $|s|$ retractions on edges, which is the meaning of the second quantifier. The most curious looks the subformula

$$\operatorname{Sim\,Ret}_{|s|-1}(h^{(s)}; f_v^{(s)})_{v\in s}.$$

It has to say that $h^{(s)}$ is a retraction on a $|s|-1$-dimensional simplex with edges $\{f_v^{(s)}\}_{v\in s}$, but we have no naturally defined order for the variables $f_v^{(s)}$. Luckily this does not matter at all as the formula $\mathrm{Simplex}_n(f_1,\dots,f_{n+1})$ is symmetric with respect to its variables (unlike Cell_n). This expresses the fact that one can permute arbitrarily the edges of any simplex by affine maps. As a conclusion, we can understand $\operatorname{Sim\,Ret}_{|s|-1}(h^{(s)}; f_v^{(s)})_{v\in s}$ as the formula $\operatorname{Sim\,Ret}_{|s|-1}(h^{(s)}; f_{v_1}^{(s)},\dots,f_{v_{|s|}}^{(s)})$, where $v_1,\dots,v_{|s|}$ is an arbitrarily chosen enumeration of s.

It is easily seen that $\mathrm{Complex}_K$ characterizes the polyhedron K in the class of all compact spaces. Really, the formula $\mathrm{Complex}_K$ is nothing else than the definition of the topological representation of the complex $\langle V,S\rangle$. Let us formulate this result as a

Proposition 3.3. *For any compact space X, $M(X)\models \mathrm{Complex}_K$ if and only if X is homeomorphic to K.*

Now we characterize all the compact n-dimensional topological manifolds in the class of all compact Hausdorff spaces ($n\in\omega$). This will be done by the following formula:

$$\begin{aligned}
\mathrm{Man}_n \equiv{} & (\forall x)\{\mathrm{Const}(x)\to \\
& (\exists h)(\exists f_1)\dots(\exists f_n)(\exists g_1)\dots(\exists g_n)[\operatorname{Cell\,Ret}_n(h; f_1\dots,f_n,g_1\dots,g_n)\,\&\\
& (\exists u,v)(u\circ v=1\,\&\,v\circ u=1\,\&\,u\circ x\neq x\,\&\\
& (\forall y)(\mathrm{Const}(y)\to(u\circ y\neq y\leftrightarrow h\circ y=y\,\&\,\bigwedge_{i=1}^{n}(f_i\circ y\neq y\,\&\,g_i\circ y\neq y))))]\}.
\end{aligned}$$

(1 in the above formula is the unit of the monoid.)

Proposition 3.4. *For any natural number $n\geq 1$ and any compact Hausdorff space X the following statements are equivalent:*

(i) $M(X)\models \mathrm{Man}_n$,
(ii) *X is an n-dimensional topological manifold.*

Proof: We only need to translate the formula Man_n into informal language. It says "for any point $x\in X$ there exists a retraction $H: X\to X$ such that the image $Y=H(X)$ is an n-cell with boundary ∂Y and the following holds: there exists a homeomorphism U of X onto itself, which moves precisely those points of X which lie in Y, but not on its border ∂Y, and moves the point x".

It is straightforward to prove that for Hausdorff spaces the above statement is equivalent to the fact that each point has a neighborhood homeomorphic to $\mathbb{R}^n$. □

0-dimensional compact manifolds (i.e. finite discrete spaces) are characterized much easier; for example by the formula

$$\mathrm{Discrete} \equiv (\forall x)[\mathrm{Const}(x) \to (\exists f)(\forall y)(\mathrm{Const}(y) \to (f \circ y \neq y \leftrightarrow y = x))].$$

Remark 3.5. Note that one can similarly characterize all the n-dimensional topological manifolds with boundary. Changes are minimal: just let the homeomorphism U move all the interior points of Y and also (the interior of) one edge of it, leaving the rest of the space fixed.

Trying to carry the same considerations for the class of all spaces containing an arc we come across a single difficulty: a retract of such a space need not contain an arc (it can be e.g. a point). This difficulty, however, is easily removed by the following well-known fact:

Lemma 3.6. *Let X be a Hausdorff space and $f : I \to X$ a nonconstant continuous map. Then $f(I)$ contains an arc (it is even arcwise connected).*

For the proof consult [2].

Consider the formula

$$\mathrm{Ret}'(h) \equiv \mathrm{Ret}(h) \,\&\, (\forall x, y)[(\mathrm{Const}(x, y) \,\&\, x \neq y) \to (\exists f)(hf = f \,\&\, fx \neq fy)].$$

Suppose X is a Hausdorff space containing an arc, $h : X \to X$, and suppose

$$\mathrm{Cl}(X) \models \mathrm{Ret}'(h).$$

Then h is a retraction onto a subspace Y such that continuous maps from X to Y separate the points in X. Lemma 3.6 then implies that Y contains an arc.

On the other hand, if X is a completely regular space containing an arc, then $\mathrm{Ret}'(h)$ holds if and only if h is a retraction on a subspace containing an arc.

Thus, substituting Ret' for Ret in the earlier formulas we obtain the following result:

Proposition 3.7.

(a) *For each compact polyhedron K there exists a monoid-theoretical first-order formula $\mathrm{Complex}'_K$ such that, for any Hausdorff space X that contains an arc, $M(X) \models \mathrm{Complex}'_K$ if and only if X is homeomorphic to K.*

(b) *For each natural number $n \in \omega$ there exists a monoid-theoretical first-order formula Man'_n such that, for any Hausdorff space X containing an arc,*
$M(X) \models \mathrm{Complex}'_K$ if and only if X is an n-dimensional topological manifold.

4. Posets

In this section, we will try to illustrate the usefulness of the methods developed in Section 2 in the category Poset of posets and monotonous maps.

Let P be a poset with order $\leq$. We will see how one can speak about the order directly in the language of the theory of monoids (applied to the monoid of monotone self-maps). The first thing to note is, that the category Poset contains constant maps. So we can again speak about elements of posets in the language of monoids and/or clones. For better readability of the formulas, variables like a, b, c, x, y, z will be always relativized to constant functions.

Suppose $a, b \in P$ are incomparable elements and let $c, d \in P$ be such that $c \geq d$. Then we can define a map

$$f(x) = \begin{cases} c, & \text{if } \ a \leq x, \\ d, & \text{otherwise.} \end{cases}$$

This is evidently a monotone map and witnesses for the following statement:

The points $a, b \in P$ are incomparable if and only if they satisfy the following formula:

$$\begin{aligned} \mathrm{Inc}(a, b) \equiv (\exists x, y)(\exists f, g)[x \neq y \,\&\, f \circ a = x \,\&\, f \circ b = y \,\&& \\ g \circ a = y \,\&\, g \circ b = x \,\&\,]. \end{aligned}$$

Denote $\mathrm{Comp}(a, b) \equiv \neg\, \mathrm{Inc}(a, b)$. Thus, $\mathrm{Comp}(a, b)$ holds if and only if a and b are comparable.

Now suppose $a < b$ and $c \leq d$. Then there is a monotonous map taking a to c and b to d. Really, one can take

$$g(x) = \begin{cases} d, & \text{if } \ b \leq x, \\ c, & \text{otherwise.} \end{cases}$$

So the formula

$$x \underset{a,b}{\leq} y \equiv (\exists f)(f \circ a = x \,\&\, f \circ b = y)$$

characterizes those pairs (x, y) which are in the same order as (a, b), in case a and b are comparable. We will also use the abbreviation

$$x \underset{a,b}{<} y \equiv x \underset{a,b}{\leq} y \,\&\, x \neq y.$$

Now we can start characterizing certain classes of posets. Non-discrete posets are characterized by the formula

$$N\,\mathrm{Diskr} \equiv (\exists a, b)(a \neq b \,\&\, \mathrm{Comp}(a, b)).$$

It is possible to characterize e.g. non-trivial lattices by

$$\begin{aligned}\mathrm{Lattice} \equiv (\exists a,b)[a \neq b \,\&\, \mathrm{Comp}(a,b) \,\&\, \\ (\forall x,y)(\exists u)(u \underset{a,b}{\leq} x \,\&\, u \underset{a,b}{\leq} y \,\&\, (\forall t)(t \underset{a,b}{\leq} x \,\&\, t \underset{a,b}{\leq} y \to t \underset{a,b}{\leq} u) \,\&\, \\ (\forall x,y)(\exists v)(x \underset{a,b}{\leq} v \,\&\, y \underset{a,b}{\leq} v \,\&\, (\forall z)(x \underset{a,b}{\leq} z \,\&\, y \underset{a,b}{\leq} z \to v \underset{a,b}{\leq} z))].\end{aligned}$$

One sees how to characterize distributive lattices, modular lattices, Boolean algebras etc. We will need the characterization of linear orders with a least element a and a greatest element b:

$$\mathrm{Linear}(a,b) \equiv \mathrm{Comp}(a,b) \,\&\, (\forall x,y)(x \underset{a,b}{\leq} y \vee y \underset{a,b}{\leq} x) \,\&\, (\forall x)(a \underset{a,b}{\leq} x \underset{a,b}{\leq} b).$$

Now, in a linearly ordered set with a least element a and a greatest element b we can characterize the continuous monotonous maps f by the following formula (cf. the formula Cont in [1]):

$$\begin{aligned}\mathrm{Cont}(f) \equiv (\forall x,y,z)\{[(a \underset{a,b}{<} x \,\&\, y \underset{a,b}{<} f\circ x) \to (\exists a')(a' \underset{a,b}{<} x \,\&\, y \underset{a,b}{<} f \circ a')] \,\&\, \\ [(x \underset{a,b}{<} b \,\&\, f\circ x \underset{a,b}{<} z) \to (\exists b')(x \underset{a,b}{<} b' \,\&\, f\circ b' \underset{a,b}{<} z)]\}.\end{aligned}$$

In the language $\mathcal{L}_c$ of the theory of clones one can similarly construct a formula $\mathrm{Cont}_2(f)$ characterizing continuous monotonous maps $f : X^2 \to X$ for a linearly ordered set X. We will not write this formula down as it is lengthy and is not particularly interesting.

A linear order is dense and complete if and only if it has no continuous and monotonous 2-valued self-maps. Thus the following formula characterizes among the linear orders those which are dense and complete (cf. Con in [1]):

$$\begin{aligned}\mathrm{Con} \equiv (\forall f))[(\mathrm{Cont}(f) \,\&\, (\exists x,y)(f\circ x \neq f \circ y)) \to \\ (\forall u,v)(\exists x)(u \neq f\circ x \neq v)].\end{aligned}$$

Now, consider the clone-theoretic formula

$$\mathrm{Cen} \equiv (\exists f_0^{(2)})[\mathrm{Cont}_2(f_0^{(2)}) \,\&\, (\forall x,y)(x \underset{a,b}{<} y \to \; x \underset{a,b}{<} \; S_0^2(f_0^{(2)}; x,y) \; \underset{a,b}{<} y)].$$

It states that one can continuously (and monotonously) choose a point in every open subinterval. The map

$$(x,y) \mapsto \frac{x+y}{2}$$

witnesses the fact that the formula Cen holds in $\mathrm{Cl}(I)$. On the other hand, it was proved in [1] that a dense and complete order with a least and greatest element, where one can continuously choose points in open subintervals, is isomorphic to the natural order of the closed unit interval (see [1], the two paragraphs before Theorem 1). In other words, the clone-theoretic formula

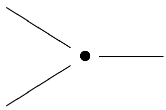
$$\mathrm{Interval} \equiv (\exists a, b)(a \neq b \,\&\, \mathrm{Linear}(a, b) \,\&\, \mathrm{Con} \,\&\, \mathrm{Cen})$$

characterizes the interval in the category Poset.

Using the methods of Section 2 one can now characterize any finite power of I (in the category Poset). Also it is possible to characterize some posets that arise as amalgamations of those posets, e.g. the "circle"

$$\{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\} \quad \text{with} \quad (x, y) \leq (x', y') \ \text{ iff } \ x \leq y \ \text{ and } \ yy' \geq 0,$$

the "triode"



ordered "from left to right", more generally each finite topological tree (i.e. a finite tree made of intervals and ordered from root to leaves) etc.

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