Relatively exact modules

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Abstract. Rim and Teply [10] investigated relatively exact modules in connection with the existence of torsionfree covers. In this note we shall study some properties of the lattice $\mathcal{E}_\tau(M)$ of submodules of a torsionfree module M consisting of all submodules N of M such that M/N is torsionfree and such that every torsionfree homomorphic image of the relative injective hull of M/N is relatively injective. The results obtained are applied to the study of relatively exact covers of torsionfree modules. As an application we also obtain some new characterizations of perfect torsion theories.

Keywords: Hereditary torsion theory τ, τ-injective module, τ-exact module, preradical, exact torsion theory, perfect torsion theory

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In what follows, R stands for an associative ring with the identity element and R-mod denotes the category of all unital left R-modules. By the word "module" we shall always mean the left R-module. If N is a submodule of a module M and $u \in M$ is an arbitrary element, then $(N : u) = \{r \in R \,|\, ru \in N\}$ denotes the (*left*) *annihilator ideal* of the element u over the submodule N. The basic properties of rings and modules can be found in [1].

A class $\mathcal{G}$ of modules is called *abstract*, if it is closed under isomorphic copies. If $\mathcal{G}$ is an abstract class of modules, then a homomorphism $\varphi : G \to M$ with $G \in \mathcal{G}$ is called a $\mathcal{G}$*-precover* of the module M, if for each homomorphism $f : F \to M$ with $F \in \mathcal{G}$ there exists a homomorphism $g : F \to G$ such that $\varphi g = f$. A $\mathcal{G}$-precover φ of M is said to be the $\mathcal{G}$*-cover*, if every endomorphism f of G with $\varphi f = \varphi$ is the automorphism of the module G. An abstract class $\mathcal{G}$ of modules is called a *precover* (*cover*) *class*, if every module has a $\mathcal{G}$-precover ($\mathcal{G}$-cover). A more detailed study of precovers and covers can be found in [13].

By a *preradical* r in the category $\mathcal{C}$ of modules with $0 \in \mathcal{C}$ is meant any subfunctor of the identity functor. This means, that r assigns to each object M of $\mathcal{C}$ its submodule $r(M)$ in such a way, that $f(r(M)) \subseteq r(N)$ for any object $N \in \mathcal{C}$ and any homomorphism $f : M \to N$. A preradical r is said to be *idempotent* if $r(r(M)) = r(M)$ for each $M \in \mathcal{C}$ and it is called a *radical* if $r(M/r(M)) = 0$ for every $M \in \mathcal{C}$.

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Recall that a *hereditary torsion theory* $\tau = (\mathcal{T}, \mathcal{F})$ for the category R-mod consists of two abstract classes $\mathcal{T}$ and $\mathcal{F}$, the *τ-torsion class* and the *τ-torsionfree class*, respectively, such that $\mathrm{Hom}\,(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$, the class $\mathcal{T}$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathcal{F}$ is closed under submodules, extensions and arbitrary direct products and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$. It is easy to see that every module M contains the unique largest τ-torsion submodule (isomorphic to T), which is called the *τ-torsion part of the module M* and which is usually denoted by $\tau(M)$. To each hereditary torsion theory τ it is associated the *Gabriel filter* $\mathcal{L}$ of left ideals of the ring R consisting of all the left ideals $I \leq R$ with $R/I \in \mathcal{T}$. Recall that τ is said to be of *finite type*, if $\mathcal{L}$ contains a cofinal subset of finitely generated left ideals. If N is a submodule of the module $M \in \mathcal{F}$, then the *τ-closure* $\mathrm{Cl}^{M}(N)$ of the submodule N in the module M is given by the formula $\mathrm{Cl}^{M}(N)/N = \tau(M/N)$. A submodule N of a module M is said to be *τ-pure* (*τ-closed*) in M, if the factor-module M/N is τ-torsionfree. Further, N is said to be *τ-dense* in M, if the factor-module M/N is τ-torsion. It is well-known that to each module M there exists an essential monomorphism $\iota : M \to E(M)$ of M into its *injective hull* $E(M)$. If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory, then a module Q is called *τ-injective*, if it is injective with respect to all short exact sequences $0 \to A \to B \to C \to 0$, where $C \in \mathcal{T}$. Note that the *Baer Test Lemma* says, that the module Q is τ-injective, if it is injective with respect to all inclusions $I \to R$, where I is a left ideal of R with $R/I \in \mathcal{T}$, i.e. with respect to all inclusions $I \to R$, where I belongs to the Gabriel filter $\mathcal{L}$ corresponding to τ. If M is an arbitrary τ-torsionfree module, then the module $E_\tau(M)$ given by the formula $E_\tau(M)/M = \tau(E(M)/M)$ is the *τ-injective hull* of the module M. The class of all τ-injective modules will be denoted by $\mathcal{I}_\tau$. Following [10] we say, that a τ-torsionfree module is *τ-exact*, if any of its τ-torsionfree homomorphic images is τ-injective. The class of all such modules will be denoted by $\mathcal{E}_\tau$. A hereditary torsion theory τ is called *exact*, if $E(Q)/Q$ is τ-torsionfree τ-injective whenever Q is so. Note that in this case the equality $\mathcal{E}_\tau = \mathcal{I}_\tau \cap \mathcal{F}$ holds. An exact torsion theory which is of finite type is called *perfect*. Finally, we denote by $Q_\tau(R)$ the τ-injective hull of the factor-module $R/\tau(R)$. For more details on preradicals and torsion theories see e.g. [9] or [8].

Rim and Teply [10] proved that a necessary condition for the existence of τ-torsionfree covers is that any directed union of τ-exact modules is τ-injective. In this note we denote by $\mathcal{E}_\tau(M)$ the set of all τ-pure submodules N of a τ-torsionfree module M having the property, that the relative injective hull $E_\tau(M/N)$ is τ-exact. We shall show that $\mathcal{E}_\tau(M)$ is a filter in the lattice $\mathcal{P}_\tau(M)$ of all τ-pure submodules of M and we characterize the members of $\mathcal{E}_\tau(M)$ as those elements $N \in \mathcal{P}_\tau(M)$, for which every τ-torsionfree homomorphic image of $E_\tau(M/N)$ is isomorphic to $E_\tau(M/K)$ for some $K \in \mathcal{P}_\tau(M)$ containing N. Using these results we shall study some properties of the idempotent preradical r on the subcategory $\mathcal{F}$ of R-mod, where $r(M)$ denotes the submodule of the module $M \in \mathcal{F}$ generated by all τ-exact submodules of M. Note that this preradical can be viewed as a direct generalization of that in the category Ab of all abelian groups which to each torsionfree abelian group assigns its largest divisible subgroup. The results are applied to obtain some new characterizations of perfect torsion theories using the τ-torsionfree and τ-exact covers.

Lemma 1. *Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of modules. If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory for the category R-mod, then the following hold:*

(i) *if $A, C \in \mathcal{I}_\tau$, then $B \in \mathcal{I}_\tau$;*
(ii) *if $B \in \mathcal{I}_\tau$ and $C \in \mathcal{F}$, then $A \in \mathcal{I}_\tau$;*
(iii) *if $A \in \mathcal{I}_\tau$ and $B \in \mathcal{F}$, then $C \in \mathcal{F}$.*

Proof: (i) Let $I \in \mathcal{L}$ together with the inclusion map $\iota : I \to R$ be arbitrary and let $f : I \to B$ be an arbitrary homomorphism. Now $C \in \mathcal{I}_\tau$ yields $g\iota = pf$ for some $g : R \to C$ and consequently $g = ph$ for some $h : R \to B$, R being projective. Thus $p(f - h\iota) = pf - g\iota = 0$ gives $f - h\iota = ik$ for a homomorphism $k : I \to A$. Since A is τ-injective, there is a homomorphism $l : R \to A$ with $l\iota = k$ and the homomorphism $il + h : R \to B$ extends f in view of the equalities $(il + h)\iota = ik + h\iota = f$.

(ii) Consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \xrightarrow{\iota} & R & \xrightarrow{\pi} & R/I & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0
\end{array}$$

with exact rows, where $I \in \mathcal{L}$, ι is the inclusion map, π is the canonical projection and $f : I \to A$ is an arbitrary homomorphism. Then $B \in \mathcal{I}_\tau$ gives $g\iota = if$ for some $g : R \to B$ and $pg\iota = pif = 0$ yields $h\pi = pg$ for some $h : R/I \to C$. However, $R/I \in \mathcal{T}$ and $C \in \mathcal{F}$ give $h = 0$ and consequently $ik = g$ for a homomorphism $k : R \to A$. Finally, $i(f - k\iota) = if - g\iota = 0$ yields $f = k\iota$, i being a monomorphism.

(iii) Denoting $L = p^{-1}(\tau(C))$, the τ-injectivity of A yields the existence of a homomorphism $f : L \to A$ such that $fj = 1_A$, j being the embedding of A into L. Thus $L = j(A) \oplus U$ and $U \cong L/j(A) \cong \tau(C) \in \mathcal{T} \cap \mathcal{F} = 0$ shows that C is τ-torsionfree. □

Lemma 2. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod and let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of modules. Then the following hold:*

(i) *if $B \in \mathcal{E}_\tau$ and $C \in \mathcal{F}$, then $A \in \mathcal{E}_\tau$;*
(ii) *if $A, C \in \mathcal{E}_\tau$, then $B \in \mathcal{E}_\tau$.*

Proof: Without loss of generality we may assume that i is the inclusion map.

(i) If K is any τ-pure submodule of A, then B/K is τ-torsionfree as an extension of A/K by C. Hence $B/K \in \mathcal{I}_\tau$ by the hypothesis and so $A/K \in \mathcal{I}_\tau$ by Lemma 1(ii).

(ii) Let K be a τ-pure submodule of the module B. Since $(A+K)/K \cong A/(A \cap K)$ lies in $\mathcal{I}_\tau$ by the hypothesis, $A+K$ is τ-pure in B by Lemma 1(iii). Thus $B/(A+K) \in \mathcal{I}_\tau$ as a τ-torsionfree homomorphic image of C and consequently $B/K \in \mathcal{I}_\tau$ by Lemma 1(i). □

Notation. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory and let $M \in R$-mod be arbitrary. We denote by $\mathcal{P}_\tau(M)$ the set of all τ-pure submodules of the module M, i.e. $\mathcal{P}_\tau(M) = \{N \leq M \mid M/N \in \mathcal{F}\}$ and by $\mathcal{E}_\tau(M) = \{N \in \mathcal{P}_\tau(M) \mid E_\tau(M/N) \in \mathcal{E}_\tau\}$ the subset of $\mathcal{P}_\tau(M)$ consisting of all submodules $N \in \mathcal{P}_\tau(M)$ such that the τ-injective hull of the factor-module M/N is τ-exact.

Proposition 3. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory. If M is a τ-torsionfree module, then $\mathcal{E}_\tau(M)$ is a filter in the lattice $\mathcal{P}_\tau(M)$.*

Proof: Let $N \in \mathcal{E}_\tau(M)$ and $K \in \mathcal{P}_\tau(M)$ with $N \subseteq K$ be arbitrary. We are going to consider the following diagram

$$\begin{array}{ccc} M/N & \xrightarrow{\ i\ } & E_\tau(M/N) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ M/K & \xrightarrow{\ j\ } & E_\tau(M/K) \end{array}$$

where i, j are inclusion maps and π is the canonical projection. The τ-density of i gives the existence of a homomorphism φ making the square commutative. Now $M/K \subseteq \operatorname{Im}\varphi$ yields that $\operatorname{Im}\varphi$ is τ-dense in $E_\tau(M/K)$, while $E_\tau(M/N) \in \mathcal{E}_\tau$ yields that $\operatorname{Im}\varphi$ is τ-pure in $E_\tau(M/K)$ by Lemma 1(iii). Thus φ is surjective and so $K \in \mathcal{E}_\tau(M)$.

Now let $N, K \in \mathcal{E}_\tau(M)$ be arbitrary. We shall consider the following diagram

$$\begin{array}{ccc} M/(N \cap K) & \xrightarrow{\ i\ } & E_\tau(M/(N \cap K)) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ M/N \oplus M/K & \xrightarrow{\ j\ } & E_\tau(M/N) \oplus E_\tau(M/K) \end{array}$$

where i and j are inclusion maps and the monomorphism α is given by the natural formula $\alpha(u + N \cap K) = (u + N, u + K)$ for each $u \in M$. There is a homomorphism β making the square commutative, the inclusion i being τ-dense. Now $j\alpha$ is injective, i is essential and consequently β is a monomorphism. Thus $\operatorname{Im}\beta \in \mathcal{I}_\tau$ yields that $\operatorname{Im}\beta$ is τ-pure in $E_\tau(M/N) \oplus E_\tau(M/K)$ by Lemma 1(iii) and Lemma 2(i) finishes the proof, the last module being τ-exact by Lemma 2(ii).

□

Proposition 4. *If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory, then for every $I \in \mathcal{E}_\mathcal{T}(R)$ and every $a \in R$ the annihilator left ideal $(I : a)$ lies in $\mathcal{E}_\mathcal{T}(R)$.*

PROOF: Clearly, $R(a+I) \leq R/I$ yields $E_\mathcal{T}(R(a+I)) \leq E_\mathcal{T}(R/I)$ and so $E_\mathcal{T}(R(a+I)) \in \mathcal{E}_\mathcal{T}$ by Lemma 1(iii) and Lemma 2(i). The isomorphism $E_\mathcal{T}(R/(I : a)) \cong E_\mathcal{T}(R(a+I))$ now gives that $(I : a) \in \mathcal{E}_\mathcal{T}(R)$. $\square$

Proposition 5. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory, let N be a τ-pure submodule of a τ-torsionfree module M and let L be any τ-torsionfree homomorphic image of $E_\mathcal{T}(M/N)$. Then there is a submodule $K \in \mathcal{P}_\mathcal{T}(M)$ containing N such that L is isomorphic to a submodule of $E_\mathcal{T}(M/K)$ containing M/K.*

PROOF: Consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K/N & \xrightarrow{j} & M/N & \xrightarrow{\sigma} & M/K & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & U & \xrightarrow{i} & E_\mathcal{T}(M/N) & \xrightarrow{\pi} & L & \longrightarrow & 0
\end{array}$$

with a given epimorphism π and $U = \operatorname{Ker}\pi$. Further, $K/N = (M/N) \cap U$, j, i, α, β are inclusion maps and σ is the natural projection. Since $\pi\beta j = \pi i \alpha = 0$, there is $\gamma : M/K \to L$ making the right square commutative and it clearly remains to verify that γ is an essential and a τ-dense monomorphism, because in this case $K \in \mathcal{P}_\mathcal{T}(M)$, M/K being isomorphic to a submodule of the τ-torsionfree module L.

Assume first, that $\gamma(v + K) = 0$ for some $v \in M$. Then $\pi\beta(v + N) = \gamma\sigma(v + N) = 0$ and so $\beta(v + N) = i(u)$ for some $u \in U$. Hence $u = v + N$ lies in K/N, $v + K = \sigma j(v + N) = 0$ and γ is a monomorphism.

Let $l \in L \setminus \gamma(M/K)$ be arbitrary. Then $l = \pi(x)$ for some $x \in E_\mathcal{T}(M/N)$ and obviously, $J = (M/N : x)$ is essential in R and lies in the Gabriel filter $\mathcal{L}$. If $r \in J$ is arbitrary, then $rx \in M/N$ yields $rl = \pi(rx) \in \pi\beta(M/N)$. However, $\pi\beta(M/N) = \gamma\sigma(M/N) = \gamma(M/K)$ and consequently $r \in (\gamma(M/K) : l)$. This means that $J \subseteq (\gamma(M/K) : l)$ and $\gamma(M/K)$ is τ-dense in L. Moreover, $(\gamma(M/K) : l) \in \mathcal{L}$ and $(0 : l) \notin \mathcal{L}$ yields the existence of an element $r \in R$ with $0 \neq rl \in \gamma(M/K)$, $\gamma(M/K)$ is essential in L and the proof is now complete. $\square$

Corollary 6. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory and let N be a τ-pure submodule of a τ-torsionfree module M. Then $N \in \mathcal{E}_\mathcal{T}(M)$ if and only if every τ-torsionfree homomorphic image of $E_\mathcal{T}(M/N)$ is isomorphic to $E_\mathcal{T}(M/K)$ for some $K \in \mathcal{E}_\mathcal{T}(M)$ containing N.*

PROOF: Assume first, that $N \in \mathcal{E}_\mathcal{T}(M)$. By the preceding proposition every τ-torsionfree homomorphic image L of $E_\mathcal{T}(M/N)$ is, up to an isomorphism, contained in $E_\mathcal{T}(M/K)$ and contains M/K for a suitable submodule $K \in \mathcal{P}_\mathcal{T}(M)$. It follows from Proposition 3 that $K \in \mathcal{E}_\tau(M)$. But then L is obviously τ-dense and τ-pure in $E_\tau(M/K)$ and hence $L = E_\tau(M/K)$. The converse is obvious. $\square$

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod. For an arbitrary τ-torsionfree module M we denote by $r(M)$ the submodule of M generated by all τ-exact submodules of M.

Theorem 7. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod and let M be an arbitrary τ-torsionfreee module. Then*

(i) *$r(M) = \sum \mathrm{Cl}^M(Ra)$, where a ranges through all the elements of the module M with $\mathrm{Cl}^M(Ra) \in \mathcal{E}_\tau$;*
(ii) *r is an idempotent preradical on the subcategory $\mathcal{F}$ of R-mod;*
(iii) *if $\mathcal{F}$ is a cover class, then $r(M) \in \mathcal{E}_\tau$ for each module $M \in \mathcal{F}$ and r is an idempotent radical on $\mathcal{F}$;*
(iv) *if $\mathcal{F}$ is a cover class, then for each module $M \in \mathcal{F}$ the embedding $\varphi : r(M) \to M$ is the τ-exact cover of M.*

Proof: (i) Letting $s(M) = \sum \mathrm{Cl}^M(Ra)$ we clearly have $s(M) \subseteq r(M)$. To prove the converse we first note, that if $N \le M$, $N \in \mathcal{E}_\tau$, and $0 \ne a \in N$, then $\mathrm{Cl}^N(Ra) \in \mathcal{E}_\tau$ by Lemma 2(i). Since $Ra \cong R/(0:a)$, we see that $(0:a) \in \mathcal{E}_\tau(R)$. So, for an arbitrary element $a \in r(M)$ we have $a = a_1 + \cdots + a_n$, where $(0 : a_i) \in \mathcal{E}_\tau(R)$ and consequently $\bigcap_{i=1}^n (0 : a_i) \in \mathcal{E}_\tau(R)$ by Proposition 3. Moreover, the inclusion $\bigcap_{i=1}^n (0 : a_i) \subseteq (0 : a)$ yields that $(0 : a) \in \mathcal{E}_\tau(R)$ by Proposition 3 again, and hence $\mathrm{Cl}^M(Ra) \in \mathcal{E}_\tau$, showing that $a \in s(M)$, as we wished to show.

(ii) If $f : M \to N$ is an arbitrary homomorphism of τ-torsionfree modules, then by (i) we have $f(r(M)) = f(\sum \mathrm{Cl}^M(Ra)) \subseteq \sum f(\mathrm{Cl}^M(Ra)) \subseteq r(N)$. Clearly, $r(M) = \sum \mathrm{Cl}^M(Ra) \subseteq r(r(M))$, which proves the idempotency of r.

(iii) By [10] (see also [7, Proposition 3]) the class $\mathcal{E}_\tau$ is closed under directed unions of submodules, and consequently under arbitrary direct sums. Since $r(M)$ is expressible as a homomorphic image of the direct sum of τ-exact modules $\mathrm{Cl}^M(Ra)$, it is τ-exact. By Lemma 2(ii) the factor-module $M/r(M)$ contains no non-zero τ-exact submodule and consequently r is a radical on the class $\mathcal{F}$.

(iv) Let $Q \in \mathcal{E}_\tau$ and $f : Q \to M$, $M \in \mathcal{F}$, be arbitrary. Then $\operatorname{Im} f \in \mathcal{F}$ yields $\operatorname{Im} f \in \mathcal{E}_\tau$ by the definition and consequently $f(Q) \subseteq r(M)$ yields that φ is an $\mathcal{E}_\tau$-precover of the module M. The rest is obvious. $\square$

To conclude this note we are going to give some new characterizations of perfect torsion theories.

Theorem 8. *The following conditions are equivalent for a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ for the category R-mod:*

(i) *τ is perfect;*
(ii) *τ is exact and $\mathcal{F}$ is a cover class;*

(iii) *every non-zero τ-torsionfree homomorphic image of a τ-injective module has a non-zero τ-exact cover and $\mathcal{F}$ is a cover class;*

(iv) *every non-zero τ-torsionfree homomorphic image of $Q_\mathcal{T}(R)$ has a non-zero τ-exact cover and $\mathcal{F}$ is the cover class;*

(v) *$Q_\mathcal{T}(R)$ is τ-exact and $\mathcal{F}$ is the cover class.*

PROOF: (i) implies (ii). By hypothesis, τ is perfect, hence it is exact and of finite type and consequently [12] and [5] applies.

(ii) implies (iii). This is trivial, since $\mathcal{E}_\mathcal{T} = \mathcal{I}_\mathcal{T} \cap \mathcal{F}$ in this case.

(iii) implies (iv) obviously.

(iv) implies (v). If $U \subsetneqq Q_\mathcal{T}(R)$ is the τ-exact cover of $Q_\mathcal{T}(R)$, then $Q_\mathcal{T}(R)/U \neq 0$ has zero τ-exact cover by Theorem 7(iii), which contradicts the hypothesis.

(v) implies (i). Let Q be an arbitrary τ-torsionfree τ-injective module. If $0 \neq a \in Q$ is arbitrary, then $Ra \in \mathcal{F}$ is a homomorphic image of $R/\tau(R)$. Thus we have the following commutative diagram

$$\begin{array}{ccc} R/\tau(R) & \xrightarrow{\ i\ } & Q_\mathcal{T}(R) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ Ra & \xrightarrow{\ j\ } & E_\mathcal{T}(Ra) \end{array}$$

where i and j are the inclusion maps and π is the natural epimorphism. Clearly, there is a homomorphism $\varphi : Q_\mathcal{T}(R) \to E_\mathcal{T}(Ra)$ making the square commutative and $\operatorname{Im}\varphi$ is τ-pure in $E_\mathcal{T}(Ra)$ by the hypothesis and Lemma 1(iii). On the other hand, $Ra \subseteq \operatorname{Im}\varphi$ means that $\operatorname{Im}\varphi$ is τ-dense in $E_\mathcal{T}(Ra)$ and consequently φ is an epimorphism showing that $E_\mathcal{T}(Ra) \in \mathcal{E}_\mathcal{T}$. From this we infer, that Q is a homomorphic image of a direct sum of τ-exact modules of the form $E_\mathcal{T}(Ra)$ and so it is τ-exact by [7, Proposition 3]. Thus $\mathcal{E}_\mathcal{T} = \mathcal{I}_\mathcal{T} \cap \mathcal{F}$, and τ is therefore exact. Moreover, by [10], any directed union of τ-torsionfree τ-injective modules is τ-injective and the torsion theory τ is of finite type by [9, Proposition 42.9]. □

Corollary 9. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category* R-mod *of all left R-modules over a left hereditary ring R. Then $\mathcal{F}$ is the cover class if and only if τ is of finite type.*

PROOF: By [9, Corollary 44.2] the torsion theory τ is exact and it suffices to use Theorem 8. □

Note. In the category Z-mod of all abelian groups with the ordinary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, $r(M)$ is simply the divisible part of the given torsionfree abelian group M.

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