# Lattices and semilattices having an antitone involution in every upper interval

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*Abstract.* We study $\vee$-semilattices and lattices with the greatest element 1 where every interval [p,1] is a lattice with an antitone involution. We characterize these semilattices by means of an induced binary operation, the so called sectionally antitone involution. This characterization is done by means of identities, thus the classes of these semilattices or lattices form varieties. The congruence properties of these varieties are investigated.

*Keywords:* semilattice, lattice, antitone involution, congruence permutability, weak regularity

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Join-semilattices whose principal filters are Boolean algebras were used by J.C. Abbott [1] for a characterization of the logic connective implication in the classical proposition logic.

A similar approach was used in [4] for a characterization of the connective implication in the logic of quantum mechanics where the principal filters are considered to be orthomodular lattices. This method was generalized in [3] to introduce and characterize lattices whose principal ideals are pseudocomplemented lattices; it enables us to extend the concept of relative pseudocomplementation also to the case of non-distributive lattices.

The aim of our paper is to generalize the mentioned approach as much as possible to obtain algebraic structures with "nice" properties (a characterization by identities, nice congruence properties, a tractable description of congruences).

Let $A$ be a set. A mapping $x \mapsto x'$ of $A$ into itself is called an *involution* whenever $x'' = x$. Let $(A; \leq)$ be an ordered set. A mapping $x \mapsto x'$ of $A$ into itself is called *antitone* whenever $a \leq b$ implies $b' \leq a'$ for all $a, b \in A$.

At first, we can list several elementary properties of antitone involutions.

**Lemma 1.** *Let $(A; \leq)$ be an ordered set. A unary mapping $x \mapsto x'$ of $A$ is an antitone involution on $(A; \leq)$ if and only if satisfies*

$$a \leq a'' \quad \text{and} \quad a' \leq b' \Rightarrow b \leq a$$

*for every $a, b \in A$.*

Proof: $a \leq a''$ for $a = c'$ gets $c' \leq c'''$ thus, by the second rule, $c'' \leq c$. Together with the first rule, we obtain $c'' = c$ for each $c \in A$ and hence this mapping is an involution on $A$. Suppose $a, b \in A$ with $a \leq b$. Then $a'' = a \leq b = b''$ and, by the second rule, $b' \leq a'$, i.e. the mapping is antitone. □

**Lemma 2.** *Let $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice and $x \mapsto x'$ be an antitone involution on $\mathcal{L}$. Then*

(i) *0'=1 and 1'=0;*

(ii) *$\mathcal{L}$ satisfies the so called DeMorgan laws:*

$$(x \vee y)' = x' \wedge y' \quad \text{and} \quad (x \wedge y)' = x' \vee y'.$$

Proof: (i) Since $x \leq 1$ for each $x \in L$, we have $1' \leq x'$ and, due to $x'' = x$, $1'$ is the least element of $\mathcal{L}$, i.e. $1' = 0$. Dually we can show $0' = 1$.

(ii) $x, y \leq x \vee y$ implies $(x \vee y)' \leq x', y'$ thus $(x \vee y)' \leq x' \wedge y'$. Further, $x' \wedge y' \leq x', y'$ yields $x = x'' \leq (x' \wedge y')'$ and $y = y'' \leq (x' \wedge y')'$ showing $x \vee y \leq (x' \wedge y')'$, i.e. $x' \wedge y' \leq (x \vee y)'$. Together we have $(x \vee y)' = x' \wedge y'$. The second law can be proved dually. □

**Lemma 3.** *Let $\mathcal{S} = (S; \vee)$ be a join-semilattice. A mapping $x \mapsto x'$ of $S$ into itself is an antitone involution whenever the following identity is satisfied:*

$$((x \vee y)' \vee y')' = y.$$

Proof: By putting $x = y$, the identity yields $y'' = y$ thus the mapping is an involution. Moreover, $y \leq x$ implies $(x' \vee y')' = y$ and hence $x' \vee y' = y'$ proving $x' \leq y'$, i.e. it is also antitone.
Conversely, we have $y \leq x \vee y$ for each $x, y \in S$ thus $(x \vee y)' \leq y'$ for an antitone mapping, i.e. $(x \vee y)' \vee y' = y'$. Since it is an involution, we obtain the identity directly. □

Let $\mathcal{S} = (S; \vee, 1)$ be a join-semilattice with the greatest element 1 and $p \in S$. A mapping $x \mapsto x^p$ of the interval $[p, 1]$ will be called a *section antitone involution* (on the interval $[p, 1]$) whenever it is an antitone involution on the ordered set $[p, 1]$ with respect to the induced order.

We can study semilattices or lattices with the greatest element 1 where every interval $[p, 1]$ has a section antitone involution $x \mapsto x^p$. Unfortunately, this unary operation $x^p$ is defined only for $x \in [p, 1]$. To avoid this discrepancy, we introduce a binary operation $x \circ y$ on $\mathcal{S}$ as follows

$$x \circ y = (x \vee y)^y.$$

Of course, $x \vee y \in [y, 1]$, thus $x \circ y$ is everywhere defined provided the semilattice $\mathcal{S} = (S; \vee; 1)$ has section antitone involutions on every interval $[y, 1]$ for $y \in S$. If it is the case, we will call the structure $\mathcal{S} = (S; \vee, \circ, 1)$ a *semilattice with sectionally antitone involutions*. If $\mathcal{L} = (L; \vee, \wedge, 1)$ is a lattice with the greatest element 1 such that $(L; \vee, \circ, 1)$ is a semilattice with sectionally antitone involutions then $\mathcal{L} = (L; \vee, \wedge, \circ, 1)$ will be called a *lattice with sectionally antitone involutions*.

**Example.** Consider the (semi)lattice $\mathcal{S}$ depicted in Figure 1:

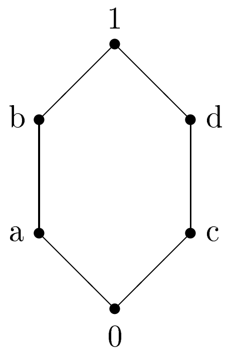


Figure 1

$$
\begin{array}{lll}
\text{Define} & 0^0 = 1,\ 1^0 = 0,\ a^0 = d,\ b^0 = c,\ c^0 = b,\ d^0 = a & \text{in } [0,1] \\
& a^a = 1,\ b^a = b,\ 1^a = a & \text{in } [a,1] \\
& c^c = 1,\ d^c = d,\ 1^c = c & \text{in } [c,1] \\
& b^b = 1,\ 1^b = b & \text{in } [b,1] \\
& d^d = 1,\ 1^d = d & \text{in } [d,1].
\end{array}
$$

Then $\mathcal{S}$ is a (semi)lattice with sectionally antitone involutions and the operation $\circ$ is determined by the table:

| $\circ$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| a | d | 1 | 1 | c | d | 1 |
| b | c | b | 1 | c | d | 1 |
| c | b | a | c | 1 | 1 | 1 |
| d | a | a | b | d | 1 | 1 |
| 1 | 0 | a | b | c | d | 1 |

Semilattices with sectionally antitone involutions can be characterized by identities in the signature $\{\vee, \circ\}$ as follows:

**Theorem 1.** *Let $\mathcal{S} = (S; \vee, \circ, 1)$ be an algebra of type $(2, 2, 0)$ such that $(S; \vee, 1)$ is a $\vee$-semilattice with the greatest element 1. Then $\mathcal{S}$ is a semilattice with sectionally antitone involutions if and only if it satisfies the identities*

(1) $(x \circ y) \circ y = x \vee y$,
(2) $((x \vee y \vee z) \circ z) \vee ((x \vee z) \circ z) = (x \vee z) \circ z$.

Proof: Let $\mathcal{S} = (S; \vee, \circ, 1)$ be a semilattice with sectionally antitone involutions where $x \circ y = (x \vee y)^y$. Then $((x \circ y) \circ y) = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$ since $(x \vee y)^y \in [y, 1]$ yields $y \leq (x \vee y)^y$. Further, $x \vee z \leq x \vee y \vee z$ and $x \vee z, x \vee y \vee z \in [z, 1]$, thus $(x \vee z)^z \geq (x \vee y \vee z)^z$ and hence $(x \vee y \vee z)^z \vee (x \vee z)^z = (x \vee z)^z$ proving the identity (2).

Conversely, let $\mathcal{S} = (S; \vee, \circ, 1)$ be an algebra satisfying (1) and (2) such that $(S; \vee, 1)$ is a join-semilattice with the greatest element 1.

For $p \in S$ we define a mapping $a \mapsto a^p$ on the interval $[p, 1]$ by the setting $a^p = a \circ p$. For $a \in [p, 1]$ we have $p \leq a$ and hence $a^{pp} = (a \circ p) \circ p = a \vee p = a$ by (1). Further, for $a, b \in [p, 1]$ with $a \leq b$ we have by (2)

$$(b \circ p) \vee (a \circ p) = ((a \vee b \vee p) \circ p) \vee ((a \vee p) \circ p) = (a \vee p) \circ p = a \circ p$$

proving $b^p \vee a^p = a^p$, i.e. $b^p \leq a^p$. Altogether, $a \mapsto a^p$ is an antitone involution on every interval $[p, 1]$ of $S$. $\square$

Since semilattices or lattices with the greatest element 1 are defined by a finite set of semilattice or lattice identities respectively, we can state an immediate consequence of Theorem 1:

**Corollary.** *The class of all semilattices (lattices) with sectionally antitone involutions considered in the signature $\{\vee, \circ, 1\}$ ($\{\vee, \wedge, \circ, 1\}$, respectively) is a finitely presented variety.*

**Remark.** Due to the identity (1), the class of all semilattices with sectionally antitone involutions is in fact a variety in the signature $\{\circ, 1\}$.

By using of the definition $x \circ y = (x \vee y)^y$, one can easily prove the following

**Lemma 4.** *Let $\mathcal{S} = (S; \vee, \circ, 1)$ be a semilattice with sectionally antitone involutions. Then $\mathcal{S}$ satisfies the identities $x \circ x = 1$, $x \circ 1 = 1$ and $1 \circ x = x$.*

Now, we will study certain congruence properties of these varieties. Recall that a variety $\mathcal{V}$ is *congruence permutable* (3-*permutable*) if $\Theta \circ \Phi = \Phi \circ \Theta$ (or $\Theta \circ \Phi \circ \Theta = \Phi \circ \Theta \circ \Phi$) for each $\mathcal{A} \in \mathcal{V}$ and every $\Theta, \Phi \in \operatorname{Con} \mathcal{A}$. If $\mathcal{V}$ is congruence permutable (3-permutable) then $\Theta \vee \Phi = \Phi \circ \Theta$ (or $\Theta \vee \Phi = \Theta \circ \Phi \circ \Theta$, respectively) holds in $\operatorname{Con} \mathcal{V}$. A variety $\mathcal{V}$ is *congruence distributive* if the lattice $\operatorname{Con} \mathcal{A}$ is distributive for every $\mathcal{A} \in \mathcal{V}$.

It is well-known that a variety $\mathcal{V}$ is congruence permutable if and only if there exist a *Mal'cev term*, i.e. a 3-ary term $p$ of $\mathcal{V}$ such that $\mathcal{V}$ satisfies the identities

$$p(x, z, z) = x \quad \text{and} \quad p(x, x, z) = z;$$

$\mathcal{V}$ is 3-permutable if and only if there exist 3-ary terms $t_1, t_2$ of $\mathcal{V}$ such that $\mathcal{V}$ satisfies the identities

$$x = t_1(x, z, z),\ t_1(x, x, z) = t_2(x, z, z),\ t_2(x, x, z) = z$$

(see e.g. [2] for some details). A variety $\mathcal{V}$ is *arithmetical* if it is both congruence permutable and congruence distributive.

**Theorem 2.** *The variety of lattices with sectionally antitone involutions is arithmetical* (*i.e. congruence permutable and distributive*). *Its Mal'cev term is*

$$p(x, y, z) = ((x \circ y) \circ z) \wedge ((z \circ y) \circ x).$$

PROOF: Since it has a majority term

$$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z),$$

it is congruence distributive. To prove arithmeticity, we need to show that it is congruence permutable. For this, it is enough to find a Mal'cev term. By using the identity (1) and the identities of Lemma 4, we compute

$$p(x, x, z) = ((x \circ x) \circ z) \wedge ((z \circ x) \circ x) = z \wedge (z \vee x) = z$$

and

$$p(x, z, z) = ((x \circ z) \circ z) \wedge ((z \circ z) \circ x) = (x \wedge z) \vee x = x$$

whence $p(x, y, z)$ is a Mal'cev term. □

**Remark.** We are able to get the *Pixley term* ensuring arithmeticity directly. For this, we should firstly compute:

$$(x \circ y) \circ x = ((x \vee y)^y \vee x)^x.$$

Since $(x \vee y)^y \vee x \in [x, 1]$, also $((x \vee y)^y \vee x)^x \in [x, 1]$ and hence $x \leq (x \circ y) \circ x$.

Now, we can set

$$t(x, y, z) = ((x \circ y) \circ z) \wedge ((z \circ y) \circ x) \wedge (x \vee z).$$

Similarly as in the proof of Theorem 2, one can see immediately that $t(x, y, z)$ is a Mal'cev term. Moreover,

$$t(x, y, x) = ((x \circ y) \circ x) \wedge ((x \circ y) \circ x) \wedge x = x$$

due to the previous computation. Hence, $t(x, y, z)$ is a Pixley term of the variety of lattices with sectionally antitone involutions.

Let $\mathcal{S} = (S; \vee, \circ, 1)$. A subset $\emptyset \neq K \subseteq S$ is called a *congruence kernel* of $\mathcal{S}$ if $K = [1]_\Theta = \{x \in S; \langle x, 1\rangle \in \Theta\}$ for some congruence $\Theta \in \operatorname{Con} \mathcal{S}$.

Recall (from [5]) that an algebra $\mathcal{A}$ with a constant 1 is *weakly regular* if every congruence $\Theta \in \operatorname{Con} \mathcal{A}$ is determined by its kernel, i.e. if $[1]_\Theta = [1]_\Phi$ implies $\Theta = \Phi$ for every $\Theta, \Phi \in \operatorname{Con} \mathcal{A}$. A variety $\mathcal{V}$ is *weakly regular* if every $\mathcal{A} \in \mathcal{V}$ has this property. The following characterization of weakly regular varieties was given by B. Csakany [5]:

**Proposition.** *A variety $\mathcal{V}$ with 1 is weakly regular if and only if there exist $n \in \mathbb{N}$ and binary terms $b_1(x,y), \dots, b_n(x,y)$ such that*

$$b_1(x,y) = \dots = b_n(x,y) = 1 \quad \textit{if and only if} \quad x = y$$

*is satisfied for every $\mathcal{A} \in \mathcal{V}$.*

Now, we can prove

**Theorem 3.** *The variety $\mathcal{V}$ of semilattices (lattices) with sectionally antitone involutions is weakly regular.*

Proof: We can take $n = 2$ and $b_1(x,y) = x \circ y$, $b_2(x,y) = y \circ x$. By Lemma 4, we have $b_1(x,x) = x \circ x = 1$, $b_2(x,x) = x \circ x = 1$. Conversely, suppose $b_1(x,y) = b_2(x,y) = 1$ for $\mathcal{S} \in \mathcal{V}$ and $x, y \in S$. Then $(x \vee y)^y = 1$ and $(y \vee x)^x = 1$. Due to Lemma 2, we have

$$x \vee y = y \quad \text{and} \quad y \vee x = x$$

whence $x = y$. By the Proposition, the variety $\mathcal{V}$ is weakly regular. □

**Theorem 4.** *The variety of semilattices with sectionally antitone involutions is congruence 3-permutable and congruence distributive.*

Proof: Consider the ternary terms $t_1(x,y,z) = (z \circ y) \circ x$ and $t_2(x,y,z) = (x \circ y) \circ z$. Then by using of the identity (1) and Lemma 4, we can compute easily

$$\begin{aligned}
&t_1(x,z,z) = (z \circ z) \circ x = 1 \circ x = x,\\
&t_1(x,x,z) = (z \circ x) \circ x = x \vee z = z \vee x = (x \circ z) \circ z = t_2(x,z,z),\\
&t_2(x,x,z) = (x \circ x) \circ z = 1 \circ z = z.
\end{aligned}$$

Hence, the variety is congruence 3-permutable. Suppose $\Theta, \Phi, \Psi \in \operatorname{Con} \mathcal{S}$ for $\mathcal{S}$ of our variety. Of course, $(\Psi \cap \Theta) \vee (\Psi \cap \Phi) \subseteq \Psi \cap (\Theta \vee \Phi)$, thus we need to prove the converse inclusion.

Suppose $a \in [1]_{\Psi \cap (\Theta \vee \Phi)}$. Thus $\langle 1, a \rangle \in \Psi \cap (\Theta \vee \Phi)$, i.e. $\langle 1, a \rangle \in \Psi$ and there exist $b, c \in S$ with $\langle 1, b \rangle \in \Theta$, $\langle b, c \rangle \in \Phi$ and $\langle c, a \rangle \in \Theta$ since in the 3-permutable variety we have $\Theta \vee \Phi = \Theta \circ \Phi \circ \Theta$. Then

$$1 = (b \circ 1) \Psi (b \circ a) \quad \text{and} \quad 1 = (c \circ 1) \Psi (c \circ a).$$

Due to transitivity, $(b \circ a) \Psi (c \circ a)$ and, hence, $a \Psi 1 \Psi (b \circ a) \Psi (c \circ a) \Psi (c \circ 1) = 1$, i.e. $a \Psi (b \circ a) \Psi (c \circ a) \Psi 1$. However,

$$\begin{aligned}
\langle 1, b \rangle \in \Theta \quad &\text{implies} \quad \langle a, b \circ a \rangle = \langle 1 \circ a, b \circ a \rangle \in \Theta,\\
\langle b, c \rangle \in \Phi \quad &\text{implies} \quad \langle b \circ a, c \circ a \rangle \Phi \quad \text{and}\\
\langle c, a \rangle \in \Theta \quad &\text{implies} \quad \langle c \circ a, 1 \rangle = \langle c \circ a, a \circ a \rangle \in \Theta,
\end{aligned}$$

thus

$$a(\Psi \cap \Theta)(b \circ a)(\Psi \cap \Phi)(c \circ a)(\Psi \cap \Theta)1$$

and hence $\langle a, 1\rangle \in (\Psi \cap \Theta) \vee (\Psi \cap \Phi)$, i.e. $a \in [1]_{(\Psi\cap\Theta)\vee(\Psi\cap\Phi)}$. Since the converse inclusion is trivial, we have shown

$$[1]_{\Psi\cap(\Theta\vee\Phi)} = [1]_{(\Psi\cap\Theta)\vee(\Psi\cap\Phi)}.$$

By Theorem 3, the variety is weakly regular, thus

$$\Psi \cap (\Theta \vee \Phi) = (\Psi \cap \Theta) \vee (\Psi \cap \Phi),$$

proving the congruence distributivity. □

Since every congruence on a lattice with sectionally antitone involutions is determined by its kernel, it is natural to ask about a description of the congruence kernel and about a procedure how to involve a congruence having a forgiven kernel. In the remaining part of the paper, we will solve these problems.

At first, we define the following terms of the variety of lattices with sectionally antitone involutions:

$$\begin{aligned}
q(x_1, x_2, y_1, y_2) &= (y_1 \circ x_2) \wedge ((y_2 \circ (x_2 \circ x_1)) \circ x_1),\\
t_1(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) &= (q(x_1, x_2, y_1, y_2) \circ q(x_3, x_4, y_3, y_4)) \circ (x_2 \circ x_4),\\
t_2(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) &= (x_2 \circ x_4) \circ (q(x_1, x_2, y_1, y_2) \circ q(x_3, x_4, y_3, y_4)),\\
t_3(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) &= (q(x_1, x_2, y_1, y_2) \wedge q(x_3, x_4, y_3, y_4)) \circ (x_2 \wedge x_4),\\
t_4(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) &= (x_2 \wedge x_4) \circ (q(x_1, x_2, y_1, y_2) \wedge q(x_3, x_4, y_3, y_4)),\\
t_5(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) &= (q(x_1, x_2, y_1, y_2) \vee q(x_3, x_4, y_3, y_4)) \circ (x_2 \vee x_4),\\
t_6(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) &= (x_2 \vee x_4) \circ (q(x_1, x_2, y_1, y_2) \vee q(x_3, x_4, y_3, y_4)).
\end{aligned}$$

One can easily compute

$$\begin{aligned}
q(x_1, x_2, 1, 1) &= x_2,\\
q(x_1, x_2, x_1 \circ x_2, x_2 \circ x_1) &= x_1.
\end{aligned}$$

Hence,

$$t_i(x_1, x_2, x_3, x_4, 1, 1, 1, 1) = 1 \quad \text{for} \quad i = 1, \dots, 6.$$

Suppose now that $\mathcal{L} = (L; \vee, \wedge, \circ, 1)$ is a lattice with sectionally antitone involutions and $\Theta \in \operatorname{Con} \mathcal{L}$. Set $I = [1]_\Theta$. Then $b \in [1]_\Theta$ if and only if $\langle b, 1\rangle \in \Theta$. Hence, for every $a_1, a_2, a_3, a_4 \in L$ and $b_1, b_2, b_3, b_4 \in I$ we have

$$\langle t_i(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4), 1\rangle \in \Theta,$$

thus $t_i(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in I$. Define $\emptyset \neq I \subseteq L$ to be an *ideal* of $\mathcal{L}$ whenever for every $a_1, a_2, a_3, a_4 \in L$ and every $b_1, b_2, b_3, b_4 \in I$ we have $t_i(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in I$ for $i = 1, \dots, 6$.

We are able to state our final result.

**Theorem 5.** *Let $\mathcal{L} = (L; \vee, \wedge, \circ, 1)$ be a lattice with sectionally antitone involutions and $\emptyset \neq I \subseteq L$. Then $I$ is a congruence kernel if and only if $I$ is an ideal of $\mathcal{L}$. If $I$ is an ideal of $\mathcal{L}$ then it is kernel of $\Theta_I \in \operatorname{Con}\mathcal{L}$ defined by*

$$\langle x, y\rangle \in \Theta_I \quad \textit{if and only if} \quad x \circ y \in I \quad \textit{and} \quad y \circ x \in I.$$

Proof: Firstly suppose $I = [1]_\Theta$ for some $\Theta \in \operatorname{Con}\mathcal{L}$. Then clearly $1 \in I$ and for every $b_1, b_2, b_3, b_4 \in I$ we have $\langle b_j, 1\rangle \in \Theta$ for $j = 1, 2, 3, 4$, thus for any $a_1, a_2, a_3, a_4 \in L$ we obtain

$$\begin{aligned}&\langle t_i(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4), 1\rangle \\ &\qquad = \langle t_i(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4), t_i(a_1, a_2, a_3, a_4, 1, 1, 1, 1)\rangle \in \Theta,\end{aligned}$$

proving $t_i(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in [1]_\Theta = I \quad (i = 1, \dots, 6)$, thus $I$ is an ideal of $\mathcal{L}$.

Conversely, let $I$ be an ideal of $\mathcal{L}$. By the definition, $I \neq \emptyset$ and hence there exists $a \in I$. One can easily compute $t_1(a, \dots, a) = 1$, thus also $1 \in I$. Define a binary relation $\Theta_I$ on $L$ as shown in the theorem and set

$$[1]_{\Theta_I} = \{x \in L; \langle x, 1\rangle \in \Theta_I\}.$$

If $a \in I$ then $1 \circ a = a \in I$ and $a \circ 1 = 1 \in I$, thus $\langle a, 1\rangle \in \Theta_I$, i.e. $I \subseteq [1]_{\Theta_I}$. If $a \in [1]_{\Theta_I}$ then $a = 1 \circ a \in I$ showing $[1]_{\Theta_I} \subseteq I$. Together, $I = [1]_{\Theta_I}$. To complete the proof we need only to show that $\Theta_I \in \operatorname{Con}\mathcal{L}$.

Evidently, $\Theta_I$ is reflexive. Suppose $\langle a, b\rangle \in \Theta_I$ and $\langle c, d\rangle \in \Theta_I$. Then $a \circ b \in I$, $b \circ a \in I$, $c \circ d \in I$ and $d \circ c \in I$. Applying the term $t_1$, we obtain

$$(a \circ c) \circ (b \circ d) = t_1(a, b, c, d, a \circ b, b \circ a, c \circ d, d \circ c) \in I.$$

Analogously,

$$(b \circ d) \circ (a \circ c) = t_2(a, b, c, d, a \circ b, b \circ a, c \circ d, d \circ c) \in I,$$

whence $\langle a \circ c, b \circ d\rangle \in \Theta_I$.

Applying $t_3, t_4$ instead of $t_1, t_2$, we conclude $\langle a \wedge c, b \wedge d\rangle \in \Theta_I$ and, for $t_5, t_6$ we obtain $\langle a \vee c, b \vee d\rangle \in \Theta_I$.

Thus $\Theta_I$ is a reflexive and compatible relation on $\mathcal{L}$. Since the variety of lattices with sectionally antitone involutions is permutable, we can apply the theorem of H. Werner [6] which yields $\Theta_I \in \operatorname{Con}\mathcal{L}$. $\square$

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