Convolution operators on the dual of hypergroup algebras

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Abstract. Let X be a hypergroup. In this paper, we define a locally convex topology β on $L(X)$ such that $(L(X),\beta)^*$ with the strong topology can be identified with a Banach subspace of $L(X)^*$. We prove that if X has a Haar measure, then the dual to this subspace is $L_C(X)^{**} = \mathrm{cl}\{F \in L(X)^{**}; F \text{ has compact carrier}\}$. Moreover, we study the operators on $L(X)^*$ and $L_0^\infty(X)$ which commute with translations and convolutions. We prove, among other things, that if $\mathrm{wap}(L(X))$ is left stationary, then there is a weakly compact operator T on $L(X)^*$ which commutes with convolutions if and only if $L(X)^{**}$ has a topologically left invariant functional. For the most part, X is a hypergroup not necessarily with an involution and Haar measure except when explicitly stated.

Keywords: Arens regular, hypergroup algebra, weakly almost periodic, convolution operators

Classification: 43A10, 43A62

1. Introduction and notations

The theory of hypergroups was initiated by Dunkl [4], Jewett [8] and Spector [19] and has received a good deal of attention from harmonic analysts. It is still unknown if an arbitrary hypergroup admits a left Haar measure, but commutative hypergroups with an involution [1] and compact hypergroups with an involution have a Haar measure. The lack of Haar measure and involution presents many difficulties, however, we succeed to get some results.

Let X be a locally compact Hausdorff space with convolution measure algebra $M(X)$ and probability measures $M_p(X)$ ([4], [5], [6]). Also, let $L(X) = \{\mu \in M(X);\ x \mapsto |\mu| * \delta_x$ is norm continuous$\}$ ([5], [15]). We assume that X is a foundation, i.e.

$$X = \mathrm{cl}(\bigcup\{\mathrm{supp}\,\mu;\ \mu \in L(X)\}).$$

It is known that $L(X)$ is an L-ideal of $M(X)$ and $L(X)$ has a positive bounded approximate identity with norm one ([5, Lemma 1]).

If $L(X)^*$, $L(X)^{**}$ are the first and second duals of $L(X)$ respectively, the first Arens product in $L(X)^{**}$ is defined by

$$\langle FG, f\rangle = \langle F, Gf\rangle, \langle Gf, \mu\rangle = \langle G, f\mu\rangle,$$

where $\mu, \nu \in L(X)$, $f \in L(X)^*$ and $F, G \in L(X)^{**}$. In addition, we define

$$\langle \mu f, \nu \rangle = \langle f, \nu * \mu \rangle, \langle f\mu, \nu \rangle = \langle f, \mu * \nu \rangle$$

where $\mu \in M(X)$, $\nu \in L(X)$ and $f \in L(X)^*$. Most of our notation in this paper is taken from [4], [14].

The paper is organized as follows. In Section 2, we introduce a locally convex topology β on $L(X)$, and prove that the strong topology on $(L(X), \beta)^*$ can be identified with a Banach subspace of $L(X)^*$, and the dual to this subspace is $L_C(X)^{**}$ (when X has Haar measure) where

$$L_C(X)^{**} = \mathrm{cl}\{F \in L(X)^{**};\ F \text{ has compact carrier}\}$$

is defined in [14].

In Section 3, we deal with the operators on $L(X)^*$ and $L_0(X)^*$ which commute with translations and convolutions, and we show that if $\mathrm{wap}(L(X))$ is left stationary, then there is a weakly compact operator T on $L(X)^*$ which commutes with convolutions if and only if $L(X)^{**}$ has a topologically left invariant functional.

2. Locally convex topology on $L(X)$

Let X be a hypergroup. If (K_n) is an increasing sequence of compact subsets of X and (a_n) is a sequence in $(0, \infty)$ with $a_n \longrightarrow \infty$, then we define

$$U((K_n), (a_n)) = \{\mu \in L(X);\ \|\mu\chi_{K_n}\| \leq a_n, n \in \mathbb{N}\}.$$

It is clear that the set of all $U((K_n), (a_n))$ is a base of neighbourhoods of zero for a locally convex topology β on $L(X)$. We write $L_0(X)^*$ for the dual $(L(X), \beta)$.

If $f \in L(X)^*$, we define

$$\|f\|_A = \sup\{|\langle f, \mu\rangle|, \mu \in L(X) \text{ and } \operatorname{supp}\mu \subseteq A,\ \|\mu\| \leq 1\}$$

where A is a Borel subset of X. Also, we take

$$L_0^\infty(X) = \{f \in L(X)^*; \|f\|_{X\setminus K} \longrightarrow 0 \text{ where } K \text{ is compact and } K \uparrow X\}$$

([12, Definition 2.4]). In this paper for $f \in L(X)^*$ and $\mu \in L(X)$, we define $\langle f\chi_A, \mu\rangle = \langle f, \mu\chi_A\rangle$ (A is a Borel subset of X).

Lemma 2.1. *Let X be a hypergroup. Then $L_0^\infty(X) = L_0(X)^*$.*

Proof: Let $f \in L_0(X)^*$, and $\epsilon > 0$ be given. There exists $U((K_n), (a_n))$ such that for $\mu \in L(X)$ with $\|\mu\chi_{K_n}\| \leq a_n$ ($n \in \mathbb{N}$), we have $|\langle f, \mu\rangle| < \epsilon$. Now, if $\mu \in L(X)$ and $\|\mu\| \leq 1$,

$$|\langle f, \mu\rangle| < \epsilon/a$$

where $a = \inf\{a_n, n \in \mathbb{N}\}$. Consequently $f \in L(X)^*$. On the other hand, there exists $n_\circ \in \mathbb{N}$ such that for all $n \geq n_\circ$ $(n \in \mathbb{N})$, $a_n \geq 1$. Therefore if $\mu \in L(X)$ with $\|\mu\| \leq 1$, then for every $n > n_\circ$ we have $\|\mu\chi_{K_n}\| \leq a_n$. Hence $\|f\|_{X\setminus K_n} < \epsilon$ $(n \geq n_\circ)$ and it follows that $f \in L_0^\infty(X)$.

To prove the converse, let $f \in L_0^\infty(X)$. There exists an increasing sequence (K_n) of compact subsets X such that $b_n = \|f\|_{X\setminus K_n} \longrightarrow 0$. Now for $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that $b_{n_i} \leq 1/(1+i)2^i$. For all $\mu \in U((K_{n_i}), (i))$, we can write

$$|\langle f, \mu\rangle| \leq \sum_{i=1}^{\infty} |\langle f\chi_{K_{n_i}\setminus K_{n_{i-1}}}, \mu\chi_{K_{n_i}\setminus K_{n_{i-1}}}\rangle|$$

where $K_{n_\circ} = \emptyset$. Hence

$$\begin{aligned} |\langle f, \mu\rangle| &\leq \sum_{i=1}^{\infty} \|f\chi_{K_{n_i}\setminus K_{n_{i-1}}}\|\|\mu\chi_{K_{n_i}\setminus K_{n_{i-1}}}\| \\ &\leq \sum_{i=1}^{\infty} \|f\|_{X\setminus K_{n_{i-1}}}\|\mu\chi_{K_{n_i}}\| \leq \|f\| + 1. \end{aligned}$$

Consequently $f \in L_0(X)^*$. □

Lemma 2.2. *Let β be as above. Then the following statements hold:*

(1) *$H \subseteq L(X)$ is β bounded if and only if H is norm bounded;*
(2) *the strong topology on $(L(X), \beta)^*$ can be identified with the norm topology on $L_0^\infty(X)$.*

Proof: (1) Let H be β bounded. If (μ_n) is a sequence in H and (α_n) is a sequence of scalars such that $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$, then $\alpha_n\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$. Indeed, we can find an increasing sequence (K_n) of compact subsets X such that $\|\mu\chi_{X\setminus K_n}\| \leq 1/\sqrt{|\alpha_n|}$ (without loss of generality we can assume that $\alpha_n \neq 0$ for all $n \in \mathbb{N}$). But H is β bounded, so there exists $m \in \mathbb{N}$ with $H \subseteq mU((K_n), (1/\sqrt{|\alpha_n|}))$. It follows that for every $n \in \mathbb{N}$, we have

$$\|\alpha_n\mu_n\| \leq \|\alpha_n\mu_n\chi_{K_n}\| + \|\alpha_n\mu_n\chi_{X\setminus K_n}\| \leq (m+1)\sqrt{|\alpha_n|}.$$

Consequently H is norm bounded ([16]). The converse is obvious.

(2) Let $B = \{\mu \in L(X); \|\mu\| < 1\}$, $f \in L_0^\infty(X)$ and $\|f\| < 1$. We consider $\delta = 1 - \|f\|$. Since B is norm bounded, B is β bounded. Hence B is weak bounded $(\sigma(L(X), (L(X), \beta)^*))$. But

$$\{g \in L_\circ^\infty(X); \rho_B(g-f) < \delta\} \subseteq \{h \in L_\circ^\infty(X); \|h\| < 1\}$$

where for $h \in L(X)^*$, $\rho_B(h) = \sup\{|\langle h, \mu\rangle|; \mu \in B\}$. So $\{f \in L_0^\infty(X); \|f\| < 1\}$ is open in the strong topology on $L_0^\infty(X)$.

Now, let A be a weak bounded subset of $(L(X), \beta)$. So A is β bounded, by (1) A is norm bounded. Therefore there exists $m \in \mathbb{N}$ such that $\|\mu\| < m$ for all $\mu \in A$. If $\epsilon > 0$, $f \in L_0^\infty(X)$ and $\rho_A(f) < \epsilon$, then

$$\{h \in L_0^\infty(X); \|h - f\| < (\epsilon - \rho_A(f))/m\} \subseteq \{g \in L_\circ^\infty(X); \rho_A(g) < \epsilon\}.$$

Consequently the strong topology is identified with the norm topology. □

Let H be a subspace of $L_0^\infty(X)$. H is called left topologically introverted if for each $F \in H^*$, $f \in H$ and $\mu \in L(X)$, both Ff and $f\mu$ are also in H.

For $\psi \in C(X)$ and $\mu \in M(X)$ we define $\langle \psi, \mu\rangle = \int \psi(x)\, d\mu(x)$. So $C(X) \subseteq M(X)^*$. Now, let $f \in C_\circ(X)$ and $\mu \in M(X)$. Then the map $\psi(x) = \langle f, \mu * \delta_x\rangle$ is in $C_\circ(X)$ [4], and

$$\int \psi(x) d\nu(x) = \int \langle f, \mu * \delta_x\rangle\, d\nu(x) = \langle f\mu, \nu\rangle$$

where $\nu \in L(X)$. So we can regard $f\mu$ as a continuous function vanishing at infinity. Consequently $C_\circ(X)$ is a left topologically introverted subspace of $L_0^\infty(X)$.

Definition 2.3. A compact subset K of X is said to be a carrier for $F \in L_0^\infty(X)^*$ (respectively $F \in L(X)^{**}$) if for all $f \in L_0^\infty(X)$ (respectively $f \in L(X)^*$) $\langle F, f\chi_K\rangle = \langle F, f\rangle$.

We know that if X is a hypergroup with an involution and Haar measure ([1], [2]), then $L^1(X)$ is an FC-algebra [11]. If X has an involution and Haar measure, then by an argument similar to the proof of ([12, Proposition 2.6]), the set of all functionals in $L_0^\infty(X)^*$ with compact carrier is dense in $L_0^\infty(X)^*$ (in the norm topology). In addition, if K_1 and K_2 are compact subsets of X, then for $\mu \in L(X) \cap M_p(X)$, $x \notin \bar{K}_2 * K_1$, we have $(\mu\chi_{K_2} * \delta_x)\chi_{K_1} = 0$ ([1, Lemma 1.2.11]). Hence for $f \in L^*(X)$, $\langle f\chi_{K_1}, \mu\chi_{K_2} * \delta_x\rangle = 0$. So for all $\nu \in L(X)$ with $\operatorname{supp}\nu \cap \bar{K}_2 * K_1 = \emptyset$,

$$\langle f\chi_{K_1}, \mu\chi_{K_2} * \nu\rangle = \int \langle f\chi_{K_1}, \mu\chi_{K_2} * \delta_x\rangle\, d\nu(x) = 0$$

(since for all $\mu \in L(X)$, $\nu \in M(X)$, we have $\mu * \nu = \int \mu * \delta_x\, d\nu(x)$ ([16, Theorem 3.20 and Theorem 3.27])). It follows that $\|f\chi_{K_1}\mu\chi_{K_2}\|_{X\setminus\bar{K}_2 * K_1} = 0$. Consequently $f\chi_{K_1}\mu\chi_{K_2} \in L_0^\infty(X)$. It is easy to see that for all $f \in L_0^\infty(X)$ and $\mu \in L(X)$ we have $f\mu \in L_0^\infty(X)$. Similarly $Ff \in L_0^\infty(X)$ whenever $F \in L_0^\infty(X)^*$ and $f \in L_0^\infty(X)$.

Lemma 2.4. *Let X be a hypergroup as above. If K_1 is a carrier for $F \in L_0^\infty(X)^*$ and K_2 is a carrier for $H \in L_0^\infty(X)^*$, then $K_1 * K_2$ is a carrier for FH.*

PROOF: Let K_1 be a carrier for $F \in L_0^\infty(X)^*$ and K_2 be a carrier for $H \in L_0^\infty(X)^*$. For $\mu, \nu \in L(X)$ and $f \in L_0^\infty(X)$, since $\mu\chi_{K_1} * \nu\chi_{K_2} = (\mu\chi_{K_1} * \nu\chi_{K_2})\chi_{K_1*K_2}$ ([1]), we have

$$\begin{aligned}\langle (f\mu\chi_{K_1})\chi_{K_2}, \nu\rangle &= \langle f\mu\chi_{K_1}, \nu\chi_{K_2}\rangle = \langle f, \mu\chi_{K_1} * \nu\chi_{K_2}\rangle \\ &= \langle f, (\mu\chi_{K_1} * \nu\chi_{K_2})\chi_{K_1*K_2}\rangle = \langle (f\chi_{K_1*K_2}\mu\chi_{K_1})\chi_{K_2}, \nu\rangle.\end{aligned}$$

So $(f\mu\chi_{K_1})\chi_{K_2} = (f\chi_{K_1*K_2}\mu\chi_{K_1})\chi_{K_2}$. But

$$\begin{aligned}\langle (Hf)\chi_{K_1}, \mu\rangle &= \langle H, (f\mu\chi_{K_1})\chi_{K_2}\rangle \\ &= \langle H, f\chi_{K_1*K_2}\mu\chi_{K_1}\rangle = \langle (Hf\chi_{K_1*K_2})\chi_{K_1}, \mu\rangle.\end{aligned}$$

Consequently

$$\langle FH, f\rangle = \langle F, (Hf)\chi_{K_1}\rangle = \langle F, (Hf\chi_{K_1*K_2})\chi_{K_1}\rangle = \langle (FH)\chi_{K_1*K_2}, f\rangle.$$

Therefore $K_1 * K_2$ is a compact carrier for FH. □

If X has an involution and Haar measure, then $L_0^\infty(X)$ is left topologically introverted and the first Arens product is well defined. Also there is an algebra isomorphism from $L_C(X)^{**}$ onto $L_\circ^\infty(X)^*$. Indeed, the restriction map is an isometric isomorphism.

We recall that a Banach algebra A is Arens regular if two Arens products on A^{**} coincide [3]. In the following theorem, we prove that if $L_\circ^\infty(X)^*$ is Arens regular, then $L(X)^{**}$ is unital.

Theorem 2.5. *Let X be a hypergroup such that the first and the second Arens multiplications are both well defined on $L_0^\infty(X)^*$. If $L_0^\infty(X)^*$ is Arens regular, then $L(X)^{**}$ is unital.*

PROOF: If $L_0^\infty(X)^*$ is Arens regular, then $L(X)$ is Arens regular. Therefore by [3] $\mathrm{wap}(L(X)) = L(X)^*$ where $\mathrm{wap}(L(X)) = \{f \in L(X)^*; \{f\mu; \mu \in L(X), \|\mu\| \leq 1\}$ is relatively weakly compact$\}$. Now, let $f \in L(X)^*$ and (e_α) be a bounded approximate identity of norm one ([5, Lemma 1]). Since $fe_\alpha \longrightarrow f$ in the weak*-topology and $f \in \mathrm{wap}(L(X))$, we have $f \in L(X)^*L(X)$. Consequently $L(X)^*$ factors on the left ([13]). It follows that $L(X)^{**}$ is unital. □

Medghalchi ([14], [15]) has defined $B = L(X)^*L(X)$ which is a Banach subspace of $L(X)^*$ and has shown B^* is a Banach algebra by Arens type product. For $\mu \in M(X)$ and $f\nu \in B$ we define $\langle \mu, f\nu\rangle = \langle f, \nu * \mu\rangle$, hence $\mu \in B^*$. We can show that if $L(X)$ is an ideal in B^*, then X is compact. Indeed if X is not compact and Σ is the family of all compact subsets of X, then Σ is a directed set under the set inclusion. Now we take $x_K \notin K$ (K is a compact subset of X). Let $m \in B^*$ be a weak*-limit of a subnet of $\{\delta_{x_K}\}$. Then for $\psi \in C_\circ(X)$, we have $\langle m, \psi\rangle = 0$. Hence $m \in C_\circ(X)^\perp$ ([14, Theorem 4]). On the other hand for all $\mu \in L(X)$, we have $\mu m \in L(X)$ and $\mu m \in C_\circ(X)^\perp$. So $\mu m = 0$ ([14, Theorem 4]). Consequently $m = 0$, which is a contradiction.

3. Convolution of operators

We know that for a locally compact abelian group G, $L(X) = L^1(G)$, $L(X)^* = L^\infty(G)$ and $f\delta_x = L_x f$ where $L_x(f)(y) = f(xy)$ ($f \in L^\infty(G)$, $x, y \in G$). Also, if $\psi \in L^1(G)$, $f\psi = \psi^\vee * f$ where $\psi^\vee(x) = \psi(x^{-1})$. The operators on $L^\infty(G)$ which commute with translations and convolutions have been studied by Lau and Pym in [12]. In [9], Larsen has studied some operators on $L^\infty(G)$, and has proved that if $\mathbb{Z}$ is the additive group of integers, then there exists $T \in M(L^\infty(\mathbb{Z}))$ ($M(L^\infty(\mathbb{Z}))$ is the set of all operators on $L^\infty(\mathbb{Z})$ which commute with translations [9]) which cannot be written as convolution with an element of $M(\mathbb{Z})$ ([9, p. 78]). Indeed, T is not weak*-weak* continuous. We show that if X is a hypergroup which has an involution and Haar measure and $T : L_0^\infty(X) \longrightarrow L_0^\infty(X)$ commutes with convolutions, i.e. $T(f\mu) = T(f)\mu$ for $f \in L_\circ^\infty(X)$ and $\mu \in L(X)$, then for some $\mu \in M(X)$ we have $T = {\lambda_\mu}^*$ where λ_μ is a left multiplier on $L(X)$ defined by $\lambda_\mu(\nu) = \mu * \nu$ for $\nu \in L(X)$. In this section, we may assume that all operators are bounded.

Theorem 3.1. *Let X be a hypergroup. Then the following statements hold:*

(1) *If $T : L(X)^* \longrightarrow L(X)^*$ is weak*-weak* continuous and $T(\delta_x f) = \delta_x T(f)$ for every $f \in L(X)^*$ and $x \in X$, then there exists a unique measure $\mu \in M(X)$ such that $T = {\lambda_\mu}^*$ and $\|T\| = \|\mu\|$. Indeed, the correspondence between T and μ defines an isometric isomorphism from $\{T; T : L(X)^* \longrightarrow L(X)^*$ is weak*-weak* continuous and $T(\delta_x f) = \delta_x T(f)$, $x \in X$, $f \in L(X)^*\}$ onto $M(X)$.*

(2) *If $T : L(X) \longrightarrow L(X)^*$ commutes with translations, i.e. $T(\mu * \delta_x) = T(\mu)\delta_x$ ($x \in X$, $\mu \in L(X)$), then there exists a unique $f \in L(X)^*$ such that $T(\mu) = f\mu$ for all $\mu \in L(X)$. In addition, $\|T\| = \|f\|$.*

(3) *Let X be a hypergroup with involution and Haar measure. If T is an operator on $L_0^\infty(X)$ such that $T(f\mu) = T(f)\mu$ for $f \in L_0^\infty(X)$ and $\mu \in L(X)$, then there exists a unique measure $\mu \in M(X)$ such that $T = {\lambda_\mu}^*$ and $\|T\| = \|\mu\|$. In addition, if T is compact then $\mu \in L(X)$. Moreover, there exists an isometric isomorphism from $\{T; T : L_0^\infty(X) \longrightarrow L_0^\infty(X), T(f\mu) = T(f)\mu$ for $f \in L_0^\infty(X)$ and $\mu \in L(X)\}$ onto $M(X)$.*

Proof: We know that $T^* : L(X)^{**} \longrightarrow L(X)^{**}$ is a bounded linear map. On the other hand, since T is weak*-weak* continuous, for $\mu \in L(X)$, $T^*(\mu) \in L(X)^{**}$ is weak* continuous. Hence $T^*(\mu) \in L(X)$ ([16, Chapter 3]). But for $x \in X$ and

$\mu \in L(X)$, $T^*(\mu * \delta_x) = T^*(\mu) * \delta_x$. Consequently, for $f \in L(X)^*$ and $\nu \in L(X)$ we have

$$\begin{aligned}\langle f, T^*(\mu * \nu)\rangle = \langle T(f), \mu * \nu\rangle = \int \langle T(f), \mu * \delta_x\rangle \, d\nu(x) = \int \langle f, T^*(\mu * \delta_x)\rangle \, d\nu(x) \\ = \int \langle f, T^*(\mu) * \delta_x\rangle \, d\nu(x) = \langle f, T^*(\mu) * \nu\rangle.\end{aligned}$$

Consequently for all $\mu, \nu \in L(X)$, we have $T^*(\mu*\nu) = T^*(\mu)*\nu$. Hence there exists a measure $\mu \in M(X)$ such that for $\nu \in L(X)$, $T^*(\nu) = \mu * \nu$ ([5, Proposition 1]). It is clear that μ is unique and $\|T^*\| = \|\mu\|$. Also, it is obvious that $T = {\lambda_\mu}^*$ and the correspondence between T and μ is an isometric isomorphism.

(2) Let $T^* : L(X)^{**} \longrightarrow L(X)^*$ be adjoint to T. For all $\mu, \nu, \eta \in L(X)$, since $T(\mu * \delta_x) = T(\mu)\delta_x$ $(x \in X)$, we have

$$\begin{aligned}\langle T(\mu * \nu), \eta\rangle = \langle T^*(\eta), \mu * \nu\rangle = \int \langle T^*(\eta), \mu * \delta_x\rangle \, d\nu(x) \\ = \int \langle T(\mu * \delta_x), \eta\rangle \, d\nu(x) = \int \langle T(\mu), \delta_x * \eta\rangle \, d\nu(x) = \langle T(\mu)\nu, \eta\rangle.\end{aligned}$$

Consequently $T(\mu * \nu) = T(\mu)\nu$.

Now, let (e_α) be a bounded approximate identity with norm one. Then without loss of generality, we may assume that $T(e_\alpha) \longrightarrow f$ $(f \in L(X)^*)$ in the weak*-topology. It is clear that $T(\mu) = f\mu$ for all $\mu \in L(X)$. Since $L(X)$ has a bounded approximate identity, f is unique. Now, let $\epsilon > 0$ be given. We take $\nu \in L(X)$ $(\|\nu\| = 1)$ such that $\|f\| \le |\langle f, \nu\rangle| + \epsilon$. Since

$$|\langle f, \nu\rangle| \le \lim_\alpha |\langle f e_\alpha, \nu\rangle| = \lim_\alpha |\langle T(e_\alpha), \nu\rangle| \le \|T\|,$$

we have $\|f\| \le \|T\| + \epsilon$. But $\|T\| \le \|f\|$. Consequently $\|T\| = \|f\|$.

(3) We know that $L_0^\infty(X)^* = L_C(X)^{**}$. Now if $T^* : L_0^\infty(X)^* \longrightarrow L_0^\infty(X)^*$ is adjoint to T, then for $\mu, \nu \in L(X)$ we have $T^*(\mu * \nu) = \mu T^*(\nu)$. But $\mu T^*(\nu) = \mu\pi(T^*(\nu))$ ([14, Proposition 6]) and $\pi(T^*(\nu)) \in M(X)$ ([14, Proposition 13]). So $T^*(\mu*\nu) \in L(X)$. Since $L(X)$ has a bounded approximate identity, by the Cohen-Hewitt factorization theorem, we have $L(X) * L(X) = L(X)$. Consequently for every $\mu \in L(X)$, $T^*(\mu) \in L(X)$. A similar proof as above can be used to show that for some $\mu \in M(X)$, $T = {\lambda_\mu}^*$, and μ is unique with $\|T\| = \|\mu\|$.

Now, if T is compact then $\lambda_\mu : L(X) \longrightarrow L(X)$ is compact. So $\mu \in L(X)$ ([5, Theorem 1]). It is obvious that the correspondence between T and μ is an isometric isomorphism. This completes our proof. $\square$

Skantharajah has proved there are some hypergroups X with $\mathrm{LIM}(L^\infty(X)) \setminus \mathrm{TLIM}(L^\infty(X)) \neq \emptyset$ ([17], [18]). In general, if G is a nondiscrete abelian group, then $\mathrm{LIM}(L^\infty(G)) \setminus \mathrm{TLIM}(L^\infty(G)) \neq \emptyset$ ([7]). If $m \in \mathrm{LIM}(L^\infty(G)) \setminus \mathrm{TLIM}(L^\infty(G))$, then the map $T : L^\infty(G) \longrightarrow L^\infty(G)$ given by $T(f) = m(f)$ commutes with translations, but T is not weak*-weak* continuous. Therefore there is no $\mu \in M(X)$ such that $T = {\lambda_\mu}^*$.

For a hypergroup X with an involution and Haar measure Wolfenstetter in [20] has defined $\mathrm{wap}(X) = \{f \in C(X); \{L_x f; x \in X\}$ is relatively weakly compact in $C(X)\}$. Also, Lasser has studied $\mathrm{ap}(X)$ [10]. In this paper, for an arbitrary hypergroup X, we define $\mathrm{wap}(L(X)) = \{f \in L(X)^*; \{f\mu; \mu \in L(X)$ and $\|\mu\| \leq 1\}$ is relatively weakly compact in $L(X)^*\}$ ([13]). It is easy to see that if $f \in \mathrm{wap}(L(X))$ and $\mu \in M(X)$, then $f\mu \in \mathrm{wap}(L(X))$. Also, the map $1 : L(X) \longrightarrow \mathbb{C}$ given by $\langle 1, \mu\rangle = \mu(X)$ is a weakly almost periodic functional on $L(X)$, i.e. $\{1\mu; \mu \in L(X)$ and $\|\mu\| \leq 1\}$ is relatively weakly compact.

Theorem 3.2. *Let X be a hypergroup and $f \in \mathrm{wap}(L(X))$. Then the following statements hold:*

1. *the weak-closure of $\{f \sum_{i=1}^n \alpha_i \delta_{x_i}; x_i \in X, \alpha_i \in \mathbb{C}, n \in \mathbb{N}, \sum_{i=1}^n |\alpha_i| \leq 1\}$ is equal to the weak-closure of $\{f\mu; \mu \in L(X), \|\mu\| \leq 1\}$.*
2. *Let T be an operator on $L(X)^*$ and $T(f\delta_x) = T(f)\delta_x$ for $f \in L(X)^*$, $x \in X$. Then $T(f\mu) = T(f)\mu$ for all $f \in \mathrm{wap}(L(X))$ and $\mu \in L(X)$.*

Proof: If $f \in \mathrm{wap}(L(X))$, then $\{f\mu; \mu \in L(X), \|\mu\| \leq 1\}$ is relatively weakly compact. Now for a bounded approximate identity (e_α) of norm one, (fe_α) has a convergence subsequence to f in the weak topology (since $fe_\alpha \longrightarrow f$ in the weak*-topology). But $B = L(X)^* L(X)$ is a Banach space, hence $f \in B$. For $x \in X$, let $m \in L(X)^{**}$ be an extension of δ_x with norm one. So there exists a net (μ_α) in $L(X)$ with $\|\mu_\alpha\| \leq 1$ such that $\mu_\alpha \longrightarrow m$ in the weak*-topology. Hence for every $\nu \in L(X)$

$$\langle \nu f, \mu_\alpha\rangle \longrightarrow \langle m, \nu f\rangle.$$

But $f \in \mathrm{wap}(L(X))$ and we may assume without loss of generality that $f\mu_\alpha \longrightarrow g$ ($g \in L(X)^*$) in the weak topology. On the other hand, $\langle m, \nu f\rangle = \langle \delta_x, \nu f\rangle = \langle f\delta_x, \nu\rangle$, and so $g = f\delta_x$. It follows that

$$\Big\{ \sum_{i=1}^n f\alpha_i\delta_{x_i}; \alpha_i \in \mathbb{C}, n \in \mathbb{N}, x_i \in X, \sum_{i=1}^n |\alpha_i| \leq 1 \Big\} \subseteq \text{weak-closure } \{f\mu; \mu \in L(X), \|\mu\| \leq 1\}.$$

To prove the converse, let $\mu \in L(X)$ and $\|\mu\| \leq 1$. By the Hahn Banach theorem, there exists a net (μ_α) in $\{\sum_{i=1}^n \alpha_i\delta_{x_i}, x_i \in X, \alpha_i \in \mathbb{C}, \sum_{i=1}^n |\alpha_i| \leq$

$1, n \in \mathbb{N}\}$ such that $\mu_\alpha \longrightarrow \mu$ in the $\sigma(B^*, B)$ topology. It is obvious to realize that

$$f\mu \in \text{weak-closure } \Big\{ \sum_{i=1}^{n} f\alpha_i \delta_{x_i}; \alpha_i \in \mathbb{C}, x_i \in X, n \in \mathbb{N}, \sum_{i=1}^{n} |\alpha_i| \leq 1 \Big\}.$$

This completes the proof.

(2) Let $f \in \operatorname{wap}(L(X))$. Since $T(f\delta_x) = T(f)\delta_x$ and $f \in B$, so $T(f) \in B$. Indeed, for $\epsilon > 0$ there exists a neighbourhood U of e such that $\|T(f)\delta_x - T(f)\| \leq \epsilon/\|T(f)\|$ $(x \in U)$. Now for $\nu \in L(X)$ $(\|\nu\| \leq 1)$ and $\mu \in L(X) \cap M_p(X)$ with $\operatorname{supp} \mu \subseteq U$, we have

$$\Big| \int \langle T(f), \delta_x * \nu \rangle - \langle T(f), \nu \rangle \, d\mu(x) \Big| < \epsilon.$$

So $|\langle T(f)\mu, \nu\rangle - \langle T(f), \nu\rangle| < \epsilon$, i.e. $\|T(f)\mu - T(f)\| < \epsilon$. But $T(f)\mu \in B$ and B is a Banach space, hence $T(f) \in B$. Now if $\mu \in L(X)$, it is easy to see that $T(f\mu) = T(f)\mu$. □

Definition 3.3. Let X be a hypergroup; $\operatorname{wap}(L(X))$ is said to be left stationary if for every $f \in \operatorname{wap}(L(X))$

$$\text{weak}^*\text{-closure } \{\mu f; \mu \in M_p(X) \cap L(X)\} \cap \{c1; c \in \mathbb{C}\} \neq \emptyset.$$

$m \in L(X)^{**}$ is said to be topologically left invariant, if $\langle m, f\mu \rangle = \langle m, f \rangle$ for all $f \in L(X)^*$ and $\mu \in L(X) \cap M_p(X)$. In the following theorem we can find a relation between the set of all weakly compact operators which commute with convolutions and the set of all topologically left invariant functionals on $L(X)^*$. It is interesting for $L^1(X)$ when X has a Haar measure.

Theorem 3.4. *Let X be a hypergroup such that $\operatorname{wap}(L(X))$ is left stationary. Then $L(X)^{**}$ has a topologically left invariant if and only if there exists a weakly compact operator $T : L(X)^* \longrightarrow L(X)^*$ such that $T(f\mu) = T(f)\mu$ for $f \in L(X)^*$ and $\mu \in L(X) \cap M_p(X)$.*

Proof: Let $m \in L(X)^{**}$ be topologically left invariant. Then the linear operator $T : L(X)^* \longrightarrow L(X)^*$ given by $T(f) = \langle m, f\rangle 1$ is a weakly compact operator and $T(f\mu) = T(f)\mu$ for all $\mu \in L(X) \cap M_p(X)$ and $f \in L(X)^*$.

Conversely, let $T : L(X)^* \longrightarrow L(X)^*$ be a weakly compact operator and $T(f\mu) = T(f)\mu$ for $f \in L(X)^*$, $\mu \in L(X) \cap M_p(X)$. So $T(L(X)^*) \subseteq \operatorname{wap}(L(X))$. Now, let $f \in \operatorname{wap}(L(X))$. There is a net (μ_α) in $L(X) \cap M_p(X)$ and $c \in \mathbb{C}$ such that $\mu_\alpha f \longrightarrow c1$ in the weak*-topology. Passing to a subnet if necessary, we can assume that (μ_α) converges to some m in $L(X)^{**}$ in the weak* topology. So, $mf = c1$. We take

$$\Sigma(f) = \{m \in L(X)^{**}; \|m\| \leq 1, \ mf = c1 \ \text{ for some } \ c \in \mathbb{C} \ \text{ and } \ \langle m, 1\rangle = 1\}.$$

For $f \in \text{wap}(L(X))$, $\Sigma(f) \neq \emptyset$. It is easy to see that $\Sigma(f)$ is weak* compact.

Now, if $f_1, f_2, \ldots, f_n \in \text{wap}(L(X))$ and $m_1 \in \bigcap_{i=1}^{n-1} \Sigma(f_i)$, then we can take $m_2 \in L(X)^{**}$ such that $m_2 m_1 f_n = c_n 1$ and $\langle m_2, 1\rangle = 1$ for some $c_n \in \mathbb{C}$ (since $m_1 f_n \in \text{wap}(L(X))$). Let $c_1, c_2, \ldots, c_{n-1} \in \mathbb{C}$ such that $m_1 f_i = c_i 1$ $(1 \leq i \leq n-1)$. We have $m_2 m_1 f_i = c_i 1$ $(1 \leq i \leq n-1)$, so that $m_2 m_1 \in \bigcap_{i=1}^{n} \Sigma(f_i)$. Consequently

$$\bigcap \{\Sigma(f); f \in \text{wap}(L(X))\} \neq \emptyset.$$

If $m_\circ \in \bigcap\{\Sigma(f); f \in \text{wap}(L(X))\}$, it is clear that $m = m_\circ m_\circ$ is a topologically left invariant on $\text{wap}(L(X))$. It follows that $m \circ T$ is a topologically left invariant on $L(X)^*$. $\square$

Acknowledgment. This work has been done while the author was visiting the Department of Mathematical Sciences of Alberta University. The author would like to thank both Professor A.T. Lau and University of Alberta for their hospitality.

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(*Received* May 15, 2003)