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Gevrey hypoellipticity for a class of degenerated quasi-elliptic operators

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Abstract. The problems of Gevrey hypoellipticity for a class of degenerated quasi-elliptic operators are studied by several authors (see [1]–[5]). In this paper we obtain the Gevrey hypoellipticity for a degenerated quasi-elliptic operator in $\mathbb{R}^2$, without any restriction on the characteristic polyhedron.

Keywords: Gevrey class, Gevrey hypoellipticity, hypoelliptic operator, degenerated quasi-elliptic operator

Classification: 35B05, 35H10, 35H35

1. Statement of the result

Let $\mathbb{R}^n$, or E^n, be the n-dimensional real Euclidean space of points $\xi = (\xi_1, \dots, \xi_n)$, $x = (x_1, \dots, x_n)$ with real components. Let $\mathbb{N}_0^n$ be the set of multi-indexes $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with nonnegative integer components. Denote

$$\mathbb{R}_0^n = \{\xi \in \mathbb{R}^n; \xi_1 \dots \xi_n \neq 0\},\ \mathbb{R}_+^n = \{\xi \in \mathbb{R}^n; \xi_j \geq 0, j = 1, \dots n\}.$$

For $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$ we set $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_j = \frac{\partial}{\partial \xi_j}$ or $D_j = -i\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$.

Let Ω be an open subset of $\mathbb{R}^n$, $\lambda \in \mathbb{R}_+^n$, $\lambda_i \geq 1$, $i = 1, \dots, n$. We denote by $G^\lambda(\Omega)$ the class of all functions $f \in C^\infty(\Omega)$ so that for any compactum $K \subset\subset \Omega$ there exists a constant $C = C(K, f)$ for which

$$\sup_{x \in K} |D^\alpha f(x)| \leq C^{|\alpha|+1} \alpha_1^{\alpha_1 \lambda_1} \dots \alpha_n^{\alpha_n \lambda_n}, \quad \forall\, \alpha \in \mathbb{N}_0^n.$$

Let in $\mathbb{R}^2$ with variables x, y,

$$P(x, D) = \sum_{\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (P)} C_\alpha x^{\alpha_3} D_x^{\alpha_1} D_y^{\alpha_2}, \tag{1}$$

be a differential operator with constant coefficients C_α. Here the sum is over a finite set of multi-indexes $(P) = \{\alpha : \alpha \in \mathbb{N}_0^3, C_\alpha \neq 0\}$.

Definition 1. The characteristic polyhedron (C.P.) $\mathcal{N}(P)$ of $P(x,D)$ is the smallest convex polyhedron in $\mathbb{R}^3_+$ containing all points $\alpha \in (P) \cup \{0\}$.

The results of Gevrey regularity for a certain class of quasi-elliptic operators degenerate on a symplectic manifold and with some restrictions on $\mathcal{N}(P)$ were obtained by V.V. Grushin, L. Rodino, L.R. Volevich, C. Parenti and others.

Let $\lambda_1 \geq 1$, $h \geq 0$ be rational numbers, $\lambda = (\lambda_1, 1)$. We denote

(2) $\mathcal{N} = \{\nu : \nu \in \mathbb{R}^3_+, \lambda_1\nu_1 + \nu_2 \leq m, \lambda_1\nu_1 + (1+h)\nu_2 - \lambda_1\nu_3 \leq m, \lambda_1\nu_3 \leq hm\}.$

We consider differential operator (1) for which C.P. have form (2). It is easy to show that $m, m/\lambda_1, hm/\lambda_1, m/(1+h)$ are naturals. After introducing some preliminary lemmas we will prove the following result, cf. Theorem 1.

Theorem. *Let the hypoelliptic differential operator $P(x,D)$ from (1) with $\mathcal{N}(P)$ in form (2) satisfy in some neighborhood U of 0*

$$\sum_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{N}(P)\cap\mathbb{N}_0^3} \|x^{\alpha_3} D_x^{\alpha_1} D_y^{\alpha_2}\psi\|_{L_2(\Omega)} \leq \|P(x,D)\psi\|_{L_2(\Omega)}, \ \psi \in C_0^\infty(U). \tag{3}$$

Then all solutions of equation $P(x,D) = f$ belong to the class $G^{(\lambda_1,1)}(V)$, with $V \subset\subset U, (0,0) \in V$, where $f \in G^{(\lambda_1,1)}(U)$.

We observe that (3) implies that $|P(x,\xi,\eta)| \neq 0$ for $x,\xi,\eta \neq 0$, analogously to the condition asked by Volevich [5] in order to ensure the hypoellipticity of $P(D)$ for $x \neq 0$, under suitable conditions Parenti-Rodino [2] that the hypoellipticity continues to hold for $x = 0$.

2. Preliminary lemmas

Let $h \geq 0$, $\lambda_1 \geq 1$ and m, j be naturals.

We denote

$$\mathcal{M}_1^j = \{\nu : \nu \in \mathbb{R}^2_+, 2\lambda_1\nu_1 + \nu_2 \leq j, \lambda_1\nu_1 \leq (1+h)m\},$$
$$\mathcal{M}_2^j = \{\nu : \nu \in \mathbb{R}^2_+, \lambda_1\nu_1 + \nu_2 \leq j - (1+h)m, \lambda_1\nu_1 \geq (1+h)m\},$$

if $j < (1-h)m$ then we take $\mathcal{M}_2^j = \emptyset$. We set $\mathcal{M}^j = \mathcal{M}_1^j \cup \mathcal{M}_2^j$,

$$\mathcal{A}_1^j = \{\nu : \nu \in \mathbb{R}^3_+, \lambda_1\nu_1 + \nu_2 \leq m + j, 2\lambda_1\nu_1 + \nu_2 \leq j + 2m, \nu_3\lambda_1 \leq hm,$$
$$\lambda_1\nu_1 + (1+h)\nu_2 - \lambda_1\nu_3 \leq m + (1+h)j, \lambda_1\nu_1 \leq (1+h)m + m\},$$
$$\mathcal{A}_2^j = \{\nu : \nu \in \mathbb{R}^3_+, \lambda_1\nu_1 + \nu_2 \leq m + j - (1+h)m,$$
$$\lambda_1\nu_1 + (1+h)\nu_2 - \lambda_1\nu_3 \leq m + (1+h)(j - (1+h)m),$$
$$\nu_3\lambda_1 \leq hm, \lambda_1\nu_1 \geq (1+h)m\},$$

if $j < (1+h)m - m/(1+h)$ then we take $\mathcal{A}_2^j = \emptyset$.

Lemma 1. *Let $h \geq 0$, $\lambda_1 \geq 1$, $m, m/\lambda_1, j$ be naturals and $\mathcal{N}$ the polyhedron in form (2). Then any multi-index $\nu \in (\mathcal{A}_1^j \setminus \mathcal{A}_1^{j-1}) \cap \mathbb{N}_0^3$ can be represented in the form $\nu = \alpha + (\beta, 0)$ where $\alpha \in \mathcal{N} \cap \mathbb{N}_0^3$, $\beta \in \mathcal{M}_1^j \cap \mathbb{N}_0^2$.*

Proof: Let $\nu \in (\mathcal{A}_1^j \setminus \mathcal{A}_1^{j-1}) \cap \mathbb{N}_0^3$. If $\nu_1 \geq m/\lambda_1$ then we take $\alpha = (m/\lambda_1, 0, \nu_3) \in \mathbb{N}_0^3$, $\beta = (\nu_1 - m/\lambda_1, \nu_2) \in \mathbb{N}_0^2$. For α and β we have

$$\lambda_1\alpha_1 + \alpha_2 = m, \lambda_1\alpha_1 + (1+h)\alpha_2 - \lambda_1\alpha_3 = m - \lambda_1\nu_3 \leq m, \lambda_1\alpha_3 = \lambda_1\nu_3 \leq hm,$$

i.e. $\alpha \in \mathcal{N}$, $2\lambda_1\beta_1 + \beta_2 = 2\lambda_1(\nu_1 - m/\lambda_1) + \nu_2 = 2\lambda_1\nu_1 + \nu_2 - 2m \leq j + 2m - 2m = j, \lambda_1\beta_1 = \lambda_1\nu_1 - m \leq (1+h)m + m - m = (1+h)m$, i.e. $\beta \in \mathcal{M}_1^j$. If $\nu_1 < m/\lambda_1$ (i.e. $\lambda_1\nu_1 \leq m - \lambda_1$) then we consider the following possible cases:

- I) $2\lambda_1\nu_1 + \nu_2 > j - 1 + 2m$ hence $\nu_2 > j - 1 + 2m - 2m + 2\lambda_1 > j$,
- II) $\lambda_1\nu_1 + \nu_2 > m + j - 1$ hence $\nu_2 > m + j - 1 - m + \lambda_1 \geq j$,
- III) $\lambda_1\nu_1 + (1+h)\nu_2 - \lambda_1\nu_3 > m + (1+h)(j-1)$ hence $(1+h)\nu_2 > m + (1+h)(j-1) - m + \lambda_1$ i.e. $\nu_2 > j - 1$.

Therefore $\nu_2 \geq j$.

We take $\alpha = (\nu_1, \nu_2 - j, \nu_3) \in \mathbb{N}_0^3$, $\beta = (0, j) \in \mathbb{N}_0^2 \cap \mathcal{M}_1^j$. We obtain $\lambda_1\alpha_1 + \alpha_2 = \lambda_1\nu_1 + \nu_2 - j \leq m + j - j = m$, $\lambda_1\alpha_1 + (1+h)\alpha_2 - \lambda_1\nu_3 = \lambda_1\nu_1 + (1+h)\nu_2 - \lambda_1\nu_3 - (1+h)j \leq m + (1+h)j - (1+h)j = m, \lambda_1\alpha_3 = \lambda_1\nu_3 \leq hm$, i.e. $\alpha \in \mathcal{N}$. $\square$

Lemma 2. *Let $h \geq 0$, $\lambda_1 \geq 1$, $m, m/\lambda_1, j$ be naturals and $\mathcal{N}$ the polyhedron in form (2). Then any multi-index $\nu \in (\mathcal{A}_2^j \setminus \mathcal{A}_2^{j-1}) \cap \mathbb{N}_0^3$ can be represented in the form $\nu = \alpha + (\beta, 0)$ where $\alpha \in \mathcal{N} \cap \mathbb{N}_0^3$, $\beta \in \mathcal{M}^j \cap \mathbb{N}_0^2$.*

Proof: Since $\nu \in (\mathcal{A}_2^j \setminus \mathcal{A}_2^{j-1}) \cap \mathbb{N}_0^3$, we have $j \geq hm$ and $\nu_1 \geq m/\lambda_1$. If $j < (1+h)m$ then $\lambda_1\nu_1 + \nu_2 \leq m + j - (1+h)m < m$ and $\lambda_1\nu_1 + (1+h)\nu_2 - \nu_3\lambda_1 \leq m + j - (1+h)m < m$ i.e. $\nu \in \mathcal{N}$. Therefore, we can take $\alpha = \nu \in \mathcal{N} \cap \mathbb{N}_0^3$ and $\beta = 0 \in \mathcal{M}^j \cap \mathbb{N}_0^2$. We can write $\nu = \alpha + \beta$.

If $j \geq (1+h)m$ then we take $\alpha = (m/\lambda_1, 0, \nu_3) \in \mathbb{N}_0^3$, $\beta = (\nu_1 - m/\lambda_1, \nu_2) \in \mathbb{N}_0^2$. Since $\lambda_1\alpha_1 + \alpha_2 = m$, $\lambda_1\alpha_1 + (1+h)\alpha_2 - \lambda_1\alpha_3 = m - \lambda_1\nu_3 \leq m$ and $\lambda_1\alpha_3 = \lambda_2\nu_3 \leq hm$, it follows that $\alpha \in \mathcal{N}$.

Let us show that $\beta \in \mathcal{M}^j$. We will consider the following possible cases:

- I) $\lambda_1\nu_1 - m \geq (1+h)m$ hence $\lambda_1\beta_1 + \beta_2 = \lambda_1\nu_1 + \nu_2 - m \leq m + j - (1+h)m - m = j - (1+h)m$, i.e. $\beta \in \mathcal{M}_2^j$,
- II) $\lambda_1\nu_1 - m \leq (1+h)m$ hence $2\lambda_1\beta_1 + \beta_2 = 2\lambda_1\nu_1 + \nu_2 - 2m = \lambda_1\nu_1 + \nu_2 + \lambda_1\nu_1 - 2m \leq m + j - (1+h)m + \lambda_1(\nu_1 - m/\lambda_1) - m \leq m + j - (1+h)m + (1+h)m - m = j$, i.e. $\beta \in \mathcal{M}_1^j$.

For $\nu_1 \geq m/\lambda_1$ we have $\alpha = (m/\lambda_1, 0, \nu_3) \in \mathcal{N} \cap \mathbb{N}_0^3$, $\beta = (\nu_1 - m/\lambda_1, \nu_2) \in (\mathcal{M}_1^j \cup \mathcal{M}_2^j) \cap \mathbb{N}_0^2$. $\square$

Lemma 3. *Let $h \ge 0$, $\lambda_1 \ge 1$, $m, m/\lambda_1, j$ be naturals and $\mathcal{N}$ the polyhedron* (2)*. If $\alpha \in \mathcal{N}$, $\beta \in \mathcal{M}_1^j \cap \mathbb{N}_0^2$ then $(\alpha_1+\beta_1-\gamma_1, \alpha_2+\beta_2, \alpha_3-\gamma_1) \in \mathcal{A}_1^{j-\gamma_1}$ for $0 \le \gamma_1 \le \min(\beta_1, \alpha_3)$.*

Proof: Since $\beta \in \mathcal{M}_1^j \cap \mathbb{N}_0^2$ and $0 \le \gamma_1 \le \min(\beta_1, \alpha_3)$ we have $j \ge 2\lambda_1\beta_1 + \beta_2 \ge 2\lambda_1\gamma_1 \ge \gamma_1$. Therefore

$$\begin{aligned} 2\lambda_1(\alpha_1+\beta_1-\gamma_1)+\alpha_2+\beta_2 &= 2(\lambda_1\alpha_1+\alpha_2)+(2\lambda_1\beta_1+\beta_2)-2\lambda_1\gamma_1 \\ &\le 2m+j-2\lambda_1\gamma_1 \le 2m+j-\gamma_1, \end{aligned}$$

$$\begin{aligned} &\lambda_1(\alpha_1+\beta_1-\gamma_1)+(1+h)(\alpha_2+\beta_2)-\lambda_1(\alpha_3-\gamma_1) \\ &\qquad = (\lambda_1\alpha_1+(1+h)\alpha_2-\lambda_1\alpha_3)+(\lambda_1\beta_1+(1+h)\beta_2) \\ &\qquad \le m+(1+h)(2\lambda_1\beta_1+\beta_2)-(2h+1)\lambda_1\beta_1 \\ &\qquad \le m+(1+h)j-(2h+1)\lambda_1\gamma_1 \le m+(1+h)(j-\gamma_1), \end{aligned}$$

$$\lambda_1(\alpha_1+\beta_1-\gamma_1) = \lambda_1\alpha_1+\lambda_1\beta_1-\lambda_1\gamma_1 \le m+(1+h)m-\lambda_1\gamma_1 \le m+(1+h)m,$$

i.e. $(\alpha_1+\beta_1-\gamma_1, \alpha_2+\beta_2, \alpha_3-\gamma_1) \in \mathcal{A}_1^{j-\gamma_1}$. □

Lemma 4. *Let $h \ge 0$, $\lambda_1 \ge 1$, $m, m/\lambda_1, j$ be naturals and $\mathcal{N}$ the polyhedron* (2)*. If $\alpha \in \mathcal{N} \cap \mathbb{N}_0^3$, $\beta \in \mathcal{M}_2^j \cap \mathbb{N}_0^2$ then $(\alpha_1+\beta_1-\gamma_1, \alpha_2+\beta_2, \alpha_3-\gamma_1) \in \mathcal{A}_2^{j-\gamma_1} \cup \mathcal{A}_1^{j-\gamma_1}$ for $0 \le \gamma_1 \le \min(\beta_1, \alpha_3)$.*

Proof: Since $\beta \in \mathcal{M}_1^j \cap \mathbb{N}_0^2$ and $0 \le \gamma_1 \le \min(\beta_1, \alpha_3)$ we have $j \ge \lambda_1\beta_1 + (1+h)m > \gamma_1$. We consider the following possible cases:

I) for $\lambda_1(\alpha_1+\beta_1-\gamma_1) \ge (1+h)m$ we obtain

a) $$\begin{aligned} \lambda_1(\alpha_1+\beta_1-\gamma_1)+(\alpha_2+\beta_2) &= (\lambda_1\alpha_1+\alpha_2)+(\lambda_1\beta_1+\beta_2)-\lambda_1\gamma_1 \\ &\le m+j-(1+h)m-\lambda_1\gamma_1 \\ &\le m+(j-\gamma_1)-(1+h)m, \end{aligned}$$

b) since $\alpha \in \mathcal{N}$, $\beta \in \mathcal{M}_2^j$ and $\gamma_1 \le \alpha_3$ we have

$$\begin{aligned} &\lambda_1(\alpha_1+\beta_1-\gamma_1)+(1+h)(\alpha_2+\beta_2)-\lambda_1(\alpha_3-\gamma_1) \\ &\qquad = (\lambda_1\alpha_1+(1+h)\alpha_2-\lambda_1\alpha_3)+(\lambda_1\beta_1+(1+h)\beta_2) \\ &\qquad \le m+(1+h)(\lambda_1\beta_1+\beta_2)-h\lambda_1\beta_1 \\ &\qquad \le m+(1+h)(j-(1+h)m)-h\lambda_1\beta_1 \\ &\qquad \le m+(1+h)(j-(1+h)m)-h(1+h)m \\ &\qquad \le m+(1+h)(j-(1+h)m)-(1+h)\lambda_1\alpha_3 \\ &\qquad \le m+(1+h)(j-(1+h)m)-(1+h)\lambda_1\gamma_1 \\ &\qquad \le m+(1+h)(j-\gamma_1-(1+h)m), \end{aligned}$$

c) $\lambda_1(\alpha_3-\gamma_1)\le \lambda_1\alpha_3\le hm$.

From a), b), c) for case I) we obtain $(\alpha_1+\beta_1-\gamma_1,\alpha_2+\beta_2,\alpha_3-\gamma_1)\in\mathcal{A}_2^{j-\gamma_1}$ or $(\alpha_1+\beta_1-\gamma_1,\alpha_2+\beta_2,\alpha_3-\gamma_1)\in\mathcal{A}_2^{j-\gamma_1}\cup\mathcal{A}_1^{j-\gamma_1}$;

II) for $\lambda_1(\alpha_1+\beta_1-\gamma_1)\le(1+h)m$ we obtain

a) $$\begin{aligned}2\lambda_1(\alpha_1+\beta_1-\gamma_1)+(\alpha_2+\beta_2)&=(2\lambda_1\alpha_1+\alpha_2)+(2\lambda_1\beta_1+\beta_2)-2\lambda_1\gamma_1\\&\le 2m+j-2\lambda_1\gamma_1\le 2m+(j-\gamma_1),\end{aligned}$$

b) $$\begin{aligned}\lambda_1(\alpha_1+\beta_1-\gamma_1)+(\alpha_2+\beta_2)&=(\lambda_1\alpha_1+\alpha_2)+(\lambda_1\beta_1+\beta_2)-\lambda_1\gamma_1\\&\le m+j-(1+h)m-\lambda_1\gamma_1\le m+j-\gamma_1,\end{aligned}$$

c) since $\alpha\in\mathcal{N}$, $\beta\in\mathcal{M}_2^j$ and $\gamma_1\le\alpha_3$,

$$\begin{aligned}\lambda_1(\alpha_1+\beta_1-\gamma_1)&+(1+h)(\alpha_2+\beta_2)-\lambda_1(\alpha_3-\gamma_1)\\&=(\lambda_1\alpha_1+(1+h)\alpha_2-\lambda_1\alpha_3)+(\lambda_1\beta_1+(1+h)\beta_2)\\&\le m+(1+h)(\lambda_1\beta_1+\beta_2)-h\lambda_1\beta_1\\&\le m+(1+h)(j-(1+h)m)-h(1+h)m\\&\le m+(1+h)(j-(1+h)m)-(1+h)\alpha_3\\&\le m+(1+h)(j-(1+h)m)-(1+h)\gamma_1\\&=m+(1+h)(j-\gamma_1-(1+h)m).\end{aligned}$$

Therefore $(\alpha_1+\beta_1-\gamma_1,\alpha_2+\beta_2,\alpha_3-\gamma_1)\in\mathcal{A}_1^{j-\gamma_1}$ or $(\alpha_1+\beta_1-\gamma_1,\alpha_2+\beta_2,\alpha_3-\gamma_1)\in\mathcal{A}_2^{j-\gamma_1}\cup\mathcal{A}_1^{j-\gamma_1}$. $\square$

3. Main results

Let $P(x,D)=\sum C_\alpha x^{\alpha_3}D_x^{\alpha_1}D_y^{\alpha_2}$ be a differential operator with C.P. in form (2). Since $m, m/\lambda_1, hm/\lambda_1, m/(1+h)$ are naturals, Lemmas 1–4 are valid for $\mathcal{N}(P)$.

For $t>0$ we denote $B_t=\{(x,y)\in\mathbb{R}^2, |x|^2+|y|^2<t^2\}$.

We use a well known result (see for example Lemma 2.1 in [3]).

Lemma 5. *Let $\rho_1>0$, $\rho>0$. Then there exists a function $\varphi\in C_0^\infty(\mathbb{R}^2)$ such that* $\operatorname{supp}\varphi\subset B_{\rho_1+\rho}$, $\varphi(x,y)=1$, $(x,y)\in B_{\rho_1}$, $0\le\varphi(x,y)\le 1$ *and*

$$\max_{x,y}|D_x^{\alpha_1}D_y^{\alpha_2}\varphi(x,y)|\le C_{\alpha_1,\alpha_2}\rho^{-(\alpha_1+\alpha_2)},$$

where C_{α_1,α_2} is independent of ρ_1 and ρ.

For $u \in C^\infty$ we denote

$$|||u,\sigma||| = \sum_{\alpha\in\mathcal{N}\cap\mathbb{N}_0^3} \|x^{\alpha_3}D_x^{\alpha_1}D_y^{\alpha_2}\|_{L_2(B_\sigma)},$$

$$|||u,\sigma|||_t = \max_{\beta_1,\beta_2\in\mathcal{M}^t\cap\mathbb{N}_0^2} |||D_x^{\beta_1}D_y^{\beta_2}u,\sigma||| \quad \text{for} \quad t>0$$

and

$$|||u,\sigma|||_t = |||u,\sigma||| \quad \text{for} \quad t<0.$$

Lemma 6. *Let $u \in C^\infty$, $\alpha \in \mathcal{N}\cap\mathbb{N}_+^3$, $0 \le \alpha_1' \le \alpha_1$, $0 \le \alpha_2' \le \alpha_2$, $\alpha_1', \alpha_2' \in \mathcal{N}$, $\rho \in (0,1)$. Then there exists a constant $C>0$ so that for any $(\beta_1,\beta_2) \in \mathcal{M}^j\cap\mathbb{N}_0^2$, $j = 1,2,\ldots,$*

$$\|x^{\alpha_3}(D_x^{\alpha_1'}D_y^{\alpha_2'}\varphi)D_x^{\alpha_1-\alpha_1'+\beta_1}D_y^{\alpha_2-\alpha_2'+\beta_2}u\|_{L_2} \le C\rho^{-(\alpha_1'+\alpha_2')}|||u,\rho_1+\rho|||_{j-(\alpha_1'+\alpha_2')},$$

where φ is from Lemma 5.

Proof: We can assume without loss of generality that $j \ge \alpha_1' + \alpha_2'$. Now we show that $\alpha = (\alpha_1-\alpha_1'+\beta_1, \alpha_2-\alpha_2'+\beta_2, \alpha_3) \in \mathcal{A}_1^{j-(\alpha_1'+\alpha_2')} \cup \mathcal{A}_2^{j-(\alpha_1'+\alpha_2')}$. From Lemmas 1, 2, α can be taken in form $\alpha = (\mu_1,\mu_2,\mu_3)+(\nu_1,\nu_2,0)$ where $(\mu_1,\mu_2,\mu_3) \in \mathcal{N}\cap\mathbb{N}_0^3$, $(\nu_1,\nu_2) \in \mathcal{M}^{j-(\alpha_1'+\alpha_2')}\cap\mathbb{N}_0^2$. Then from Lemma 5 we obtain

$$\begin{aligned}
&\|x^{\alpha_3}(D_x^{\alpha_1'}D_y^{\alpha_2'}\varphi)D_x^{\alpha_1-\alpha_1'+\beta_1}D_y^{\alpha_2-\alpha_2'+\beta_2}u\|_{L_2}\\
&\qquad \le C_{\alpha_1',\alpha_2'}\rho^{-(\alpha_1'+\alpha_2')}\|x^{\mu_3}(D_x^{\mu_1}D_y^{\mu_2}(D_x^{\nu_1}D_y^{\nu_2}u)\|_{L_2(B_{\rho_1+\rho})}\\
&\qquad \le C\rho^{-(\alpha_1'+\alpha_2')}|||u,\rho_1+\rho|||_{j-(\alpha_1'+\alpha_2')},
\end{aligned}$$

where $C = \max_{\alpha_1'\le m,\alpha_2'\le m} C_{\alpha_1',\alpha_2'}$. □

Corollary 1. *Let $U \in C^\infty$. Then there exists $C>0$ such that for all $(\beta_1,\beta_2) \in \mathcal{M}^j\cap\mathbb{N}_0^2$, $j \ge 1$,*

$$\|[P,\varphi]D_x^{\beta_1}D_y^{\beta_2}u\|_{L_2} \le C\sum_{i=1}^{m}\rho^{-i}|||u,\rho_1+\rho|||_{j-i},$$

where φ is from Lemma 5.

Proof: The proof follows from Lemma 6, if we note that $[P,\varphi]$ is representable by linear combination of terms in form

$$x^{\alpha_3}(D_x^{\alpha_1'}D_y^{\alpha_2'}\varphi)D_x^{\alpha_1-\alpha_1'}D_y^{\alpha_2-\alpha_2'},$$

where $(\alpha_1,\alpha_2,\alpha_3) \in \mathcal{N}\cap\mathbb{N}_+^3$, $0 \le \alpha_1' \le \alpha_1$, $0 \le \alpha_2' \le \alpha_2$, and $\alpha_1'+\alpha_2' \ge 1$. □

Lemma 7. *Let $U \in C^\infty$ and $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{N} \cap \mathbb{N}_+^3$, $(\beta_1, \beta_2) \in \mathcal{M}^j \cap \mathbb{N}_0^2$, $0 \le \gamma_1 \ge \beta_1$, $\gamma_1 \in \mathcal{N}$. Then*

$$\|(D_x^{\gamma_1} x^{\alpha_3}) D_x^{\alpha_1+\beta_1-\gamma_1} D_y^{\alpha_2+\beta_2} u\|_{L_2(B_{\rho_1+\rho})} \le C |\!|\!| u, \rho_1 + \rho |\!|\!|_{j-\gamma_1} \tag{4}$$

with some constant $C > 0$.

Proof: Inequality (4) is trivial for $\gamma_1 > \alpha_3$. Let $\gamma_1 \le \alpha_3$. Then from Lemmas 3, 4 we obtain that $(\alpha_1 + \beta_1 - \gamma_1, \alpha_2 + \beta_2, \alpha_3 - \gamma_1) \in \mathcal{A}_2^{j-\gamma_1} \cup \mathcal{A}_1^{j-\gamma_1}$. From Lemmas 1, 2, this multi-index can be taken in form $(\alpha_1', \alpha_2', \alpha_3 - \gamma_1) + (\beta_1', \beta_2', 0)$, where $(\alpha_1', \alpha_2', \alpha_3 - \gamma_1) \in \mathcal{N} \cap \mathbb{N}_+^3$, $(\beta_1', \beta_2') \in \mathcal{M}^{j-\gamma_1}$. If we take $C \ge \alpha_3!/\gamma_1!$ then the proof is complete. □

Corollary 2. *Let $U \in C^\infty$. Then there exists a constant $C > 0$ so that for any multi-index $(\beta_1, \beta_2) \in \mathcal{M}^j \cap \mathbb{N}_0^2$, $j = 1, 2, \ldots$,*

$$\|[P, D_x^{\beta_1} D_y^{\beta_2}]u\|_{L_2(B_{\rho_1+\rho})} \le C \sum_{i=1}^{j} j!/(j-i)! |\!|\!| u, \rho_1 + \rho |\!|\!|_{j-i}.$$

Proof: The proof follows from Lemma 7 if we note that $[P, D_x^{\beta_1} D_y^{\beta_2}]$ is representable by a linear combination of terms in form $(D_x^{\gamma_1} x^{\alpha_3}) D_x^{\alpha_1+\beta_1-\gamma_1} D_y^{\alpha_2+\beta_2}$ where $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{N} \cap \mathbb{N}_0^3$, $1 \le \gamma_1 \le \min(\beta_1, \alpha_3)$ and the number of the nonzero terms in which x^{α_3} is differentiated γ_1 times is less than $C_j^{\gamma_{\alpha_3}}$ ($C_j^{\gamma_{\alpha_3}}$ are binomial coefficients). □

Theorem 1. *Let the hypoelliptic differential operator $P(x, D)$ from (1) with $\mathcal{N}(P)$ in form (2) satisfy in any neighborhood U of 0*

$$\sum_{(\alpha_1,\alpha_2,\alpha_3) \in \mathcal{N}(P) \cap \mathbb{N}_0^3} \|x^{\alpha_3} D_x^{\alpha_1} D_y^{\alpha_2} \psi\|_{L_2} \le \|P(x, D)\psi\|_{L_2}, \quad \psi \in C_0^\infty(U).$$

Then all solutions of equation $P(x, D) = f$ belong to the class $G^{(\lambda_1, 1)}(V)$, with $V \subset\subset U$, $(0, 0) \in V$, where $f \in G^{(\lambda_1, 1)}(U)$.

Proof: We take $U = B_3$, $V = B_1$. Let $\rho > 0$, $\rho_1 > 1$, $\rho_1 + \rho < 2$, then for any multi-indices $\beta \in \mathbb{N}_0^2$ from (3) we obtain

$$|\!|\!| D_x^{\beta_1} D_y^{\beta_2} u, \rho_1 |\!|\!| \le |\!|\!| \varphi D_x^{\beta_1} D_y^{\beta_2} u, 2 |\!|\!| \le C \|P(x, D)(\varphi D_x^{\beta_1} D_y^{\beta_2} u)\|_{L_2},$$

where φ is from Lemma 5. Since

$$\begin{aligned} &\|P(x, D)(\varphi D_x^{\beta_1} D_y^{\beta_2} u)\|_{L_2} \\ &\qquad = \|\varphi D_x^{\beta_1} D_y^{\beta_2} f\|_{L_2} + \|[P, \varphi] D_x^{\beta_1} D_y^{\beta_2} u\|_{L_2} + \|[P, D_x^{\beta_1} D_y^{\beta_2}]u\|_{L_2(B_{\rho_1+\rho})} \end{aligned}$$

then for any natural j

$$|||u,\rho|||_j \le \max_{(\beta_1,\beta_2)\in\mathcal{M}^j\cap\mathbb{N}_0^2}\{\|D_x^{\beta_1}D_y^{\beta_2}f\|_{L_2(B_2)} + \|[P,\varphi]D_x^{\beta_1}D_y^{\beta_2}u\|_{L_2} + \|[P,D_x^{\beta_1}D_y^{\beta_2}]u\|_{L_2(B_{\rho_1+\rho})}\}.$$

From condition $f \in G^{(\lambda_1,1)}(U)$ and from Corollary 1, 2 for $j \ge 1$ with some constant $C_1 = C_1(f)$ we obtain

$$(5)\quad |||u,\rho_1|||_j \le C(C_1^{j+1}j! + \sum_{i=1}^{m}\rho^{-i}|||u,\rho_1+\rho|||_{j-i} + \sum_{i=1}^{j} j!/(j-i)!|||u,\rho_1+\rho|||_{j-i}).$$

For any natural s, $j \le s$ we denote

$$\omega_{s,j} = s^{-j}|||u, 2-(j+1)/s|||_j.$$

Applying (5) with $\rho = 1/s$, $\rho_1 = 2 - j/s$, we obtain

$$(6)\qquad \omega_{s,j} \le C(C_2^j + \sum_{i=1}^{m}\omega_{s,j-i} + \sum_{i=1}^{j}\omega_{s,j-i})$$

with some constant C_2. From (6) we obtain by induction $\omega_{s,j} \le C_3^{j+1}$ for $j \le s$ with some constant $C_3 > 1$. For $j = s$ we obtain $|||u, 2-(s+1)/s|||_s \le C_3^{s+1}s^s$, $s = 1, 2, \ldots$. Since

$$\{\nu : \nu \in \mathbb{R}_+^2, \lambda_1\nu_1 + \nu_2 \le k-(1+h)m\} \subset \mathcal{M}^k \subset \{\nu : \nu \in \mathbb{R}_+^2, \lambda_1\nu_1 + \nu_2 \le k\}$$

for any $k \ge (1+h)m$, the proof is complete. □

References

[1] Grushin V.V., *On a class of elliptic pseudodifferential operators degenerate on a submanifold*, Mat. Sbornik **84** (1971), 163–1295; Math. USSR Sbornik **13** (1971), 155–185.

[2] Parenti C., Rodino L., *Parametrices for a class of pseudo differential operators, I, II*, Ann. Mat. Pura Appl. **125** (1980), 221–278.

[3] Rodino L., *Gevrey hypoellipticity for a class of operators with multiple characteristics*, Asterisque **89-90** (1981), 249–262.

[4] Tartakoff D.S., *Elementary proofs of analytic hypoellipticity for Δ_b and δ-Neumann problem*, in Analytic Solution of Partial Differential Equations, Asterisque **89-90** (1981), 85–116.

[5] Volevich L.R., *Local regularity of the solutions of the quasi-elliptic systems* (in Russian), Mat. Sbornik **59** (1962), 3–52.

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