Multiplicity of positive solutions for some quasilinear Dirichlet problems on bounded domains in $\mathbb{R}^n$

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Abstract. We show that, under appropriate structure conditions, the quasilinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $1 < p < +\infty$, admits two positive solutions u_0, u_1 in $W_0^{1,p}(\Omega)$ such that $0 < u_0 \leq u_1$ in Ω, while u_0 is a local minimizer of the associated Euler-Lagrange functional.

Keywords: p-Laplacian, positive solutions, sub- and supersolutions, local minimizers, Palais-Smale condition

Classification: 35J20, 35J60, 35J70

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with boundary of class C^2 and consider the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the so-called p-Laplace operator with $1 < p < +\infty$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e. continuous in u for a.e. $x \in \Omega$ and measurable in x for all $u \in \mathbb{R}$.

Questions concerning the effect of the nonlinear term $f(x,u)$ on the existence and multiplicity of solutions of (1.1) have been extensively investigated in recent years. A comprehensive review of the existing literature is beyond the present scope and the interested reader should consult the survey in [2]. Confining ourselves to the class of *positive* solutions, it is essential, however, to report the results which are closely related to the theme discussed in the present article. These pertain, in particular, to the model case provided by the function

$$f(x,u) = \lambda|u|^{r-2}u + |u|^{s-2}u,$$

where $1 < r \leq p < s$ and $\lambda > 0$ is a real parameter. As a matter of fact, with the aid of variational techniques it was shown in [3] that when $1 < r < p = 2 < s \leq 2^* := \frac{2n}{n-2}$, there exists a constant $\Lambda > 0$ such that problem (1.1) admits at least two positive solutions in $W_0^{1,p}(\Omega)$ for all $\lambda \in (0, \Lambda)$, at least one positive solution if $\lambda = \Lambda$ and no solution if $\lambda > \Lambda$. This multiplicity result was then extended via topological degree arguments in [1] for the quasilinear case $p \neq 2$ with $1 < r < p < s < p^* := \frac{np}{n-p}$, albeit for the special class of radial solutions. Note that when $1 < r < p < s < \infty$, the existence of one positive solution, without any symmetry assumptions on the domain Ω, was established in [8] via Sattinger's iteration scheme [17]. Nevertheless, this method cannot yield more solutions. The issue of existence and multiplicity in the nonradial setting and with $p \neq 2$ was studied in [7] via an extension to p-Laplace equations of a theorem by Brezis and Nirenberg [10] which concerns the relationship between local minimizers of the associated Euler-Lagrange functional in the $W_0^{1,p}$ and C_0^1 topologies. More specifically, by applying arguments similar to those used in the semilinear case, it was shown in [7] that one positive solution can be obtained as a local minimizer of the above functional while a second positive solution can then be found by means of a variant of the Mountain-Pass Theorem.

In this paper we are concerned with the issue of multiplicity as above, but in the context of a much larger class of nonlinearities. Our approach remains variational in nature and combines several ideas from [3], [10] and [13]. In particular, we show the existence of two positive solutions u_0, u_1 which are ordered; i.e. $0 < u_0 \leq u_1$ in Ω. Note that this property has been established so far only in the semilinear case $p = 2$ where, in fact, due to the linearity of the principal part of (1.1), the ordering is strict (i.e. $0 < u_0 < u_1$ in Ω), [3].

Let us finally mention that the critical semilinear case $1 < r \leq p = 2 < s = 2^*$ was originally studied in the pioneering paper of Brezis and Nirenberg [9] and their results were then extended to the quasilinear case in [13] for $1 < r = p < s = p^*$ and in [6] for $1 < r < p < s = p^*$.

2. Existence and multiplicity of positive solutions

Throughout this section we are concerned with the problem of finding positive solutions for (1.1), assuming that the lower order nonlinearity f satisfies the structure conditions:

(H1) $f(x,u)$ is nondecreasing in u with $f(x,0) = 0$ for a.e. $x \in \Omega$.

(H2) There exists $C > 0$ such that $|f(x,u)| \leq C(1 + |u|^{k-1})$ for a.e. $x \in \Omega$, where
 (i) $k \in (1, p^*]$ if $p < n$, with $p^* := \frac{np}{n-p}$,
 (ii) $k \in (1, +\infty)$ if $p \geq n$.

(H3) $\liminf_{s \to 0^+} \frac{f(x,s)}{s^{p-1}} > \lambda_1$ for a.e. $x \in \Omega$, where λ_1 is the principal eigenvalue of $-\Delta_p$ on Ω with zero Dirichlet boundary conditions.

Before we proceed, a few preliminary facts that will be used repeatedly in the sequel are in order. First, a basic ingredient in our approach is provided by the following proposition which concerns the boundary regularity of weak solutions of (1.1).

Theorem 1. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the quasilinear Dirichlet problem* (1.1) *where $f(x,u)$ conforms with* (H2). *Then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*

A proof of Theorem 1 in the case where $1 < p \leq n$ and $f(x,u)$ is continuous in $\overline{\Omega} \times \mathbb{R}$ can be found in [13]. A different proof covering the present situation, as well as the full range of the exponent p, is provided in the Appendix.

Consider now the Euler-Lagrange functional associated with (1.1),

$$\Phi(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x,u) \, dx, \tag{2.1}$$

where

$$F(x,u) := \int_0^u f(x,t) \, dt. \tag{2.2}$$

As is well known, on account of (H2), $\Phi(\cdot)$ defines a continuous functional from $W_0^{1,p}(\Omega)$ to $\mathbb{R}$ which is also weakly lower semicontinuous unless $p < n$ and $k = p^*$. Moreover, it is easy to show that any minimizer $u \in W_0^{1,p}(\Omega)$ of $\Phi(\cdot)$ is a weak solution of (1.1). Even more, according to the following remarkable theorem, any local minimizer of $\Phi(\cdot)$ in the C_0^1-topology must also be a local minimizer in the $W_0^{1,p}$-topology.

Theorem 2. *Let* (H2) *hold and assume that there exist $w \in W_0^{1,p}(\Omega)$ and $\rho > 0$ such that*

$$\Phi(w) \leq \Phi(w+v) \quad \textit{for every} \quad v \in C_0^1(\overline{\Omega}) \quad \textit{with} \quad \|v\|_{C^1} \leq \rho. \tag{2.3}$$

Then there exists $\rho' > 0$ such that

$$\Phi(w) \leq \Phi(w+z) \quad \textit{for every} \quad z \in W_0^{1,p}(\Omega) \quad \textit{with} \quad \|z\|_{W^{1,p}} \leq \rho'. \tag{2.4}$$

As already mentioned in the introduction, this rather surprising result was first proved by Brezis and Nirenberg when $p = 2$ in [10] and then it was extended for all $p \in (1, +\infty)$ by Azorero, Alonso and Manfredi in [7]. It should be pointed out here, however, that this property may not hold for a general functional since it is the special structure of (2.1) which plays an essential role in the proof.

Finally, the following lemma is essentially a variant regarding the monotonicity of the $-\Delta_p$ operator and can be easily proved via Hölder's inequality.

Lemma 3. *Let $1 < p < +\infty$. Then for any $u, v \in W_0^{1,p}(\Omega)$ the following inequality holds*

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right](\nabla u - \nabla v)\,dx \geq \left(\|u\|^{p-1} - \|v\|^{p-1}\right)\left(\|u\| - \|v\|\right) \geq 0,$$

where $\|u\| := \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$.

Definition 4. A nonnegative function $w \in C^1(\overline{\Omega})$ is said to be a *strict supersolution* (resp. *strict subsolution*) for (1.1) if $-\Delta_p w > f(x,w)$ in Ω (resp. $-\Delta_p w < f(x,w)$ in Ω) and $w = 0$ on $\partial\Omega$.

Observe that, on account of the strong maximum principle of Vázquez [18] and (H1), a strict supersolution is necessarily positive everywhere in Ω.

Our first result is the following

Theorem 5. *Suppose that* (H1), (H2) *and* (H3) *hold. Assume further that a strict supersolution $\overline{u}$ for* (1.1) *exists. Then, problem* (1.1) *admits a positive solution u_0 which is also a local minimizer of $\Phi(\cdot)$ in the $W_0^{1,p}$-topology.*

PROOF: Let φ_1 be the eigenfunction corresponding to the principal eigenvalue λ_1 of $-\Delta_p$ on Ω with zero Dirichlet boundary conditions, normalized so that $\|\varphi_1\|_\infty = 1$. Since $\lambda_1 > 0$ and $\varphi_1(\cdot) > 0$ in Ω (see [5]), in view of (H3), there exists $\varepsilon > 0$ such that $\underline{u} = \varepsilon\varphi_1$ is a strict subsolution of (1.1). Moreover, by virtue of the strong maximum principle ([18]) it is straightforward to check that if ε is chosen sufficiently small then

$$\underline{u} < \overline{u}, \quad \text{in } \Omega. \tag{2.5}$$

Let us now define

$$\widehat{f}(x,t) := \begin{cases} f(x,\overline{u}(x)), & \text{if } t > \overline{u}(x), \\ f(x,t), & \text{if } \underline{u}(x) \leq t \leq \overline{u}(x), \\ f(x,\underline{u}(x)), & \text{if } t < \underline{u}(x), \end{cases} \tag{2.6}$$

and consider the problem

$$\begin{cases} -\Delta_p u = \widehat{f}(x,u(x)), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.7}$$

with the associated Euler-Lagrange functional

$$\widehat{\Phi}(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega\int_0^u \widehat{f}(x,t)\,dt\,dx.$$

From (H2) and (2.6), it is easily seen that $\widehat{\Phi}(\cdot)$ is bounded from below and weakly lower semicontinuous in $W_0^{1,p}(\Omega)$. Therefore, the infimum of $\widehat{\Phi}(\cdot)$ is achieved at some point $u_0 \in W_0^{1,p}(\Omega)$ which is a solution of (2.7). In particular, $u_0 \in C^1(\overline{\Omega})$ by Theorem 1. We claim that $\underline{u} \leq u_0 \leq \overline{u}$ in Ω. Indeed, let us define the set

$$\Omega_0 := \{x \in \Omega : u_0(x) < \underline{u}(x)\},$$

and assume that it is nonempty. Since Ω_0 is open, it must have positive measure. Furthermore, in view of (2.6),

$$-\Delta_p u_0 = f(x, \underline{u}(x)), \quad x \in \Omega_0, \tag{2.8}$$

while

$$-\Delta_p \underline{u} < f(x, \underline{u}(x)), \quad x \in \Omega_0. \tag{2.9}$$

Hence, by multiplying (2.8) and (2.9) with $\underline{u} - u_0$ and integrating over Ω_0, we get

$$\int_{\Omega_0} |\nabla u_0|^{p-2} \nabla u_0 \nabla(\underline{u} - u_0)\, dx = \int_{\Omega_0} f(x, \underline{u}(x))(\underline{u} - u_0)\, dx,$$

and

$$\int_{\Omega_0} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla(\underline{u} - u_0)\, dx < \int_{\Omega_0} f(x, \underline{u}(x))(\underline{u} - u_0)\, dx,$$

which combined yield

$$\int_{\Omega_0} \left\{|\nabla \underline{u}|^{p-2} \nabla \underline{u} - |\nabla u_0|^{p-2} \nabla u_0\right\} \nabla(\underline{u} - u_0)\, dx < 0.$$

However, the last inequality contradicts Lemma 3 and so Ω_0 must be empty. The proof of $u_0 \leq \overline{u}$ in Ω is analogous. Because now $f(x, u)$ is nondecreasing in u for a.e. $x \in \Omega$, on account of (2.5), (2.6) and (2.7), we have

$$0 < -\Delta_p \underline{u} < f(x, \underline{u}) \leq -\Delta_p u_0 = f(x, u_0) \leq f(x, \overline{u}) < -\Delta_p \overline{u}, \qquad x \in \Omega, \tag{2.10}$$

and so, by the strong comparison principle in [13], we eventually deduce that

$$\underline{u}(x) < u_0(x) < \overline{u}(x), \qquad x \in \Omega, \tag{2.11}$$

while

$$\frac{\partial \overline{u}}{\partial \nu}(x) < \frac{\partial u_0}{\partial \nu}(x) < \frac{\partial \underline{u}}{\partial \nu}(x), \quad x \in \partial\Omega, \tag{2.12}$$

where ν denotes the exterior unit normal at $x \in \partial\Omega$. Moreover, by virtue of the strong maximum principle in [18],

$$\frac{\partial \underline{u}}{\partial \nu}(x) < 0, \quad x \in \partial\Omega. \tag{2.13}$$

Note that this inequality holds under the assumption that the boundary $\partial\Omega$ satisfies the so-called interior sphere condition. However, this condition is automatically true here because $\partial\Omega$ was taken to be of class C^2. In the sequel we shall show that there exists $\delta > 0$ such that

$$\underline{u}(x) + \delta \operatorname{dist}(x, \partial\Omega) \leq u_0(x) \leq \overline{u}(x) - \delta \operatorname{dist}(x, \partial\Omega), \quad x \in \Omega. \tag{2.14}$$

Note first that, since $\partial\Omega$ is compact, an immediate implication of (2.13) is the existence of positive constants β, σ such that

$$|\nabla \underline{u}(x)| > \beta > 0, \tag{2.15}$$

for all x in the annular region

$$\mathcal{R} := \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq \sigma\}.$$

Furthermore, (2.12), (2.13) and (2.15) imply that there exists a constant $\gamma > 1$ and a continuous function $\mu(\cdot)$ such that

$$\frac{\partial u_0}{\partial \nu}(x) = \mu(x)\frac{\partial \underline{u}}{\partial \nu}(x), \quad x \in \partial\Omega, \tag{2.16}$$

with

$$\mu(x) > \gamma > 1. \tag{2.17}$$

Since now $u_0 = \underline{u} = 0$ on $\partial\Omega$, the projections of $\nabla u_0(x)$ and $\nabla \underline{u}(x)$ on the hyperplane which is tangent to $\partial\Omega$ at x must be equal to zero. Consequently, (2.16) reduces to

$$\frac{\partial u_0}{\partial x_i}(x) = \mu(x)\frac{\partial \underline{u}}{\partial x_i}(x), \quad i = 1, \ldots, n, \quad x \in \partial\Omega. \tag{2.18}$$

On the other hand, by the mean value theorem we can write, as in [13],

$$-\Delta_p u_0 + \Delta_p \underline{u} = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}(u_0 - \underline{u})\right), \quad x \in \Omega, \tag{2.19}$$

where

$$\begin{aligned} a_{ij}(x) := |t_i \nabla u_0 + (1-t_i)\nabla \underline{u}|^{p-4} \Bigg(& \delta_{ij}\, |t_i \nabla u_0 + (1-t_i)\nabla \underline{u}|^2 \\ & + (p-2)\left(t_i \frac{\partial u_0}{\partial x_i} + (1-t_i)\frac{\partial \underline{u}}{\partial x_i}\right)\left(t_i \frac{\partial u_0}{\partial x_j} + (1-t_i)\frac{\partial \underline{u}}{\partial x_j}\right)\Bigg), \quad x \in \Omega, \end{aligned}$$

and $t_i \in (0,1)$, $i = 1, \dots n$. By setting now

$$d_i(x) := |\nabla u_0|^{p-2}\,\frac{\partial u_0}{\partial x_i} - |\nabla \underline{u}|^{p-2}\,\frac{\partial \underline{u}}{\partial x_i}, \quad i = 1, \dots, n, \quad x \in \overline{\Omega},$$

and using (2.17), (2.18), we have

$$d_i(x) = \frac{\mu^{p-1} - 1}{\mu - 1}\, |\nabla \underline{u}|^{p-2}\,\frac{\partial}{\partial x_i}(u_0 - \underline{u}), \quad i = 1, \dots, n, \quad x \in \partial\Omega. \tag{2.20}$$

But since

$$-\Delta_p u_0 + \Delta_p \underline{u} = -\sum_i \frac{\partial}{\partial x_i} d_i(x), \quad x \in \Omega, \tag{2.21}$$

on combining (2.15), (2.17), (2.19), (2.20) and (2.21), we deduce by continuity that the second order differential operator appearing on the righthand side of (2.19) is uniformly elliptic in the region $\mathcal{R}$. Hence, in view of (2.10), the extension of the classical Hopf's lemma (see [18]) implies the existence of $\delta_1 > 0$ such that $u_0(x) - \underline{u}(x) \geq \delta_1 \operatorname{dist}(x, \partial\Omega)$ for all $x \in \mathcal{R}$. In a similar fashion it can be shown that there exists $\delta_2 > 0$ such that $\overline{u}(x) - u_0(x) \geq \delta_2 \operatorname{dist}(x, \partial\Omega)$ for all $x \in \mathcal{R}$. The validity of (2.14) for every $x \in \Omega$ then follows by using (2.11) and choosing $\delta > 0$ appropriately. Let now $u \in C_0^1(\Omega)$ with $\|u - u_0\|_{C_0^1} \leq \delta$. Then, $\underline{u} \leq u \leq \overline{u}$ in Ω by (2.14). At the same time, $\Phi = \widehat{\Phi}$ on the set

$$\big\{u \in C_0^1(\Omega) : \|u - u_0\|_{C_0^1} \leq \delta\big\}.$$

Therefore, u_0 is a local minimizer of $\Phi(\cdot)$ in $C_0^1(\Omega)$ and by Theorem 2, also a local minimizer of $\Phi(\cdot)$ in $W_0^{1,p}(\Omega)$. Consequently, u_0 is a positive solution of problem (1.1). $\square$

Remark 6. The assumption for the existence of a strict supersolution $\overline{u}$ in Theorem 5 appears to be very essential. Its importance can also be verified by consulting the proofs of the related theorems in [1], [3] and [8] where strict supersolutions are actually constructed. On the other hand, when a strict supersolution $\overline{u}$ for (1.1) is known, it follows from (H3) that a strict subsolution $\underline{u}$ can easily be found with $\underline{u} < \overline{u}$.

Our next result provides the existence of a second positive solution u_1 of (1.1), with $u_0 \leq u_1$ in Ω, if more conditions on the structure of the nonlinearity $f(x,u)$ are imposed. In particular, our strategy involves the use of the Mountain-Pass Theorem for a modified functional $\Psi(\cdot)$ which satisfies the Palais-Smale condition and is unbounded from below under the assumptions:

(H2)$'$ The same growth condition in (H2) holds but with $1 < k < p^*$ if $p < n$.

(H4) There exist $\varrho > 0$ and $\theta \in (0, \frac{1}{p})$ such that

$$F(x,u) \leq \theta f(x,u)u \quad \text{when} \quad |u| \geq \varrho. \tag{2.22}$$

(H5) There exist $\eta > 0$ and $r > p$ such that $\liminf_{s\to+\infty} \frac{f(x,s)}{s^{r-1}} > \eta$ for a.e. $x \in \Omega$.

Theorem 7. *Suppose that* (H1), (H2)$'$, (H3), (H4) *and* H(5) *hold. Assume further that* (1.1) *possesses a strict supersolution* $\overline{u}$*. Then problem* (1.1) *admits two solutions* u_0, u_1 *in* $W_0^{1,p}(\Omega)$ *such that* $0 < u_0 \leq u_1$ *in* Ω.

Proof: Let u_0 be the solution obtained in Theorem 5 and consider the problem of finding $v \in W_0^{1,p}(\Omega)$ such that $v \not\equiv 0$ and

$$\begin{cases} -\Delta_p(u_0+v) = f(x,u_0+v), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{2.23}$$

For this set

$$k(x,t) := \begin{cases} f(x,u_0+t), & \text{if } t \geq 0, \\ f(x,u_0), & \text{if } t < 0, \end{cases} \tag{2.24}$$

$$K(x,v) := \int_0^v k(x,t)\,dt, \tag{2.25}$$

and define the functional

$$\Psi(v) := \frac{1}{p}\|u_0+v\|^p - \int_\Omega K(x,v)\,dx. \tag{2.26}$$

We claim that $v = 0$ is a local minimizer of $\Psi(\cdot)$ in $W_0^{1,p}(\Omega)$. Indeed, if v^+ and v^- denote the positive and negative parts of v respectively, we have

$$\begin{aligned}
\int_\Omega K(x,v)\,dx &= \int_{\{v\geq 0\}} K(x,v^+)\,dx + \int_{\{v<0\}} K(x,v^-)\,dx \\
&= \int_\Omega \int_0^{v^+} k(x,t)\,dt\,dx + \int_\Omega \int_0^{v^-} k(x,t)\,dt\,dx \\
&= \int_\Omega \int_0^{v^+} f(x,u_0+t)\,dt\,dx + \int_\Omega \int_0^{v^-} f(x,u_0)\,dt\,dx \\
&= \int_\Omega \int_{u_0}^{u_0+v^+} f(x,t)\,dt\,dx + \int_\Omega f(x,u_0)v^-\,dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Psi(v) &= \frac{1}{p}\left\|u_0+v^+\right\|^p + \frac{1}{p}\left\|u_0+v^-\right\|^p - \frac{1}{p}\left\|u_0\right\|^p \\
&\qquad - \int_\Omega \int_{u_0}^{u_0+v^+} f(x,t)\,dt\,dx - \int_\Omega f(x,u_0)v^-\,dx \\
&= \Phi(u_0+v^+) + \int_\Omega \int_0^{u_0+v^+} f(x,t)\,dt\,dx + \frac{1}{p}\left\|u_0+v^-\right\|^p - \frac{1}{p}\left\|u_0\right\|^p \\
&\qquad - \int_\Omega \int_{u_0}^{u_0+v^+} f(x,t)\,dt\,dx - \int_\Omega f(x,u_0)v^-\,dx \\
&= \Phi(u_0+v^+) + \frac{1}{p}\left\|u_0+v^-\right\|^p - \frac{1}{p}\left\|u_0\right\|^p \\
&\qquad + \int_\Omega \int_0^{u_0} f(x,t)\,dt\,dx - \int_\Omega f(x,u_0)v^-\,dx.
\end{aligned}$$

Moreover, since u_0 solves (1.1),

$$\int_\Omega |\nabla u_0|^{p-2}\,\nabla u_0 \nabla v^-\,dx = \int_\Omega f(x,u_0)v^-\,dx,$$

and so

$$\Psi(v) = \Phi(u_0+v^+) - \Phi(u_0) + \frac{1}{p}\left\|u_0+v^-\right\|^p - \int_\Omega |\nabla u_0|^{p-2}\,\nabla u_0\nabla v^-\,dx.$$

On the other hand, by the strict convexity of the mapping $\xi \longmapsto |\xi|^p$ for any $p > 1$, the following inequality holds

$$|\xi_2|^p \geq |\xi_1|^p + p\,|\xi_1|^{p-2}\,\xi_1\cdot(\xi_2-\xi_1)\,, \qquad \xi_1,\xi_2\in\mathbb{R}^n, \tag{2.27}$$

which yields

$$\Psi(v) \geq \Phi(u_0 + v^+) - \Phi(u_0) + \frac{1}{p}\|u_0\|^p .$$

But since u_0 is a local minimizer of $\Phi(\cdot)$ in $W_0^{1,p}(\Omega)$, this implies

$$\Psi(v) \geq \frac{1}{p}\|u_0\|^p = \Psi(0),$$

if $\|v\|$ is small enough, thereby proving the claim. At the same time it is easily checked that, on account of (H2)$'$ and (H4), the functional $\Psi(\cdot)$ satisfies the Palais-Smale condition (see [4]). Moreover, by using (H5), $\Psi(tu_0) \to -\infty$ as $t \to +\infty$ and so there exists $t_0 > 0$ such that $\Psi(t_0 u_0) < 0$. Hence, by applying the Ghoussoub-Preiss version of the Mountain-Pass Theorem [12] we get the existence of a second critical point $v_0 \not\equiv 0$ of $\Psi(\cdot)$. In particular, $v_0 \in C^1(\overline{\Omega})$ by virtue of Theorem 1. We shall now show that $v_0 \geq 0$. Indeed, since $\Psi'(v_0) = 0$, we have

$$\int_\Omega |\nabla u_0 + \nabla v_0|^{p-2}(\nabla u_0 + \nabla v_0)\nabla z\, dx = \int_\Omega k(x, v_0) z\, dx, \qquad z \in W_0^{1,p}(\Omega),$$

and by choosing $z = v_0^-$,

$$\begin{aligned}\int_\Omega |\nabla u_0 + \nabla v_0^-|^{p-2}(\nabla u_0 + \nabla v_0^-)\nabla v_0^-\, dx &= \int_\Omega k(x, v_0^-) v_0^-\, dx \\ &= \int_\Omega f(x, u_0) v_0^-\, dx.\end{aligned} \tag{2.28}$$

At the same time, since $\Phi'(u_0) = 0$,

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \nabla v_0^-\, dx = \int_\Omega f(x, u_0) v_0^-\, dx, \tag{2.29}$$

and so, on combining (2.28) and (2.29),

$$\int_\Omega \left\{ |\nabla u_0 + \nabla v_0^-|^{p-2}(\nabla u_0 + \nabla v_0^-) - |\nabla u_0|^{p-2}\nabla u_0 \right\} \nabla v_0^-\, dx = 0, \tag{2.30}$$

which, by applying Lemma 3, yields

$$\left\| u_0 + v_0^- \right\| = \|u_0\| . \tag{2.31}$$

On the other hand, by applying (2.27) with $\xi_1 = \nabla u_0 + \nabla v_0^-$, $\xi_2 = \nabla u_0$ and using (2.31), we get

$$\int_\Omega |\nabla u_0 + \nabla v_0^-|^{p-2}(\nabla u_0 + \nabla v_0^-)\nabla v_0^-\, dx \geq 0, \tag{2.32}$$

while by doing the same with $\xi_1 = \nabla u_0$, $\xi_2 = \nabla u_0 + \nabla v_0^-$,

$$\int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \nabla v_0^- \, dx \le 0. \tag{2.33}$$

Thus, from (2.29), (2.30), (2.32) and (2.33) we conclude that

$$\int_\Omega f(x, u_0) v_0^- \, dx = 0,$$

which, since $f(x, u_0) > 0$, implies $v_0^- = 0$; i.e. $v_0 \ge 0$. Hence, $u_1 := u_0 + v_0$ is a second positive solution of (1.1). The proof is complete. □

Appendix

Proof of Theorem 1: We proceed by examining separately three different ranges of the exponent p and showing first that $u \in L^\infty(\Omega)$.

(I) Let $1 < p < n$. Clearly, $u \in L^{p^*}(\Omega)$ by Sobolev's inequality. We now distinguish two cases:

Case 1: $1 < k < p^*$.

Then, by virtue of Theorem 7.1, Chapter IV, of [14], we immediately conclude that $u \in L^\infty(\Omega)$.

Case 2: $k = p^*$.

Here Theorem 7.1, Chapter IV, of [14] cannot be applied directly. Hence, motivated by [10], we proceed by decomposing $f(\cdot, u(\cdot))$ as follows:

$$f(x, u(x)) = a(x)|u(x)|^{p-2}u(x) + b(x),$$

where

$$a(x) := \begin{cases} \frac{f(x,u(x))}{|u(x)|^{p-2}u(x)}, & |u(x)| > 1, \\ 0, & |u(x)| \le 1, \end{cases} \tag{2.34}$$

and

$$b(x) := \begin{cases} 0, & |u(x)| > 1, \\ f(x, u(x)), & |u(x)| \le 1. \end{cases} \tag{2.35}$$

Then, in view of (H2), it is easily seen that $b \in L^\infty(\Omega)$ while $a(\cdot)$ satisfies the growth estimate

$$|a(x)| \le C_1(1 + |u(x)|^{p^*-p}),$$

which, since $u \in L^{p^*}(\Omega)$, implies that $a \in L^{n/p}(\Omega)$.

For any $m \in \mathbb{N}$ we set

$$u_m := \begin{cases} m & \text{if } u \geq m, \\ u & \text{if } |u| < m, \\ -m & \text{if } u \leq -m. \end{cases}$$

Clearly, if $r \geq 2$ then $|u_m|^{r-2}u_m \in W_0^{1,p}(\Omega)$ and so by multiplying $(1.1)_1$ with $|u_m|^{r-2}u_m$ and integrating over Ω we get

$$(r-1)\sum_{i=1}^{n}\int_\Omega \left(|\nabla u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\right)\left(|u_m|^{r-2}\frac{\partial u_m}{\partial x_i}\right)$$
$$= \int_\Omega a\ |u|^{p-2}u|u_m|^{r-2}u_m + \int_\Omega b|u_m|^{r-2}u_m,$$

which, since $uu_m \geq 0$ a.e. in Ω, gives

$$(2.36)\qquad (r-1)\int_\Omega |\nabla u_m|^p|u_m|^{r-2} \leq \int_\Omega a^+|u|^{p-1}\ |u_m|^{r-1} + \int_\Omega b|u_m|^{r-2}u_m,$$

where a^+ denotes the positive part of $a(\cdot)$. At the same time, it can be easily verified that

$$(2.37)\qquad \int_\Omega |\nabla u_m|^p|u_m|^{r-2} = \left(\frac{p}{p+r-2}\right)^p \int_\Omega \left|\nabla\left(|u_m|^{\frac{r-2}{p}}\ u_m\right)\right|^p.$$

Moreover, by Sobolev's inequality

$$(2.38)\qquad \left\||u_m|^{\frac{p+r-2}{p}}\right\|_{L^{p^*}} \leq C_S\left\|\nabla\,|u_m|^{\frac{p+r-2}{p}}\right\|_{L^p} = C_S\left\|\nabla\left(|u_m|^{\frac{r-2}{p}}\ u_m\right)\right\|_{L^p}$$

where C_S is the best Sobolev constant. Hence, on combining (2.36), (2.37) and (2.38), we deduce that

$$(2.39)\qquad \left\||u_m|^{\frac{p+r-2}{p}}\right\|_{L^{p^*}}^p \leq c_1\left(\int_\Omega a^+|u|^{p-1}\ |u_m|^{r-1} + \int_\Omega b|u_m|^{r-2}u_m\right)$$

where $c_1 > 0$ is a constant depending only on p, r and C_S. Fix now $k > 0$ and let $\Omega_1 := \{x \in \Omega : a^+(x) \leq k\}$ and $\Omega_2 := \{x \in \Omega : a^+(x) > k\}$. Since $|u_m| \leq |u|$ a.e. in Ω, (2.39) gives

$$(2.40)\qquad \left\||u_m|^{\frac{p+r-2}{p}}\right\|_{L^{p^*}}^p$$
$$\leq kc_1\int_{\Omega_1}|u|^{p+r-2} + c_1\int_{\Omega_2}a^+|u|^{p+r-2} + c_1\int_\Omega b|u_m|^{r-2}u_m.$$

Because Ω is bounded, there exists a constant $c_2 > 0$, depending only on Ω, p and r, such that

$$\|u_m\|_{L^{r-1}} \leq c_2 \|u_m\|_{L^{p+r-2}},$$

and so

$$\int_\Omega b|u_m|^{r-2}u_m \leq c_2 \|b\|_{L^\infty} \|u_m\|^{r-1}_{L^{p+r-2}} . \tag{2.41}$$

On the other hand, by virtue of Hölder's inequality,

$$\begin{aligned}\int_{\Omega_2} a^+|u|^{p+r-2} &\leq \left(\int_{\Omega_2} \left|a^+\right|^{\frac{n}{p}}\right)^{\frac{p}{n}} \left(\int_{\Omega_2} |u|^{\frac{n}{n-p}(p+r-2)}\right)^{\frac{n-p}{n}} \\ &\leq \left\|a^+\right\|_{L^{\frac{n}{p}}(\Omega_2)} \|u\|^{p+r-2}_{L^{\frac{p^*}{p}(p+r-2)}}.\end{aligned} \tag{2.42}$$

Thus, in view of (2.41) and (2.42), inequality (2.40) yields

$$\begin{aligned}&\|u_m\|^{p+r-2}_{L^{\frac{p^*}{p}(p+r-2)}} \\ &\quad\leq kc_1 \int_\Omega |u|^{p+r-2} + c_1 \|a^+\|_{L^{\frac{n}{p}}(\Omega_2)} \|u\|^{p+r-2}_{L^{\frac{p^*}{p}(p+r-2)}} + c_3 \|b\|_{L^\infty} \|u_m\|^{r-1}_{L^{p+r-2}},\end{aligned}$$

and by choosing $k > 0$ large enough so that $c_1 \left\|a^+\right\|_{L^{\frac{n}{p}}(\Omega_2)} \leq \frac{1}{2}$,

$$\|u_m\|^{p+r-2}_{L^{\frac{p^*}{p}(p+r-2)}} \leq 2kc_1 \int_{\Omega_1} |u|^{p+r-2} + 2c_3\|b\|_{L^\infty}\|u_m\|^{r-1}_{L^{p+r-2}}.$$

Assuming now that $u \in L^{p+r-2}(\Omega)$, if we allow $m \to \infty$ in the last inequality, we get

$$\|u\|^{p+r-2}_{L^{\frac{p^*}{p}(p+r-2)}} \leq 2kc_1\|u\|^{p+r-2}_{L^{p+r-2}} + 2c_3 \|b\|_{L^\infty} \|u\|^{r-1}_{L^{p+r-2}}, \tag{2.43}$$

which implies that $u \in L^{\frac{p^*}{p}(p+r-2)}(\Omega)$. Hence, by starting from $r = p^* - p + 2$ and bootstrapping (2.43) we easily deduce that $u \in L^s(\Omega)$ for every $s \in [p, +\infty)$. Therefore, $u \in W_0^{1,p}(\Omega) \cap L^s(\Omega)$ for every $s \in [p^*, +\infty)$ and so, by virtue of Theorem 7.1, Chapter IV, of [14], we deduce again that $u \in L^\infty(\Omega)$.

(II) Suppose now $p = n$. Then, $u \in L^q(\Omega)$ for any $q \in [1, +\infty)$ by the Sobolev embedding. Hence, on account of (H2), $f(\cdot, u(\cdot)) \in L^q(\Omega)$ for any $q \in [1, +\infty)$ and so by a standard bootstrap procedure in the spirit of Moser [16] we infer that $u \in L^\infty(\Omega)$ (see e.g. the proof of Proposition 2.1 in [11]).

(III) Finally, let $p > n$. Then, $u \in L^\infty(\Omega)$ directly by the Sobolev embedding. The assertion of the proposition now follows by applying Theorem 1 of [15].

□

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