Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant

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In memoriam Jindřich Nečas

Abstract. First we recall a Faber-Krahn type inequality and an estimate for $\lambda_p(\Omega)$ in terms of the so-called Cheeger constant. Then we prove that the eigenvalue $\lambda_p(\Omega)$ converges to the Cheeger constant $h(\Omega)$ as $p \to 1$. The associated eigenfunction u_p converges to the characteristic function of the Cheeger set, i.e. a subset of Ω which minimizes the ratio $|\partial D|/|D|$ among all simply connected $D \subset\subset \Omega$. As a byproduct we prove that for convex Ω the Cheeger set ω is also convex.

Keywords: isoperimetric estimates, eigenvalue, Cheeger constant, p-Laplace operator, 1-Laplace operator

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Let $p \in (1,\infty)$, and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded simply connected domain with sufficiently smooth boundary. A well-known result in nonlinear partial differential equations states that the following eigenvalue problem

$$\begin{aligned} \Delta_p u + \lambda |u|^{p-2} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

has a positive (weak) solution in $W_0^{1,p}(\Omega)$, which is unique modulo scaling, in other words the corresponding eigenvalue is simple. A simple proof of this long-known fact was recently given in [3]. These functions are also called first eigenfunctions of the p-Laplace operator, and $\Delta_p v := \operatorname{div}(|\nabla v|^{p-2}\nabla v)$. To avoid ambiguity, we shall normalize it by prescribing $\|u\|_\infty = 1$.

One can characterize the eigenvalue and eigenfunction by

$$\lambda_p(\Omega) := \min_{0 \neq v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx}\,, \tag{2}$$

with u as minimizer. As mentioned above, in this paper the eigenfunction is scaled to $\|u\|_\infty = 1$. The functional in the right hand side is usually called Rayleigh quotient. The eigenfunction, or rather a multiple of it, can also be characterized as a critical point, and in fact a minimizer, of the functional

$$(3) \qquad J_p(v) = \int_\Omega |\nabla v|^p \, dx \quad \text{on} \quad K := \{ \, v \in W_0^{1,p}(\Omega) \mid \|v\|_{L^p(\Omega)} = 1 \, \}.$$

Upper bounds for λ_p can be obtained by choosing particular test functions v in (2), but lower bounds are more challenging.

Theorem 1. *Among all domains of given n-dimensional volume the ball minimizes every λ_p, in other words*

$$(4) \qquad \lambda_p(\Omega) \geq \lambda_p(\Omega^*),$$

where Ω^* is the n-dimensional ball of same volume as Ω.

As noted in [20, p. 224] and [12, p. 3353], this is a straightforward consequence of results in [14]. To prove the theorem, one replaces the first eigenfunction u_p on any domain Ω by its Schwarz symmetrization $(u_p)^*$ and notes that the Rayleigh quotient does not increase under this operation. Moreover, $(u_p)^*$ is in $W_0^{1,p}(\Omega^*)$ and thus an admissible function in (2). Therefore it provides an upper bound for $\lambda_p(\Omega^*)$. Theorem 1 was apparently rediscovered in [2] and [12], but $\lambda_p(\Omega^*)$ does not seem to be explicitly known unless $p = 2$.

Remark 2. If the Euclidean modulus of ∇v in (2) is replaced by its ℓ_p norm then Theorem 1 must be modified in the sense that Ω^* is a ball in $\mathbb{R}^n$ equipped with the $\ell_{p'}$-norm, see [4]. Analogous isoperimetric inequalities for (linear) operators of fourth order are discussed in [18].

In order to state the next result we need to define the **Cheeger constant** $h(\Omega)$ of a domain Ω. Cheeger defines it in [6, p. 196] for manifolds with or without boundary, and in this paper we are only interested in the case with boundary. In this case

$$(5) \qquad h(\Omega) := \inf_D \frac{|\partial D|}{|D|}$$

with D varying over all smooth subdomains of Ω whose boundary ∂D does not touch $\partial\Omega$, and with $|\partial D|$ and $|D|$ denoting $(n-1)$- and n-dimensional Lebesgue measure of ∂D and D. For ease of notation we call the expression $Q(D) := |\partial D|/|D|$ the **Cheeger quotient** of D and a subset ω of Ω for which $Q(\omega) = h(\Omega)$ a **Cheeger domain** of Ω. The existence, (non)uniqueness and regularity of Cheeger domains is discussed in Theorem 8 and the remarks following it.

In his celebrated paper Cheeger proved the case $p = 2$ of the following theorem.

Theorem 3 ([22]). *For every $p \in (1, \infty)$ the first eigenvalue can be estimated from below via*

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p . \tag{6}$$

The proof can be found in the appendix of [22], but since it is short, we repeat it here for the reader's convenience. Suppose first that $w \in C_0^\infty(\Omega)$ is a positive function and set $A(t) := \{ x \in \Omega \mid w(x) > t \}$. Then by the coarea formula and by Cavalieri's principle

$$\begin{aligned} \int_\Omega |\nabla w| \, dx &= \int_{-\infty}^{\infty} |\partial A(t)| \, dt = \int_{-\infty}^{\infty} \frac{|\partial A(t)|}{|A(t)|} \; |A(t)| \, dt \\ &\geq \inf_{D \subset\subset \Omega} \frac{|\partial D|}{|D|} \int_{-\infty}^{\infty} \; |A(t)| \, dt = h(\Omega) \int_\Omega |w| \, dx. \end{aligned} \tag{7}$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,1}(\Omega)$, relation (7) holds also for any $w \in W_0^{1,1}(\Omega)$. For $p > 1$ take any $v \in W_0^{1,p}(\Omega)$ and define $\Phi(v) = |v|^{p-1}v$. Then Hölder's inequality implies

$$\int_\Omega |\nabla \Phi(v)| \, dx = p \int_\Omega |v|^{p-1} |\nabla v| \, dx \leq p \; \|v\|_p^{p-1} \; \|\nabla v\|_p \, , \tag{8}$$

so that $w = \Phi(v) \in W_0^{1,1}(\Omega)$. Therefore (7) applies and

$$\int_\Omega |\nabla \Phi(v)| \, dx \geq h(\Omega) \int_\omega |v|^p \, dx, \tag{9}$$

or rather, using (8),

$$h(\Omega) \leq \frac{\int_\Omega |\nabla w| \, dx}{\int_\Omega |w| \, dx} \leq \frac{p \; \|v\|_p^{p-1} \; \|\nabla v\|_p}{\int_\Omega |v|^p \; dx} = p \; \frac{\|\nabla v\|_p}{\|v\|_p} \, . \tag{10}$$

Since $v \in W_0^{1,p}(\Omega)$ is arbitrary, we obtain (6). This proves Theorem 3. □

Remark 4. Cheeger's original result [6] treated also manifolds without boundary, and an extension of this result from $p = 2$ to general p was done in [23], resulting in the same relation (6).

Remark 5. Since Cheeger's constant is known for special domains, Theorem 3 provides concrete numbers. If Ω is a ball B_R of radius R in n-space, then $h(B_R) = \frac{n}{R}$, the Cheeger domain ω of Ω coincides with Ω and

$$\lambda_p(B_R) \geq \left(\frac{n}{Rp} \right)^p . \tag{11}$$

Note that (after taking the p-th root) the right hand side of (11) goes to zero as $p \to \infty$, while the left hand side goes to $1/R$, see [13]. It is the limit $p \to 1$, for which (11) becomes sharp.

If Ω is a stadium, that is the convex hull of two balls of same radius, then $h(\Omega) = Q(\Omega)$ and the Cheeger domain ω coincides again with Ω.

However, if Ω is a plane square $S_a = (-a,a)^2$, then a longer but straightforward calculation gives

$$h(S_a) = \frac{4-\pi}{(4-2\sqrt{\pi})a} \approx \frac{1.886226925}{a}.$$

Note that $|\partial S_1|/|S_1| = 2$ and $|\partial B_1|/|B_1| = 2$ are larger than 1.886226925. The Cheeger domain ω that minimizes the quotient in (5) for S_1 is a square with its corners rounded off by circular arcs of radius $\rho = (4-2\sqrt{\pi})/(4-\pi) \approx 0.5301589043$. Its area $|\omega|$ is $4-(4-2\sqrt{2})^2/(4-\pi) \approx 3.758728766$. This was shown in [15, p. 22], but not stated in context with the name Cheeger. Now Theorem 3 states

$$\lambda_p(S_a) \geq \left(\frac{1.886}{pa}\right)^p. \tag{12}$$

For $p = 2$ and $a = 1$ this is $\lambda_2(S_1) = 2\pi^2 \approx 19.73920881 \geq 0.8894531$, not a very sharp estimate; and for $p \to \infty$ the estimate (12) is trivially $\lambda_\infty(S_1) = 1 \geq 0$. More instructive details on how to obtain such results will be given below after Remark 7.

Corollary 6. *As $p \to 1$, the first eigenvalue $\lambda_p(\Omega)$ of the p-Laplacian converges to Cheeger's constant $h(\Omega)$.*

The lower bound (6) converges to $h(\Omega)$ as $p \to 1$. Therefore it suffices to give an upper bound for $\lambda_p(\Omega)$ with the same limit as $p \to 1$. To this end we choose a smooth subdomain $D_k \subset\subset \Omega$, such that $|\partial D_k|/|D_k| - h(\Omega) \leq 1/k$ and approximate the characteristic function of D_k by a function $v(x)$ with the following properties: $v \equiv 1$ on D_k, $v \equiv 0$ outside an ε-neighborhood of D_k and $|\nabla v| = 1/\varepsilon$ on an ε-layer outside D_k. For sufficiently small ε this function is in $W_0^{1,\infty}(\Omega)$. Plugged into (2) it provides the upper bound

$$\lambda_p(\Omega) \leq \frac{|\partial D_k|}{|D_k|}\varepsilon^{1-p}. \tag{13}$$

Now one sends first $p \to 1$, then $k \to \infty$ to complete the proof of Corollary 6.

Remark 7. If we define $\lambda_1(\Omega) := \lim_{p\to 1+} \lambda_p(\Omega) (= h(\Omega))$ we can ask for the solvability of the formal limit problem

$$\begin{aligned} -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) &= \lambda_1(\Omega) \qquad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{14}$$

Suppose there exists a classical solution u of (14). Then also $f(u)$ solves (14) for any Lipschitz-continuous f. Moreover, if we assume that $|\nabla u|$ is nonzero in the neighborhood of a point $x \in \Omega$, then in intrinsic coordinates the p-Laplace operator can be written as

$$\Delta_p u = (p-1)|\nabla u|^{p-4} < D^2 u \nabla u, \nabla u > -|\nabla u|^{p-2}(n-1)H(x)|\nabla u|.$$

Here $H(x)$ is the mean curvature of the level surface of u in x. As $p \to 1$, the equation from (14) turns into $(n-1)H(x) = h(\Omega)$, that is every level set $\Omega_c := \{x \in \Omega; u(x) > c\}$ has a boundary with the same constant mean curvature $h(\Omega)$ independent of c. Therefore (14) cannot have a classical solutions, because its level sets would be strictly nested in the sense that $\Omega_c \subset\subset \Omega_d$ for $c > d$. This reasoning goes back to [16, p. 355]. Problem (14) was already addressed in [15], where it was shown that one cannot even expect solutions in $BV(\Omega)$ for a constant positive right hand side different from $h(\Omega)$.

A natural problem arises: Does the variational problem (5) have a minimizing domain and what does it look like? An analysis of (7) reveals that (5) is equivalent to solving

$$\min_{v \in BV(\Omega),\ \|v\|_\infty = 1} H(v) := \frac{\int_\Omega |Dv|\,dx + \int_{\partial\Omega} |v|\,ds}{\int_\Omega |v|\,dx} \tag{15}$$

in the following sense: Any solution of (15) has the property that almost all of its level sets are Cheeger sets. Moreover, if ω minimizes h, then χ_ω solves (15). Here $|Dv|$ denotes the distributional derivative of v. For a definition of $BV(\Omega)$ and its norm we refer to [10]. Because of the coarea formula the numerator of (15) can also be written as $\int_0^\infty P(\Omega_t, \mathbb{R}^n)\,dt$, where $\Omega_t := \{x \in \Omega |\ |v(x)| > t\}$ and $P(\Omega_t, \mathbb{R}^n) = \mathcal{H}^{n-1}(\partial\Omega_t)$ denote a level set and its perimeter in $\mathbb{R}^n$.

Solving (15) is straightforward. There exists a sequence of domains $D_k \subset\subset \Omega$ which minimizes h. Let $v_k = \chi_{D_k}$. Then the sequence v_k is bounded in $BV(\Omega)$ and after passing to a subsequence it converges strongly in $L^1(\Omega)$, see [10, p. 17]. This and the weak lower semicontinuity of the BV-norm along this sequence (see [10, p. 7]) show, that it converges to a solution $v_\infty = \chi_\omega$ of (15). Thus we have shown:

Theorem 8. *Problems* (5) *and* (15) *have a solution* ω *and* χ_ω, *but* ω *is not in the admissible class for* (5) *because it touches the boundary* $\partial\Omega$. *Therefore* $\tilde{u}_1(x) := c\chi_\omega$ *with* ω *as a Cheeger domain can be considered to be the first eigenfunction of the operator* $-\Delta_1$ *and the Cheeger constant* $h(\omega)$ *is the associated eigenvalue.*

For the special case that Ω is a ball this was recently published in [8], see also [15, (3.18)]. The existence proof was already given. To see that ω cannot be a compact subset of Ω, one blows it up by a factor larger than one until it touches the boundary. This would decrease h, a contradiction to ω being optimal.

Remark 9. On the regularity of a Cheeger domain
Several qualitative properties can be shown. Once the existence of a Cheeger domain ω is known, we can look for its shape. Given the volume constraint $|D| = |\omega|$, we can look for subdomains of Ω which minimize surface area $|\partial D|$. Variational problems of this type were studied in [11] and [29], where it was shown, that the boundaries of the optimal domain are as smooth as $\partial\Omega$ where they touch it, and analytic except on a set of $(n-8)$ dimensional measure where they do not touch $\partial\Omega$. Moreover, for C^1 domains they are globally C^1. For $n = 2$ this means that the boundaries have only finitely many singular points. Clearly ω solves this variational problem, and $\partial\omega \cap \Omega$ is a surface of constant mean curvature $h(\Omega)$. In two dimensions it must consist of circular arcs. This is how we arrived at (12), by minimizing $Q(D)$ among "squares with rounded corners".

Remark 10. On the convexity of a Cheeger domain for convex Ω
The boundary of the Cheeger set splits into two parts. $\partial\omega \cap \Omega$ has constant mean curvature $h(\Omega)$, and $\partial\omega \cap \partial\Omega$ has the same mean curvature as $\partial\Omega$. From this it is evident, that for convex Ω the Cheeger set ω is convex if $n = 2$ and at least mean-convex, i.e. $\partial\omega$ has nonnegative mean curvature, if $n \geq 3$. But more can be said if Ω is convex. In this case, a result of Sakaguchi [27] states that the positive eigenfunctions u_p which minimize (2) are all logconcave, i.e. $\ln u_p$ is concave for every $p > 1$. The functions u_p are admissible in (15) and uniformly bounded in $BV(\Omega)$. Therefore after passing to a subsequence they converge in $L^1(\Omega)$ to a logconcave limit u_1, and $H(u_1) \leq \liminf_{p\to 1^+} H(u_p)$. Suppose we can show that

$$u_1(x) = \chi_\omega(x) \tag{16}$$

with ω a Cheeger domain. Then ω must be convex because u_1 is logconcave. To prove (16) we observe that u_1 minimizes (15), because $H(u_p) = \lambda_p$ and $\lambda_p \to h(\Omega)$, see Corollary 2. Therefore by the argument following (15) almost every level set of u_1 is a Cheeger domain. Now another argument yields that the level sets Ω_t all coincide for $t \in (0, 1]$. In two dimensions the Cheeger set is unique by Remark 12 and in general dimension n one notes that the level sets of u_1 are all convex and must be nested. But the fact that they are Cheeger domains implies that they cannot be strictly nested, see the proof of Theorem 8. This observation can then be used to reach a contradiction. More details on this and related results will be given in [9].

Remark 11. On monotone dependence between Ω and $h(\Omega)$
The variational characterization (5) of the Cheeger constant implies the monotone dependence $h(\Omega_1) \geq h(\Omega_2)$ if $\Omega_1 \subset \Omega_2$. However, strict inclusion of the domains $\Omega_1 \subset\neq \Omega_2$ does not always imply strict inequality $h(\Omega_1) > h(\Omega_2)$ of the corresponding Cheeger constants. As one example imagine the square S_1 from Remark 5 and modify it near one of the corners. Then both ω and the Cheeger constant are not affected by this modification.

Remark 12. On uniqueness of the Cheeger domain
Suppose that Ω is the union of two disjoint squares of length $2a$, which are connected by a thin pipe that enters each square in a corner. Then (in the notation of Remark 5) $h(\Omega) = h(S_a)$, and now there are at least two disjoint rounded squares ω_1 and ω_2 (and maybe even their union) which qualify for a Cheeger domain, since $Q(\omega_1) = Q(\omega_2) = h(\Omega) = Q(\omega_1 \cup \omega_2)$. However, for convex Ω and $n = 2$ the Cheeger domain is unique, provided it is large enough. This follows from [29, Theorem 3.14]. In fact, let us call H_Ω the union of all largest balls in Ω. Then (at least for convex plane Ω) the Cheeger domain is unique if $|\omega|$ is at least as large as $|H_\Omega|$. But the fact that $|\omega| \geq |H_\Omega|$ follows from Remarks 11 and 13. If $n \geq 3$ and Ω convex, and if Ω satisfies a "great circle condition", then Theorem 3.13 of [29] applies to yield uniqueness of the Cheeger domain. In fact the volume of $|\omega|$ is at least as large as that of a largest ball inside Ω. If $|\omega| < |H_\Omega|$, then $|\omega|$ is the convex hull C of two largest balls, but elementary calculations show that $h(H_\Omega) < h(C)$, so that $|\omega|$ must be at least equal to $|H_\Omega|$.

Remark 13. On guessing the Cheeger domain
Once $h(\Omega)$ is determined, we can take a ball of radius $(n-1)/h$ and sweep Ω with it. Then one may wonder if $D := \bigcup_{x\in\Omega,\, d(x,\partial\Omega)>(n-1)/h} B(x, \frac{n-1}{h(\Omega)})$ is a good candidate for the minimizer of (5). If $n = 2$ this is clearly the case, but for $n \geq 3$ it is not true in general. Consider a large parallelepiped P and sweep it from inside with balls of radius h. The resulting set has mean curvature h near the rounded corners and $h/(n-1)$ near the rounded edges of P.

Remark 14. On parabolic equations
Cheeger domains play an important role in the qualitative study of certain quasilinear parabolic equations, see [17, (1.5) and section 4] and [24]. Suppose that $u(x,t)$ solves the differential equation

$$u_t - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\nabla u|^2}}\right) = 1 \qquad \text{in } \Omega \times (0,\infty)$$

under vanishing initial and boundary data, and suppose that $h(\Omega) < 1$. Then solutions of these equations grow in time with speed proportional to $1 - h(\Omega)$ on the Cheeger domain ω. In fact, since there are no classical stationary solutions, the growth and "detachment" of the solution u from its homogeneous boundary data can only be understood by passing to a viscosity limit in a more regular problem. We refer to [17] for details of the analysis and to [24] for numerical confirmation of the predictions from [17].

Corollary 15. *The following inequalities hold and become equalities if Ω is a ball.*

$$\lambda_1(\Omega) \geq n \left(\frac{\omega_n}{|\Omega|}\right)^{1/n} \tag{17}$$

$$\lim_{p\to\infty} \big(\lambda_p(\Omega)\big)^{1/p} \geq \left(\frac{\omega_n}{|\Omega|}\right)^{1/n} \tag{18}$$

To see (17) one combines Theorem 1, Corollary 6 and Remark 5. For the square S_a from Remark 5 estimate (17) boils down to $1.886226925/a \approx \lambda_1(S_a) \geq \sqrt{\pi}/a \approx 1.772453851/a$, a fairly good estimate. In a similar way the estimate (18) follows from Theorem 1 and the result in [13] that $\lim_{p\to\infty}\big(\lambda_p(\Omega)\big)^{1/p} = \max\{\mathrm{dist}(x,\partial\Omega)\ :\ x\in\Omega\}$. It is remarkable that the left hand sides in (17) and (18) depend only on the geometry of Ω.

Remark 16. On related results
The equation $-\Delta_p u = f(x)$ and the limiting behaviour of solutions as $p \to 1$ were studied in [13] for $f \equiv 1$ and in [7] for more general f. In both cases the solution was shown to converge to zero as $p \to 1$, provided f is sufficiently small or the Cheeger constant is sufficiently large. [28] addresses a multivalued differential inclusion of type $\kappa\Delta_p u + u \in f(x) + \partial I_{[-1,1]}(u)$ with small positive κ in case of one space dimension as $p \to 1$ and under Neumann boundary conditions.

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