On minimal-α-spaces

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Abstract. An α-space is a topological space in which the topology is generated by the family of all α-sets (see [N]). In this paper, minimal-$\alpha\mathcal{P}$-spaces (where $\mathcal{P}$ denotes several separation axioms) are investigated. Some new characterizations of α-spaces are also obtained.

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1. Introduction

The family of all topologies on a set X is a complete atomic lattice. There has been a considerable amount of interest in topologies which are minimal or maximal in this lattice with respect to certain topological properties.

Given a topological property $\mathcal{P}$, we say that a topology on a set X is $\mathcal{P}$-*minimal* if every weaker topology on X does not possess property $\mathcal{P}$.

Throughout this paper, the word "space" will mean topological space, the topology on a space X is denoted by $\tau(X)$, int_τ and cl_τ (or int_X and cl_X when no confusion is possible about the topology on X) will denote respectively the interior and the closure operators with respect to $\tau(X)$ and if σ is a topology on the underlying set of X, then σ is called an *expansion* (respectively a *compression*) of $\tau(X)$ if $\tau(X) \subseteq \sigma$ (resp. $\sigma \subseteq \tau(X)$).

A subset R of a space X is called *regular open* if $\mathrm{int}_\tau(\mathrm{cl}_\tau(R)) = R$. The family of all regular open sets of X is denoted by $RO(X)$ and forms a base for a topology $\tau_s(X)$ on X which is a compression of $\tau(X)$ and it is called the *semiregularization* of X. We say that X is semiregular if $\tau_s(X) = \tau(X)$.

The notion of an α-set was introduced in 1965 by Njåstad [N]. Given a space X, we say that $A \subseteq X$ is an α-*set* if $A \subseteq \mathrm{int}_\tau(\mathrm{cl}_\tau(\mathrm{int}_\tau(A)))$. It is easy to prove that the family $\alpha\tau(X)$ of all α-sets of X is a topology on X which will be called the α-*topology* induced by $\tau(X)$ and it consists of all the subsets A of X such that there exists some open set $U \in \tau(X)$ such that $U \subseteq A \subseteq \mathrm{int}_\tau(\mathrm{cl}_\tau(U))\}$. The members of $\alpha\tau(X)$ will be called the α-*open* sets of X while their complemens

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will be called the *α-closed* sets of X. Evidently, every open set is an α-set and hence the α-topology is an expansion of $\tau(X)$. We say that X is an *α-space* if $\tau(X) = \tau_\alpha(X)$.

Obviously, the partial ordered set of all α-topologies contain both a maximum (the discrete topology) and a minimum (the trivial topology) element. If $\mathcal{P}$ is a topological property, we denote by $\alpha\mathcal{P}$ the class of α-spaces which satisfy property $\mathcal{P}$. In this paper, we investigate the minimal-$\alpha\mathcal{P}$ property with particular regard to the properties $\mathcal{P} = T_0, T_1, T_2$ and $T_{2\frac{1}{2}}$.

2. Basic facts on α-spaces

The α-topology $\alpha\tau(X)$ of a space $(X, \tau(X))$ has some interesting similarities with the notion of *semiregularization* $\tau_s(X)$ (see, for example, [PW]).

Lemma 1 ([N]). *For any α-open set A and any α-closed set C of a space X, we have:*

(1) $\mathrm{cl}_{\alpha\tau}(A) = \mathrm{cl}_\tau(A)$;
(2) $\mathrm{int}_{\alpha\tau}(C) = \mathrm{int}_\tau(C)$;
(3) $\mathrm{int}_{\alpha\tau}(\mathrm{cl}_{\alpha\tau}(A)) = \mathrm{int}_\tau(\mathrm{cl}_\tau(A))$.

Proposition 2 ([N]). *For any space X, we have $\alpha(\alpha\tau(X)) = \alpha\tau(X)$.*

Proof: Evidently $\alpha\tau(X) \subseteq \alpha(\alpha\tau(X))$. Let $B \in \alpha(\alpha\tau(X))$. Then, there is some $A \in \alpha\tau(X)$ such that $A \subseteq B \subseteq \mathrm{int}_{\alpha\tau}(\mathrm{cl}_{\alpha\tau}(A))$. So, there exists some $U \in \tau(X)$ such that $U \subseteq A \subseteq \mathrm{int}_\tau(\mathrm{cl}_\tau(U))$. Hence, $\mathrm{int}_\tau(\mathrm{cl}_\tau(U)) = \mathrm{int}_\tau(\mathrm{cl}_\tau(A))$ and, by Lemma 1(2), we have

$$U \subseteq A \subseteq B \subseteq \mathrm{int}_{\alpha\tau}(\mathrm{cl}_{\alpha\tau}(A)) = \mathrm{int}_\tau(\mathrm{cl}_\tau(A)) = \mathrm{int}_\tau(\mathrm{cl}_\tau(U)).$$

This proves that $B \in \alpha\tau(X)$. $\square$

Definition 1 ([N]). A space X is called an *α-space* if $\tau(X) = \alpha\tau(X)$ or, equivalently, if $\tau(X) = \alpha(\sigma(X))$ for some topology $\sigma(X)$ on X.

Proposition 3. *Let $(X, \tau(X))$ be a space. The following are equivalent:*

(1) *X is an α-space;*
(2) $\tau(X) = \{U \cup \{p\} : U \in \tau(X),\, p \in \mathrm{int}(\mathrm{cl}(U))\}$;
(3) $\{U \cup \{p\} : U \in \tau(X)$ *is dense in X and* $p \in X\} \subseteq \tau(X)$.

Proof: (1)$\Rightarrow$(2) Evidently, $\tau(X) \subseteq \{U \cup \{p\} : U \in \tau(X), p \in \mathrm{int}_\tau(\mathrm{cl}_\tau(U))\}$ (it suffices to take $p \in U$). Conversely, let $U \in \tau(X)$ and $p \in \mathrm{int}_\tau(\mathrm{cl}_\tau(U))$. Then $U \subseteq U \cup \{p\} \subseteq U \cup \mathrm{int}_\tau(\mathrm{cl}_\tau(U)) = \mathrm{int}_\tau(\mathrm{cl}_\tau(U))$. Since X is an α-space, $U \cup \{p\} \in \tau(X)$.

(2)$\Rightarrow$(3) Obvious because, when $\mathrm{cl}_\tau(U) = X$, every $p \in \mathrm{int}_\tau(\mathrm{cl}_\tau(U)) = X$.

(3) $\Rightarrow$(1) Suppose that $\{U \cup \{p\} : U \in \tau(X)$ is dense in X and $p \in X\} \subseteq \tau(X)$ and let $A \in \alpha\tau(X)$, i.e. $A \subseteq \mathrm{int}_\tau(\mathrm{cl}_\tau(\mathrm{int}_\tau(A)))$. The set $D = \mathrm{int}_\tau(A) \cup \mathrm{int}_\tau(X \setminus \mathrm{int}_\tau(A))$ is open and dense because

$$\begin{aligned}
\mathrm{cl}(D) &= \mathrm{cl}_\tau\left(\mathrm{int}_\tau(A)\right) \cup \mathrm{cl}_\tau\left(\mathrm{int}_\tau(X \setminus \mathrm{int}_\tau(A))\right) \\
&= \mathrm{cl}_\tau\Big(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))\Big) \cup \mathrm{cl}_\tau\Big(X \setminus \mathrm{cl}_\tau(\mathrm{int}_\tau(A))\Big) \\
&= \mathrm{cl}_\tau\Big(\mathrm{cl}_\tau(\mathrm{int}_\tau(A)) \cup \big(X \setminus \mathrm{cl}_\tau(\mathrm{int}_\tau(A))\big)\Big) \\
&= \mathrm{cl}_\tau(X) \\
&= X.
\end{aligned}$$

So, for any $p \in A$, by hypothesis we have that $D \cup \{p\} \in \tau(X)$.

Now, we consider the open set:

$$\begin{aligned}
W_p &= \big(D \cup \{p\}\big) \cap \mathrm{int}_\tau(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))) \\
&= \Big(\mathrm{int}_\tau(A) \cup \mathrm{int}_\tau(X \setminus \mathrm{int}_\tau(A)) \cup \{p\}\Big) \cap \mathrm{int}_\tau(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))) \\
&= \Big(\mathrm{int}_\tau(A) \cap \mathrm{int}_\tau\left(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))\right)\Big) \\
&\quad \cup \Big(\mathrm{int}_\tau\left(X \setminus \mathrm{int}_\tau(A)\right) \cap \mathrm{int}_\tau\left(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))\right)\Big) \\
&\quad \cup \Big(\{p\} \cap \mathrm{int}_\tau\left(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))\right)\Big) \\
&= \mathrm{int}_\tau(A) \cup \{p\}
\end{aligned}$$

as

$$\begin{aligned}
&\left(\mathrm{int}_\tau(X \setminus \mathrm{int}_\tau(A))\right) \cap \mathrm{int}_\tau\left(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))\right) \\
&\quad = \Big(X \setminus \mathrm{cl}_\tau\left(\mathrm{int}_\tau(A)\right)\Big) \cap \mathrm{int}_\tau\left(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))\right) \\
&\quad \subseteq \Big(X \setminus \mathrm{int}_\tau\left(\mathrm{cl}_\tau\left(\mathrm{int}_\tau(A)\right)\right)\Big) \cap \mathrm{int}_\tau\left(\mathrm{cl}_\tau(\mathrm{int}_\tau(A))\right) = \emptyset.
\end{aligned}$$

Thus, for every $p \in A$, $W_p = \mathrm{int}_\tau(A) \cup \{p\} \in \tau(X)$ and so, also the union $\bigcup_{p\in A} W_p = \mathrm{int}_\tau(A) \cup A = A$ is an open set of X. This proves that $\alpha\tau(X) \subseteq \tau(X)$ and hence that X is an α-space. $\square$

In [N] it is proved the following:

Proposition 4. *A space X is an α-space if and only if all the nowhere dense sets are closed sets.*

The following proposition improves the previous one.

Proposition 5. *A space (X, τ) is an α-space if and only if every nowhere dense set is discrete.*

Proof: Suppose that X is an α-space and that N is a nowhere dense subset of X. By Proposition 4, N is a closed set and so $X \setminus N$ is an open dense set. For any $p \in N$, by Proposition 3(3), the set $U = (X \setminus N) \cup \{p\}$ is open in X. Since $U \cap N = \{p\}$, it follows that N is discrete.

Conversely, let U be an open dense set of X and $p \in X$. Then $X \setminus U$ is a nowhere dense set and by hypothesis, it is discrete. So, if $p \in X \setminus U$, there exists some $V \in \tau$ such that $V \cap (X \setminus U) = \{p\}$. Hence $U \cup \{p\} = U \cup V \in \tau(X)$. Since the case when $p \in U$ is trivial, by Proposition 3(3), it is proved that X is an α-space. □

It is shown in [Lo] that the operator α is not monotonic, i.e. that, in general, for two topologies $\tau(X)$ and $\sigma(X)$ on a set X, $\tau(X) \subseteq \sigma(X)$ does not imply $\alpha\tau(X) \subseteq \alpha\sigma(X)$. However, we have the following:

Lemma 6. *Let $\tau(X)$ and $\sigma(X)$ be topologies on a set X such that $\tau(X) \subseteq \sigma(X)$ and $\tau_s(X) = \sigma_s(X)$. Then $\alpha\tau(X) \subseteq \alpha\sigma(X)$.*

Proof: Let $A \in \alpha\tau(X)$. Then there exists some $U \in \tau(X)$ such that $U \subseteq A \subseteq \mathrm{int}_\tau(\mathrm{cl}_\tau(U))$. So, being

$$\mathrm{int}_\sigma(\mathrm{cl}_\sigma(U)) = \mathrm{int}_{\sigma_s}(\mathrm{cl}_{\sigma_s}(U)) = \mathrm{int}_{\tau_s}(\mathrm{cl}_{\tau_s}(U)) = \mathrm{int}_\tau(\mathrm{cl}_\tau(U)),$$

we have $U \subseteq A \subseteq \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U))$ with $U \in \tau \subseteq \sigma$, that is $A \in \alpha\sigma(X)$. □

Let us recall that a space $(X, \tau(X))$ is called:

- *T_2-closed* (resp. $T_{2\frac{1}{2}}$-closed) if it is closed in every Hausdorff (resp. $T_{2\frac{1}{2}}$) space containing X as a subspace,
- *minimal-T_2* (resp. minimal-$T_{2\frac{1}{2}}$) if it is a T_2- (resp. $T_{2\frac{1}{2}}$-) space and there is no strictly coarser T_2 (resp. $T_{2\frac{1}{2}}$) topology on the same set X.

The following properties are well-known and will be used later.

Proposition 7. *A Hausdorff space is T_2-closed if and only if every open ultrafilter on X is fixed.*

Proposition 8. *A space $(X, \tau(X))$ is Hausdorff if and only if its semiregularization $(X, \tau_s(X))$ is Hausdorff.*

Corollary 9. *A space $(X, \tau(X))$ is T_2-closed if and only if its semiregularization $(X, \tau_s(X))$ is minimal-T_2.*

Proposition 10. *Every regular closed subspace of a T_2-closed space is T_2-closed.*

Proposition 11. *A Hausdorff space X is minimal-T_2 if and only if it is semiregular and T_2-closed.*

Proposition 12. *If $\sigma(X)$ and $\tau(X)$ are two topologies on a set X such that $\tau_s(X) \subseteq \sigma(X) \subseteq \tau(X)$ then $\tau_s(X) = \sigma_s(X)$.*

Proof: Let R be a regular open set of $(X, \tau(X))$. Then there exists some $U \in \tau(X)$ such that $R = \mathrm{int}_\tau\big(\mathrm{cl}_\tau(U)\big)$. So, $R \in \tau_s(X) \subseteq \sigma(X)$. Since $\tau_s(X) \subseteq \sigma(X)$ and using a well-known property of the closure of the semiregularization (see [PW]), we have that

$$\mathrm{cl}_\sigma(R) \subseteq \mathrm{cl}_{\tau_s}(R) = \mathrm{cl}_\tau(R)$$

while, being $\sigma(X) \subseteq \tau(X)$, it follows

$$\mathrm{int}_\sigma\big(\mathrm{cl}_\sigma(R)\big) \subseteq \mathrm{int}_\tau\big(\mathrm{cl}_\tau(R)\big) = R.$$

Obviously, being $R \in \sigma(X)$, we also have that $R \subseteq \mathrm{int}_\sigma\big(\mathrm{cl}_\sigma(R)\big)$ and hence that $R = \mathrm{int}_\sigma\big(\mathrm{cl}_\sigma(R)\big)$. Since $R \in \sigma(X)$, this means that R is a regular open set of $(X, \sigma(X))$. On the other hand, let S be a regular open set of $(X, \sigma(X))$. Then there exists some $V \in \sigma(X)$ such that $S = \mathrm{int}_\sigma\big(\mathrm{cl}_\sigma(V)\big)$. Since $S \in \sigma(X) \subseteq \tau(X)$, we have that

$$\mathrm{cl}_\tau(S) \subseteq \mathrm{cl}_\sigma(S)$$

Hence, being $\tau_s(X) \subseteq \sigma(X)$ and by some well-known properties of the interior of the semiregularization it follows that

$$\mathrm{int}_\tau\big(\mathrm{cl}_\tau(S)\big) = \mathrm{int}_{\tau_s}\big(\mathrm{cl}_\tau(S)\big) \subseteq \mathrm{int}_\sigma\big(\mathrm{cl}_\sigma(S)\big) = S.$$

Obviously, being $S \in \tau(X)$, we also have that $S \subseteq \mathrm{int}_\tau\big(\mathrm{cl}_\tau(S)\big)$ and hence that $S = \mathrm{int}_\tau\big(\mathrm{cl}_\tau(S)\big)$, that is S is a regular open set of $(X, \tau(X))$. Thus, the topologies generated by these families of regular open sets, i.e. the semiregularization of $(X, \tau(X))$ and $(X, \sigma(X))$ coincide and we can conclude that $\tau_s(X) = \sigma_s(X)$. $\square$

Proposition 13. *Let $\mathcal{U}$ be a free open ultrafilter on a Hausdorff space X and p be a fixed point in X. Then, there exists a Hausdorff topology $\tau_{\mathcal{U}}$ on X such that $\alpha_{\mathcal{U}}(X) \subsetneq \alpha\tau(X)$.*

Proof: Let us consider the family $\tau_{\mathcal{U}}(X) = \{U \in \tau(X) : p \in U \Rightarrow U \in \mathcal{U}\}$. It is a simple routine to verify that $\tau_{\mathcal{U}}(X)$ forms a topology on X such that $\tau_{\mathcal{U}}(X) \subseteq \tau(X)$.

The space $(X, \tau_{\mathcal{U}}(X))$ is T_2. In fact, for every $x \neq y$ in X, since $(X, \tau(X))$ is Hausdorff, there are $U, V \in \tau(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. If $p \notin U \cup V$, $U, V \in \tau_{\mathcal{U}}(X)$ and we are done. Otherwise, if, for example, $p \in U$, we have $p \notin V$ and $V \in \tau(X)$. Furthermore, since $\mathcal{U}$ is free with respect to $(X, \tau(X))$, there exist some $N \in \tau(X)$ and some $W \in \mathcal{U}$ such that $y \in N$ and $W \cap N = \emptyset$.

Thus $W \subseteq U \cup W$ implies $U \cup W \in \mathcal{U}$ and hence $U \cup W \in \tau_{\mathcal{U}}(X)$, while being $p \notin V \cap N \in \tau(X)$ it follows that $V \cap N \in \tau_{\mathcal{U}}(X)$. So, $U \cup W$ and $V \cap N$ are two open neighborhoods of x and y respectively in $\tau_{\mathcal{U}}(X)$ such that

$$\begin{aligned}\big(U \cup W\big) \cap \big(V \cap N\big) &= \big(U \cap (V \cap N)\big) \cup \big(W \cap (V \cap N)\big)\\ &\subseteq (U \cap V) \cup (W \cap N)\\ &= \emptyset \cup \emptyset = \emptyset\end{aligned}$$

and this proves that the space $(X, \tau_{\mathcal{U}}(X))$ is Hausdorff.

Since it is immediate to see that every neighborhood of p in $\tau_{\mathcal{U}}(X)$ belongs to $\mathcal{U}$ and $(X, \tau_{\mathcal{U}}(X))$ is T_2, it follows that p is the unique convergence point of $\mathcal{U}$ with respect to $\tau_{\mathcal{U}}(X)$.

In order to show that $\alpha\tau_{\mathcal{U}}(X) \subseteq \alpha\tau(X)$, we observe first that, for every $U \in \tau_{\mathcal{U}}(X)$, it results:

$$\text{(1)} \qquad \mathrm{cl}_{\tau_{\mathcal{U}}}(U) = \begin{cases} \mathrm{cl}_{\tau}(U) & \text{if } U \notin \mathcal{U},\\ \mathrm{cl}_{\tau}(U) \cup \{p\} & \text{if } U \in \mathcal{U}.\end{cases}$$

In fact, $\tau(X) \subseteq \tau_{\mathcal{U}}(X)$ implies, in any case, $\mathrm{cl}_{\tau_{\mathcal{U}}}(U) \subseteq \mathrm{cl}_{\tau}(U)$.

Now, consider the case $U \notin \mathcal{U}$ and suppose, by contradiction that there is some $x \in \mathrm{cl}_{\tau_{\mathcal{U}}}(U)$ such that $x \notin \mathrm{cl}_{\tau}(U)$. So, there exists some neighborhood N of x in $\tau(X)$ such that $N \cap U = \emptyset$.

Since $\mathcal{U}$ is an open ultrafilter on $(X, \tau(X))$, $U \notin \mathcal{U}$ implies that $X \setminus \mathrm{cl}_{\tau}(U) \in \mathcal{U}$. Now, $W = \big(X \setminus \mathrm{cl}_{\tau}(U)\big) \cup N$ is an open neighborhood of x with respect to $\tau_{\mathcal{U}}(X)$ (because $W \in \tau(X)$, $x \in N \subseteq W$, and $X \setminus \mathrm{cl}_{\tau}(U) \subseteq W$ implies $W \in \mathcal{U}$) but it results

$$W \cap U = \Big(\big(X \setminus \mathrm{cl}_{\tau}(U)\big) \cup N\Big) \cap U = \Big(\big(X \setminus \mathrm{cl}_{\tau}(U)\big) \cap U\Big) \cup \big(N \cap U\big) = \emptyset \cup \emptyset = \emptyset$$

which is a contradiction to $x \in \mathrm{cl}_{\tau_{\mathcal{U}}}(U)$.

Let us consider the case $U \in \mathcal{U}$. Evidently $p \in \mathrm{cl}_{\tau_{\mathcal{U}}}(U)$ as for every open neighborhood N of p in $\tau_{\mathcal{U}}(X)$, it follows that $N \in \mathcal{U}$ and hence $N \cap U \in \mathcal{U}$ implies $N \cap V \neq \emptyset$. Conversely, suppose, by contradiction, that there is some $x \in \mathrm{cl}_{\tau_{\mathcal{U}}}(U)$ such that $x \neq p$ and $x \notin \mathrm{cl}_{\tau}(U)$. Then, there exists some open neighborhood N of p in $\tau(X)$ such that $N \cap U = \emptyset$ and, it must be $N \notin \tau_{\mathcal{U}}(X)$, that is $p \in N \notin \mathcal{U}$. Since $(X, \tau(X))$ is T_2 and $x \neq p$, there is some open neighborhood G of $x \in \tau(X)$ such that $p \notin G$. Thus, $p \notin N \cap G \in \tau(X)$ implies that $N \cap G \in \tau_{\mathcal{U}}(X)$. So, $N \cap G$ is an open neighborhood of x in $\tau_{\mathcal{U}}(X)$ such that $\big(N \cap G\big) \cap U \subseteq N \cap U = \emptyset$. A contradiction to $x \in \mathrm{cl}_{\tau_{\mathcal{U}}}(U)$. Applying the usual duality rules to formula (1), we also obtain that, for every $U \in \tau_{\mathcal{U}}(X)$, it results:

$$\text{(2)} \qquad \mathrm{int}_{\tau_{\mathcal{U}}}\big(\mathrm{cl}_{\tau_c U}(U)\big) = \begin{cases} \mathrm{int}_{\tau}\big(\mathrm{cl}_{\tau}(U)\big) \setminus \{p\} & \text{if } U \notin \mathcal{U},\\ \mathrm{int}_{\tau}\big(\mathrm{cl}_{\tau}(U)\big) & \text{if } U \in \mathcal{U}.\end{cases}$$

In fact, if $U \notin \mathcal{U}$, as $\mathcal{U}$ is an open ultrafilter on $(X, \tau(X))$, $X \setminus \mathrm{cl}_\tau(U) \in \mathcal{U}$ and so we have that:

$$\begin{aligned}
\mathrm{int}_{\tau_\mathcal{U}}\left(\mathrm{cl}_{\tau_c U}(U)\right) &= \mathrm{int}_{\tau_\mathcal{U}}\left(\mathrm{cl}_\tau(U)\right) \\
&= X \setminus \mathrm{cl}_{\tau_\mathcal{U}}\Big(X \setminus \left(\mathrm{cl}_\tau(U)\right)\Big) \\
&= X \setminus \mathrm{cl}_\tau\Big(X \setminus \left(\mathrm{cl}_\tau(U) \cup \{p\}\right)\Big) \\
&= \Big(X \setminus \mathrm{cl}_\tau\left(X \setminus \mathrm{cl}_\tau(U)\right)\Big) \cap \left(X \setminus \{p\}\right) \\
&= \mathrm{int}_\tau\left(\mathrm{cl}_\tau(U)\right) \setminus \{p\}.
\end{aligned}$$

If $U \in \mathcal{U}$, since $\mathcal{U}$ is an open ultrafilter on $(X, \tau(X))$, it necessarily results $X \setminus \mathrm{cl}_\tau(U) \notin \mathcal{U}$ and we have:

$$\begin{aligned}
\mathrm{int}_{\tau_\mathcal{U}}\left(\mathrm{cl}_{\tau_\mathcal{U}}(U)\right) &= X \setminus \mathrm{cl}_{\tau_\mathcal{U}}\left(X \setminus \mathrm{cl}_{\tau_\mathcal{U}}(U)\right) \\
&= X \setminus \mathrm{cl}_\tau\left(X \setminus \mathrm{cl}_{\tau_\mathcal{U}}(U)\right) \\
&= \mathrm{int}_\tau\left(\mathrm{cl}_{\tau_\mathcal{U}}(U)\right) \\
&= \mathrm{int}_\tau\left(\mathrm{cl}_\tau(U) \cup \{p\}\right) \\
&= \mathrm{int}_\tau\left(\mathrm{cl}_\tau(U)\right),
\end{aligned}$$

where the last equality is due to the fact that $p \in \mathrm{cl}_\tau(U)$ (because for every neighborhood N of p in $\tau(X)$, we have $N \in \mathcal{U}$ and so, being $U \in \mathcal{U}$, it follows that $N \cap U \in \mathcal{U}$ and $N \cap U \neq \emptyset$).

Now, for every $A \in \alpha\tau_\mathcal{U}(X)$, there exists some $U \in \tau_\mathcal{U}(X)$ such that $U \subseteq A \subseteq \mathrm{int}_{\tau_\mathcal{U}}\left(\mathrm{cl}_{\tau_\mathcal{U}}(U)\right)$ and by formula (2), it immediately follows, in both cases, that $U \subseteq A \subseteq \mathrm{int}_\tau\left(\mathrm{cl}_\tau(U)\right)$ with $U \in \alpha\tau(X) \subseteq \tau(X)$ and so that $A \in \alpha\tau(X)$. Thus $\alpha\tau_\mathcal{U}(X) \subseteq \alpha\tau(X)$. To finish the proof, we will show that $\alpha\tau_\mathcal{U}(X) \neq \alpha\tau(X)$. Since $\mathcal{U}$ is free, p is not an adherent point for $\mathcal{U}$ and so there exist some neighborhood V of p in $\tau(X)$ and some $U \in \mathcal{U}$ such that $U \cap V = \emptyset$. Thus $V \notin \mathcal{U}$ and, being $p \in V$, by definition of $\tau_\mathcal{U}(X)$, it follows that $V \notin \tau_\mathcal{U}(X)$. Evidently $V \in \alpha\tau(X)$ (as $V \in \tau(X)$) but $V \notin \alpha\tau_\mathcal{U}(X)$. Suppose, by contradiction, that there exists some $W \in \tau_\mathcal{U}(X)$ such that $W \subseteq V \subseteq \mathrm{int}_{\tau_\mathcal{U}}\left(\mathrm{cl}_{\tau_\mathcal{U}}(W)\right)$. Since $V \notin \mathcal{U}$, it follows a fortiori that $W \notin \mathcal{U}$ and so, by formula (2), we have that $W \subseteq V \subseteq \mathrm{int}_\tau\left(\mathrm{cl}_\tau(W)\right) \setminus \{p\}$ and thus that $p \notin V$. A contradiction. This proves that $\alpha\tau_\mathcal{U}(X) \subsetneqq \alpha\tau(X)$ and concludes our proof. □

3. Minimal-αT_0-spaces

Definition 2. Let $\mathcal{P}$ be a topological property. A space X is said to be an *$\alpha\mathcal{P}$-space* if it is an α-space and property $\mathcal{P}$ holds.

Definition 3. Let $\mathcal{P}$ be a topological property. A space X is called *minimal-$\alpha\mathcal{P}$* if it is an $\alpha\mathcal{P}$-space and there is no strictly coarser $\alpha\mathcal{P}$-topology on the same set.

Since the unique T_1-minimal-space is the cofinite topology which is an α-space (by Proposition 3), it is evident that the class of minimal-αT_1-spaces coincides with the well-known class of minimal-T_1.
One would suspect that if X is minimal-T_0, then $(X, \alpha\tau(X))$ is minimal-αT_0. Here is a counterexample.

Example 14. Consider $\mathbb{R}$ with $\tau(\mathbb{R}) = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Then $\mathbb{R}$ is minimal-T_0. If $\emptyset \neq U \in \tau(\mathbb{R})$, $\operatorname{cl}_{\mathbb{R}}(U) = \mathbb{R}$, then $\operatorname{int}_{\mathbb{R}}\left(\operatorname{cl}_{\mathbb{R}}(U)\right) = \mathbb{R}$. So, if $a < b$ and $c = a-1$, then $(-\infty, c) \cup \{a\}$, $(-\infty, c) \cup \{b\} \in \alpha\tau(\mathbb{R})$. In particular, it follows that $(\mathbb{R}, \alpha\tau(\mathbb{R}))$ is T_1. If σ is the cofinite topology on $\mathbb{R}$, then $\sigma \subset \tau(\mathbb{R})$.

Lemma 15. *Let X be a set, $p \in X$, and $\tau(X) = \{U : p \in U$ and $X \setminus U$ is finite$\} \cup \{\emptyset\}$. Then X is minimal-αT_0.*

Proof: Clearly, X is T_0. If $\emptyset \neq U \in \tau(X)$, then $p \in U$, $\operatorname{cl}_X U = X$, and $\operatorname{int}_X \operatorname{cl}_X U = X$. Now $U \cup \{q\}$ is open for all $q \in X$, by Proposition 3, since X is αT_0. Let $\sigma \subseteq \tau(X)$ and (X, σ) be αT_0. If $\emptyset \neq U \in \tau(X)$, then $p \in U$, and $X \setminus U = \{q_1, \cdots, q_n\}$ is finite. Since $\emptyset \neq V \in \sigma \subseteq \tau(X)$ implies $p \in V$, then for each $1 \leq i \leq n$, there is a $V_i \in \sigma$ such that $p \in V_i \subseteq X \setminus \{q_i\}$. For $T = \bigcap\{V_i : 1 \leq i \leq n\}$, $p \in T \in \sigma$ and $T \subseteq U$. As (X, σ) is αT_0 and $\operatorname{cl}_\sigma T = X$, then $U = \bigcup\{T \cup \{q\} : q \in U \setminus T\} \in \sigma$. □

In the topology just considered, every point different from p is closed. Now, let us consider the case in which some point of X is not closed.

Lemma 16. *Let X be an αT_0-space, and $p \in X$ such that $\operatorname{cl}_X\{p\} \neq \{p\}$. Then $\operatorname{cl}_X(\{p\})$ is a regular-closed set and $p \in \operatorname{int}_X\left(\operatorname{cl}_X(\{p\})\right)$.*

Proof: Let $U = X \setminus \operatorname{cl}_X(\{p\})$. It suffices to show that U is regular-open. If $p \in \operatorname{cl}_X(U)$, then $\operatorname{cl}_X(U) = X$ and $\operatorname{int}_X\left(\operatorname{cl}_X(U)\right) = X$. So, for $q \in \operatorname{cl}_X(\{p\}) \setminus \{p\}$, $U \cup \{q\}$ is open and $(U \cup \{q\}) \cap \{p\} = \emptyset$. That is, $q \notin \operatorname{cl}_X(\{p\})$, a contradiction. So, $p \notin \operatorname{cl}_X(U)$. For $V = X \setminus \operatorname{cl}_X(U)$, we have that $p \in V \subseteq \operatorname{cl}_X(\{p\})$. Thus, $\operatorname{cl}_X(V) = \operatorname{cl}_X(\{p\})$. □

In order to obtain a characterization of minimal-T_0-spaces, we need some other lemmas.

Lemma 17. *Let X be an αT_0-space, and $p, q \in X$ such that $\operatorname{cl}_X(\{p\}) \neq \{p\}$, $\operatorname{cl}_X(\{q\}) \neq \{q\}$, and $p \neq q$. Then $p \notin \operatorname{cl}_X(\{q\})$ and $q \notin \operatorname{cl}_X(\{p\})$.*

Proof: Since X is T_0, then $\operatorname{cl}_X(\{p\}) \neq \operatorname{cl}_X(\{q\})$. Assume that $q \in \operatorname{cl}_X(\{p\})$. Then $p \notin \operatorname{cl}_X(\{q\})$ as $\operatorname{cl}_X(\{p\}) \neq \operatorname{cl}_X(\{q\})$. So, $\operatorname{cl}_X(\{q\}) \subseteq \operatorname{cl}_X(\{p\})$. By Lemma 16, $q \in \operatorname{int}_X\left(\operatorname{cl}_X(\{q\})\right) \subseteq \operatorname{int}_X\left(\operatorname{cl}_X(\{p\})\right) \subseteq \operatorname{cl}_X(\{p\})$. As $p \notin \operatorname{cl}_X(\{q\})$, it follows that $p \notin \operatorname{int}_X\left(\operatorname{cl}_X(\{q\})\right)$. Therefore, $q \notin \operatorname{cl}_X(\{p\})$, a contradiction.

□

Lemma 18. *Let X be a minimal-αT_0-space, and $p, q \in X$ such that $\mathrm{cl}_X(\{p\}) \neq \{p\}$, $\mathrm{cl}_X(\{q\}) \neq \{q\}$, and $p \neq q$. Then $\mathrm{int}_X\big(\mathrm{cl}_X(\{p\})\big) = \{p\}$ and $\mathrm{int}_X\big(\mathrm{cl}_X(\{q\})\big) = \{q\}$.*

PROOF: Let $\tau = \tau(X)$ be the topology on X, $r \in \mathrm{int}_X\big(\mathrm{cl}_X(\{p\})\big) \setminus \{p\}$ and consider the topology $\sigma = \{U \in \tau : r \in U \text{ implies } q \in U\}$ on X. Clearly, $\sigma \subseteq \tau$. There are a number of cases to verify that (X, σ) is T_0 but each case is straightforward. Next we show that (X, σ) is an α-space. Let U be open and dense in (X, σ) and $t \in X \setminus U$. We want to show that $U \cup \{t\} \in \sigma$. If $r \in U$, then U is dense and open in (X, τ) and hence, $U \cup \{t\} \in \sigma$. Suppose that $r \notin U$. Our first goal is to show that U is dense in τ. If $\mathrm{cl}_\tau(U) \neq \mathrm{cl}_\sigma(U)$, then $r \in \mathrm{cl}_\sigma(U) \setminus \mathrm{cl}_\tau(U)$. Then $p \notin U$ and $\mathrm{cl}_\tau(\{p\}) \cap U = \emptyset$. In particular, $\mathrm{int}_\tau\big(\mathrm{cl}_\tau(\{p\})\big) \cap U = \emptyset$. There is an $V \in \tau$ such that $p \in V$ and $r \notin V$. Now $p \in \mathrm{int}_\tau\big(\mathrm{cl}_\tau(\{p\})\big) \cap V \in \sigma$ and $\mathrm{int}_\tau\big(\mathrm{cl}_\tau(\{p\})\big) \cap V \cap U = \emptyset$. So, U is not dense in (X, σ), a contradiction. Thus, U is dense in τ. Also the above proof shows that $p \in U$. If $q \notin U$, then $\mathrm{cl}_\tau(\{q\}) \cap U = \emptyset$. In particular, $\mathrm{int}_\tau\big(\mathrm{cl}_\tau(\{q\})\big) \cap U = \emptyset$. But $q \in \mathrm{int}_\tau\big(\mathrm{cl}_\tau(\{q\})\big) \in \sigma$, a contradiction. Thus, $q \in U$. With both $p, q \in U$, we have that $U \cup \{r\} \in \sigma$. For $t \neq r$, then $U \cup \{t\} \in \sigma$ as $U \cup \{t\} \in \tau$ and $r \notin U \cup \{t\}$. □

Proposition 19. *Let X be a minimal-αT_0-space, and $P = \big\{p \in X : \mathrm{cl}_X(\{p\}) \neq \{p\}\big\}$ such that $|P| \geq 2$. Then P is dense in X and if $V \in \tau(X)$ and $V \setminus P \neq \emptyset$, then $P \subseteq V$.*

PROOF: Let $\tau = \tau(X)$ be the topology of the space X. First we show that $Q = \{q \in X : \{q\} \in \tau\}$ is dense. Clearly, $Q \supseteq P$. Fix $r \in X \setminus \mathrm{cl}_X(Q)$. Note that the topology $\sigma = \{U \in \tau : r \in U \text{ implies } Q \subseteq U\} \subseteq \tau$. If we show that (X, σ) is αT_0, it will follow that Q is dense. Since $\{q\} \in \sigma$ for all $q \in Q$, q can be T_0-separated from all $p \in X \setminus \{q\}$ in σ. As $(X \setminus \mathrm{cl}_X(Q)) \cup Q \in \sigma$ and $(X \setminus \mathrm{cl}_X(Q)) \cup Q \cup \{t\} \in \sigma$ for $t \in \mathrm{cl}_X(Q) \setminus Q$, a point $t \in \mathrm{cl}_X(Q) \setminus Q$ can be T_0-separated from all $p \in X \setminus \{t\}$ in σ. Let $s, t \in X \setminus \mathrm{cl}_X(Q)$. As τ is T_0, there is some $V \in \tau$ such that $s \in V$ and $t \notin V$ or vice versa. Now, $s \in V \cup Q \in \sigma$ and $t \notin V \cup Q$. This completes the proof that (X, σ) is T_0. Next we show that (X, σ) is an α-space. Let U be an open and dense subset of (X, σ). Note that $Q \subseteq U$. If $\mathrm{cl}_\tau(U) \neq \mathrm{cl}_\sigma(U)$, then $r \in \mathrm{cl}_\sigma(U) \setminus \mathrm{cl}_\tau(U)$. There is an $V \in \tau$ such that $r \in V$ and $V \cap Q = \emptyset$. As $r \notin P$, $\mathrm{cl}_\tau(\{r\}) = \{r\}$, $\emptyset \neq V \setminus \{r\} \in \sigma$ and $V \setminus \{r\} \cap U = \emptyset$, a contradiction as U is dense in (X, σ). This shows that $\mathrm{cl}_\tau(U) = X$. As τ is an α topology, for $t \in X, U \cup \{t\} \in \tau$. As $Q \subseteq U$, $U \cup \{t\} \in \sigma$. This completes the proof that (X, σ) is an αT_0-space. Finally, we show that $Q = P$. Assume that $q \in Q \setminus P$. The topology $\sigma = \{U \in \tau : q \in U \text{ implies } P \subseteq U\} \subseteq \tau$. Similar to the above, it is straightforward to show that (X, σ) is T_0 and an α-space, a contradiction as τ is minimal-αT_0. Thus, $P = Q$. □

Proposition 20. *Let X be a minimal-αT_0-space, and $P = \{p \in X : \mathrm{cl}_X(\{p\}) \neq \{p\}\}$ such that $|P| \geq 2$. Then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$.*

Proof: This is an obvious consequence of Proposition 19. □

Proposition 21. *Let X be a set and $P \subsetneq X$ such that $|P| \geq 2$. Let $\tau(X)$ be generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$. Then X is a minimal-αT_0-space.*

Proof: Let $\tau = \tau(X)$ be the topology on X and $\sigma \subseteq \tau$ be an αT_0 topology. For $p \in P, (X \setminus P) \cup \{p\} = \mathrm{cl}_\tau(\{p\}) \subseteq \mathrm{cl}_\sigma(\{p\})$. Thus, for all $p \in P$, $\mathrm{cl}_\sigma(\{p\}) \neq \{p\}$. By Lemma 17, for $q \in P \setminus \{p\}$, $q \notin \mathrm{cl}_\sigma(\{p\})$. Hence $\mathrm{cl}_\sigma(\{p\}) = (X \setminus P) \cup \{p\} = \mathrm{cl}_\tau(\{p\})$. Thus, $P \setminus \{p\} \in \sigma$ for all $p \in P$. By Lemma 16, $p \in \mathrm{int}_\sigma\left(\mathrm{cl}_\sigma(\{p\})\right)$. As $|P| \geq 2$, let $q \in P \setminus \{p\}$. Then $\mathrm{int}_\sigma\left(\mathrm{cl}_\sigma\{q\}\right) \cap P \setminus \{p\} = \{q\} \in \sigma$ and $P \in \sigma$. As $X = \mathrm{cl}_\tau(P) \subseteq \mathrm{cl}_\sigma(P)$, $X = \mathrm{int}_\sigma\left(\mathrm{cl}_\sigma(P)\right)$. Since (X, σ) is an αT_0-space, it follows that for $q \in X \setminus P$, $P \cup \{q\} \in \sigma$. This shows that $\sigma = \tau$. □

Finally, we obtain the characterization of αT_0-spaces.

Theorem 22. *Let X be an αT_0-space and $P = \{p \in X : \mathrm{cl}_X(\{p\}) \neq \{p\}\}$. Then X is minimal-αT_0 iff $P \neq \emptyset$ and*

(i) *if $P = \{p\}$, then $\tau(X) = \{U : p \in U$ and $X \setminus U$ is finite$\} \cup \{\emptyset\}$, or*
(ii) *if $|P| \geq 2$, then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$.*

Proof: $(\Longleftarrow)$ It follows from Lemma 18 and Proposition 20.
$(\Longrightarrow)$ It follows from Lemma 15 and Proposition 21. □

4. Minimal-αT_2-spaces

Proposition 23. *Let X be an αT_2-space. Then X is a minimal-αT_2-space if and only if $(X, \tau_s(X))$ is minimal-T_2 and $\tau(X) = \alpha\tau_s(X)$.*

Proof: $(\Longrightarrow)$ Suppose that X is a minimal-αT_2-space. Thus $\alpha\tau(X) = \tau(X)$. Then the space $(X, \tau(X))$ is T_2-closed. In fact, if, by contradiction, it is not, by Proposition 7, there exists some free open ultrafilter $\mathcal{U}$ on $(X, \tau(X))$ and by Proposition 13, there exists a strictly coarser αT_2 topology $\tau_{\mathcal{U}}$. A contradiction to the α-minimality of $(X, \tau(X))$. Hence, by Corollary 9, $(X, \tau_s(X))$ is minimal-T_2. Furthermore, being $\tau_s(X) \subseteq \tau(X)$ and, obviously, $\left(\tau_s(X)\right)_s = \tau_s(X)$, by Lemma 6, we have that $\alpha\tau_s(X) \subseteq \alpha\tau(X)$, i.e. $\alpha\tau_s(X) \subseteq \tau(X)$ where $\alpha\tau_s(X)$ is Hausdorff by Proposition 11 as $\tau(X)$ is Hausdorff and the T_2 axioms are expansive. Since $(X, \tau(X))$ is minimal-αT_2, we conclude that $\alpha\tau_s(X) = \tau(X)$.

$(\Longleftarrow)$ Let us suppose that $(X, \tau_s(X))$ is minimal-T_2 and $\tau(X) = \alpha\tau_s(X)$. Then $(X, \tau(X))$ is H-closed by Corollary 9. Now, let $\sigma(X)$ be an αT_2 topology on X such that

$$\sigma(X) \subseteq \tau(X).$$

For every R regular open set of $(X,\tau(X))$, $X\setminus R$ is a regular closed set and hence, by Proposition 10, it is a T_2-closed subspace of the Hausdorff space $(X,\sigma(X))$. Thus $X\setminus R$ is a closed set of $(X,\sigma(X))$ and $R\in\sigma(X)$. This proves that

$$\tau_s(X)\subseteq\sigma(X).$$

Hence, by Proposition 12, $\tau_s(X)=\sigma_s(X)$ and so, applying Lemma 6 to $\tau_s(X)\subseteq\sigma(X)$, we have that $\alpha\tau_s(X)\subseteq\alpha\sigma(X)$. Since $\sigma(X)$ is an α topology and, by hypothesis $\alpha\tau_s(X)=\tau(X)$, it follows that $\tau(X)\subseteq\sigma(X)$ and so that $\tau(X)=\sigma(X)$. This shows that $(X,\tau(X))$ is a minimal-αT_2-space and concludes our proof. □

Corollary 24. *Let $(X,\tau(X))$ be a space with a dense set D of isolated points. Then:*

(1) *the α-topology $\alpha\tau(X)$ coincides with the topology generated by $\tau(X)\cup\{D\cup\{x\}:x\in X\}$, i.e. it is a simple extension of the subspace D;*
(2) *X is an α-space if and only if $X\setminus D$ is discrete;*
(3) *if X is a semiregular α-space, it results:*

$$\big\{\sigma:\sigma \text{ is a topology on } X \text{ such that } \sigma_s=\tau(X)\big\}=\{\tau(X)\}.$$

Proof: Straightforward applications of Propositions 3, 5 and 6. □

Example 25. Let us consider the set $X=\mathbb{R}\times[0,+\infty[$. It is easy to verify that

$$\tau_s(X)=\big\{U\subseteq X:(x,0)\in U\ \Rightarrow\ \exists\,\epsilon>0\ \text{ such that }\ ]x-\epsilon,x+\epsilon[\times[0,\epsilon[\subseteq U\big\}$$

defines a Tychonoff topology on X and that $D=\mathbb{R}\times]0,+\infty[$ is a dense set of isolated points. Now, if we consider another topology on X,

$$\begin{aligned}\tau(X)=\big\{U\subseteq X:\ &(x,0)\in U\setminus\mathbb{Q}\times\{0\}\Rightarrow\exists\epsilon>0\\ &\text{such that }\ ]x-\epsilon,x+\epsilon[\times[0,\epsilon[\subseteq U\\ &\text{and }\ (x,0)\in U\cap\mathbb{Q}\times\{0\}\Rightarrow\exists\epsilon>0 \text{ such that}\\ &\quad]x-\epsilon,x+\epsilon[\times]0,\epsilon[\cup\big(]x-\epsilon,x+\epsilon[\cap\mathbb{Q}\times\{0\}\big)\subseteq U\big\},\end{aligned}$$

it is a simple routine to check that $\tau_s(X)$ is the semiregularization of $\tau(X)$ and that $\tau(X)\neq\tau_s(X)$. Furthermore, since $C=X\setminus(\mathbb{R}\times]0,+\infty[)$ is a closed nowhere dense subset with respect to both $\tau(X)$ and $\tau_s(X)$, by Corollary 24 follows that $\alpha\tau(X)=\alpha\tau_s(X)$ coincides with the topology generated by $\tau(X)\cup\big\{(\mathbb{R}\times]0,+\infty[)\cup\{(x,0)\}:x\in\mathbb{R}\big\}$ and so that $\tau(X)\neq\alpha\tau(X)$. Thus, we have that

$$\tau_s(X)\subsetneq\tau(X)\subsetneq\alpha\tau(X)=\alpha\tau_s(X).$$

Example 26 (A Tychonoff α-space which is not minimal-T_2)**.** Let us recall that two sets are *almost disjoint* if their intersection is a finite set. It is a simple routine to show (by using Zorn's Lemma) that there exists a maximal almost disjoint family $\mathcal{M}$ of subsets of $\mathbb{N}$. The space generated on the set $\psi = \mathbb{N} \cup \mathcal{M}$ by the base

$$\mathcal{B} = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \big\{\{M\} \cup S : M \in \mathcal{M} \text{ and } S \text{ is a cofinite subset of } M\big\}$$

is a 0-dimensional (and hence Tychonoff) but not normal (and hence not minimal-T_2). This space is known in literature as the ψ-space (see 1N, [PW]). Since, every closed nowhere dense set of ψ is discrete, by Proposition 5, it is evident that ψ is an α-space.

Example 27 (A minimal-T_2, α-space)**.** Let us consider the set

$$Z = \left\{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{m}\right) : n \in \mathbb{N},\, m \in \mathbb{Z}\right\}$$

with the topology $\tau(Z)$ induced by the usual topology on $\mathbb{R}^2$. Let $X = Z \cup \{a, b\}$ and define a topology $\tau(X)$ on X by saying that a subset $U \subset Z$ is open if $U \cap Z \in \tau(Z)$ and if $a \in U$ (respectively, $b \in U$) there exists some $k \in \mathbb{N}$ such that $\big\{\big(\frac{1}{n}, \frac{1}{m}\big) : n \in \mathbb{N},\, m \geq k\big\} \subseteq U$ (respectively, $\big\{\big(\frac{1}{n}, -\frac{1}{m}\big) : n \in \mathbb{N},\, m \geq k\big\} \subseteq U$). It is well-known (see 4.8(d), [PW]) that the space X is Urysohn, not compact and minimal-T_2. Furthermore, since every its nowhere dense subset (namely, $\big\{\big(\frac{1}{n}, 0\big) : n \in \mathbb{N}\big\} \cup \{a, b\}$) is discrete, it follows from Proposition 5 that X is an α-space.

Let us note that, by 24(3), in both of the above spaces, we have that $\big\{\sigma : \sigma$ is a topology on X such that $\sigma_s = \tau(X)\big\} = \{\tau(X)\}$. Thus, the question remains: *is there some semiregular α-space $\big(X, \tau(X)\big)$ such that $\{\tau(X)\} \subsetneqq \big\{\sigma : \sigma$ is a topology on X such that $\sigma_s = \tau(X)\big\}$?*

We now provide an example of a Tychonoff α-space X such that

$$\{\tau(X)\} \subsetneqq \{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\}.$$

Example 28. Recall that a measurable subset A of $\mathbb{R}$ has density 1 if

$$A = \{x \in \mathbb{R} : \lim_{h \to 0} \frac{m(A \cap [x-h, x+h])}{2h} = 1\}.$$

The set of $\{A \subseteq \mathbb{R} : A$ measurable with density $1\}$ forms a topology $\delta(\mathbb{R})$, called the density topology, on the set $\mathbb{R}$. The space $\big(\mathbb{R}, \delta(\mathbb{R})\big)$ is a Tychonoff space without isolated points, strictly finer than the usual topology $\tau(\mathbb{R})$, and has the property that every nowhere dense subset is closed and discrete (see 2.7 in [T]). In particular, $(\mathbb{R}, \delta(\mathbb{R}))$ is an α-space by Proposition 4. We need one additional fact about $(\mathbb{R}, \delta(\mathbb{R}))$. Note that for $x \in U \in \delta(\mathbb{R})$, $U \cap (x, \infty) \neq \emptyset$ and $U \cap (-\infty, x) \neq \emptyset$. In particular, the $\delta(\mathbb{R})$ open neighborhood filter is contained in two distinct $\delta(\mathbb{R})$ open ultrafilters on $\mathbb{R}$. Before continuing with this example, we need a result about absolutes of spaces.

For a Hausdorff space X and let $EX = \{\mathcal{U} : \mathcal{U}$ is a convergent, open ultrafilter on $X\}$. For $U \in \tau(X)$, let $O(U) = \{\mathcal{U} \in EX : U \in \mathcal{U}\}$. For $U, V \in \tau(X)$, it is easy to verify (see [PW]) that $O(\emptyset) = \emptyset, O(X) = EX, O(U \cap V) = O(U) \cap O(V), O(U \cup V) = O(U) \cup O(V), EX \backslash O(U) = O(X \backslash \operatorname{cl}_X(U))$, and $O(U) = O(\operatorname{int}_X \operatorname{cl}_X(U))$. EX with the topology generated by $\{O(U) : U \in \tau(X)\}$ is an extremally disconnected Tychonoff space, called the *absolute* of X. The function $k : EX \to X$ defined by $k(\mathcal{U})$ is the unique convergent point of $\mathcal{U}$ is called a covering function and has the properties that k is irreducible, θ-continuous, perfect and onto. If X is regular, then k is also continuous. If $D \subseteq EX$ such that $k[D] = X$, then D is dense in EX.

Proposition 29. *Let X be a regular α-space and Y a subspace of EX such that for each $x \in X$, $|k^{\leftarrow}((x))| = 2$. Then Y is an α-space.*

Proof: Let $\overline{k} = k|Y$. The function $\overline{k} = k|Y : Y \to X$ is continuous and onto. So, Y is dense in EX and extremally disconnected. Let N be a nowhere dense subset of Y. Suppose $U = \operatorname{int}_{EX}\left(\operatorname{cl}_{EX}(N)\right) \neq \emptyset$. Then, as Y is dense, $\emptyset \neq U \cap Y \subseteq \operatorname{cl}_{EX}(N) \cap Y = \operatorname{cl}_Y(N)$, contradicting that N is nowhere dense in Y. Thus, N is nowhere dense in EX. By 6.5d(2) in [PW], $k[N] = \overline{k}[N]$ is nowhere dense in X and hence discrete in X. Let $p \in N$. There is an open set V in X such that $V \cap \overline{k}[N] = \{k(p)\}$. So, $N \cap \overline{k}^{\leftarrow}[V] = \overline{k}^{\leftarrow}(\{p\})$. But $\overline{k}^{\leftarrow}(\{p\})$ has only two points and there is an open set W in Y such that $N \cap \overline{k}^{\leftarrow}[V] \cap W = \{p\}$. This shows that N is discrete in Y and Y is an α-space. $\square$

We are ready to apply Proposition 29 to the regular α-space $(\mathbb{R}, \delta(\mathbb{R}))$. First note that the covering function $k : E\mathbb{R} \to (\mathbb{R}, \delta(\mathbb{R}))$ has the property that $|k^{\leftarrow}(r)| \geq 2$ for each $r \in \mathbb{R}$ since each open neighborhood filter is contained in two distinct $\delta(\mathbb{R})$ open ultrafilters on $\mathbb{R}$. Let $Y \subseteq E\mathbb{R}$ have the property that $|k^{\leftarrow}(r) \cap Y| = 2$ for each $r \in \mathbb{R}$, and let $Z \subset Y$ be such that $|k^{\leftarrow}(r) \cap Z| = 1$ for each $r \in \mathbb{R}$. Now, both Y and Z are dense in $E\mathbb{R}$. By Proposition 29, Y is a Tychonoff α-space. Let σ be the topology on Y generated by $\tau(Y) \cup \{Z\}$. Clearly, $\tau(Y) \subsetneq \sigma$, and it is straightforward to show that $\sigma_s = \tau(Y)$. The space Y is the desired space.

5. A problem

Since the property to be $T_{2\frac{1}{2}}$-closed does not pass to the regular open subspaces, we cannot use a result like Proposition 10 to prove a proposition similar to Proposition 23. So, we leave the following as an unsolved problem:

Conjecture. Let X be an $\alpha T_{2\frac{1}{2}}$-space. Then X is a minimal-$\alpha T_{2\frac{1}{2}}$-space if and only if $(X, \tau_s(X))$ is semiregular, $T_{2\frac{1}{2}}$-closed and $\tau(X) = \alpha\tau_s(X)$.

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