Inert subgroups of uncountable locally finite groups

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Abstract. Let G be an uncountable universal locally finite group. We study subgroups $H < G$ such that for every $g \in G$, $|H : H \cap H^g| < |H|$.

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0. Introduction

A locally finite group G is *universal locally finite* if every finite group is embedded into G and any isomorphism between finite subgroups of G can be realized by an inner automorphism of G. There is a unique countable universal locally finite group and there are 2^κ universal locally finite groups of cardinality $\kappa > \omega$ [2] and [6].

We say that an infinite subgroup $H \leq G$ is *inert* in G if for every conjugate H^g the group $H \cap H^g$ is of index $< |H|$ in H (and then also in H^g). The main result of the paper is devoted to existence of inert subgroups in uncountable universal locally finite groups. The problem of description of inert subgroups in locally finite groups is connected with the question of V. Belyaev *whether an uncountable locally finite group G can be nontrivially topologized* [1]. In Section 1 we show that G can be nontrivially topologized if it has an infinite inert subgroup H which is residually finite.

The main result of the paper is Theorem 7 (Section 2) which gives two constructions of uncountable universal locally finite groups with inert residually finite subgroups. This shows that there are several universal locally finite groups of cardinality ω_1 with inert residually finite subgroups. In [6] Macintyre and Shelah show by a model-theoretic method that there are 2^{ω_1} universal locally finite groups of cardinality ω_1. In Section 3 we study to what extent this method can be applied to prove that there are 2^{ω_1} universal locally finite groups of cardinality ω_1 with inert residually finite subgroups.

We use standard notation. By $\mathbf{F}_2$ we denote the two element field. A group G is *residually finite*, if for any $g \in G \setminus \{1\}$ there is a normal subgroup $H < G$ of finite index with $g \notin H$. In our arguments we use several facts about universal locally finite groups from [2], [6] and [4]. For example we will frequently apply the fact that the union of an increasing chain of universal locally finite groups is a universal locally finite group [4].

The author is grateful to the referee for helpful remarks and suggestions. In particular, the proof of Theorem 7(1) belongs to the referee. It substantially simplifies the argument of the author from earlier versions of the paper.

1. Groups which can be topologized

We now explain the connection with topological groups mentioned above.

Definition. Let κ be an infinite cardinal. We say that a subgroup H of a group G is κ-inert if $\kappa \leq |H|$ and for every $g \in G$, $|H^g : H \cap H^g|$ (and then also $|H : H \cap H^g|$) is less than κ.

Let H be any subgroup of a group G such that $|H| < \kappa$. Then every non-trivial $K < G$ such that $|K| < \kappa$, $H < N_G(K)$ and $H \cap K = 1$ is called a κ-H-signalizer.

If in this definition $\kappa = \omega$, then κ-H-signalizers are called finite H-signalizers. Notice that any conjugate of an inert subgroup is inert.

Definition. We say that infinite groups H and F are coincident if $|H : H \cap F| < |H|$ and $|F : H \cap F| < |F|$.

An inert subgroup H is coincident with any conjugate H^g. It is easy to see that coincident groups have the same cardinality. In the following lemma we give further properties of the relation of coincidency. Statements (1) and (2) will be used below.

Lemma 1. (1) *Let H, F and K be infinite groups. Suppose F and K are both coincident with H. Then the intersection $F \cap K$ is also coincident with H.*

(2) *The relation of coincidency is an equivalence relation.*

(3) *If H and F are subgroups of G, H is inert in G and F is coincident with H, then F is inert.*

Proof: (1) If $|H : H \cap (F \cap K)|$ is infinite, we will prove that $|H : H \cap F \cap K| \leq \max\{|H : H \cap F|, |H : H \cap K|\}$. Notice that any $A \subseteq H$ of cardinality $> \max\{|H : H \cap F|, |H : H \cap K|\}$ contains some $B \subseteq A$ of cardinality $> \max\{|H : H \cap F|, |H : H \cap K|\}$ of the same coset with respect to $H \cap F$, i.e. $(\forall g, h \in B)(gh^{-1} \in H \cap F)$. On the other hand the set B contains some $C \subseteq B$ of cardinality $> \max\{|H : H \cap F|, |H : H \cap K|\}$ of the same coset with respect to $H \cap K$. As a result we have that the elements of A cannot represent pairwise distinct cosets with respect to $H \cap K \cap F$.

Assume that $|F \cap K : (F \cap K) \cap H|$ is infinite. By the previous paragraph $|F \cap K| = |H| = |K| = |F|$. Now it suffices to prove that $|F \cap K : F \cap K \cap H| \leq \min\{|F : F \cap H|, |K : H \cap K|\}$. Let $A \subseteq F \cap K$ be any set of cardinality $> \min\{|F : F \cap H|, |K : K \cap H|\}$. Then there are two distinct $g, h \in A$ such that $gh^{-1} \in F \cap H$ or $gh^{-1} \in K \cap H$. Therefore $gh^{-1} \in F \cap K \cap H$.

(2) Assume that F is coincident with H and H is coincident with K. If $|F : F \cap K|$ is infinite we prove $|F : F \cap K| \leq \max\{|F : F \cap H|, |H : H \cap K|\}$ by an argument similar to the proof of part 1.

(3) Lemma 1(3) can be deduced from Lemma 1(2) as follows. Since H is inert, H is coincident with H^g. By the assumption F is coincident with H as well as H^g is coincident with F^g. Since the relation of coincidency is an equivalence relation, F is coincident with F^g. □

The following proposition is known for countable groups [1].

Proposition 2. (1) *Let G be a group of regular cardinality κ such that every subgroup F with $|F| < \kappa$ has a κ-F-signalizer. Then there is an inert subgroup $H < G$ such that $|H| = \kappa$ and $\bigcap\{K \lhd H : |H : K| < |H|\}$ is trivial.*

(2) *Let G be an infinite group. If G has an infinite inert subgroup H such that $\bigcap\{K < H : |H : K| < |H|\}$ is trivial, then G is non-trivially topologized.*

Proof: (1) Let $|G| = \kappa$ and $G = \{g_\alpha : \alpha < \kappa\}$. We define a sequence $(H_\alpha)_{\alpha<\kappa}$ of subgroups of G inductively in the following way. We put $H_0 = \langle g_0 \rangle$. Suppose that $\beta < \kappa$ and for every $\alpha < \beta$ we have already defined H_α in such a way that $|H_\alpha| < \kappa$. Since κ is regular then the group $D_\beta = \langle \{H_\alpha : \alpha < \beta\}, \{g_\alpha : \alpha < \beta\} \rangle$ is also of cardinality less than κ. Now we define H_β as a κ-D_β-signalizer. Finally we put $H = \langle \{H_\alpha : \alpha < \kappa\} \rangle$. Then H is an inert subgroup of G of cardinality κ. To see that $\bigcap\{K \lhd H : |H : K| < |H|\}$ is trivial, take any $h \in H$ and $\beta < \kappa$ with $h \in \langle \{H_\alpha : \alpha < \beta\} \rangle$. Since $\langle \{H_\delta : \delta \geq \beta\} \rangle$ is normal in H, it suffices to show that $h \notin \langle \{H_\delta : \delta \geq \beta\} \rangle$. If the latter is not true, find the least γ such that $h \in \langle \{H_\delta : \beta \leq \delta \leq \gamma\} \rangle$. Since $H_\delta < N_G(H_\gamma)$, where $\beta \leq \delta < \gamma$, the element h can be presented as $h' \cdot h''$ with $h' \in \langle \{H_\delta : \beta \leq \delta < \gamma\} \rangle$ and $h'' \in H_\gamma \setminus \{1\}$. Then $h'' \in H_\gamma \cap D_\gamma$, which contradicts the choice of H_γ.

(2) Let H be an inert subgroup such that $\bigcap\{K < H : |H : K| < |H|\}$ is trivial. Consider the group $\mathrm{Sym}(S)$, where $S = \{gK : g \in G, K < G$ and K is coincident with $H\}$. For $C \subseteq S$ and $g \in \mathrm{Sym}(S)$ define the set $A^g_C = \{h \in \mathrm{Sym}(S) : h|_C = g|_C\}$. The topology given on $\mathrm{Sym}(S)$ by the system of basic open sets of the form A^g_C, where $|C| < \omega$, is Hausdorff as well as the inherited (with respect to the natural embedding into $\mathrm{Sym}(S)$) topology on G. To see that the topology is non-trivial we should prove that for every family $\{g_t K_t \in S : t < n\}, n \in \omega$, the set of all $h \in H$ with $hg_t K_t = g_t K_t$, $t < n$, forms a subgroup of index $< |H|$ in H. Since

$$\{h \in H : (\forall\, t < n)(hg_t K_t = g_t K_t)\} = H \cap \bigcap_{t<n} g_t K_t g_t^{-1},$$

by Lemma 1(1) it suffices to show that any $g_t K_t g_t^{-1}$ is coincident with H. As K is coincident with H, any $g_t K_t g_t^{-1}$ is coincident with $g_t H g_t^{-1}$. Since the latter is coincident with H (H is inert), H and $g_t K g_t^{-1}$ are coincident by Lemma 1(2). □

Notice that in the argument of the second statement if $|H|$ is a strongly inaccessible cardinal κ and H is κ-inert, then there is another topology on $\mathrm{Sym}(S)$ given by the system of basic open sets of the form A^g_C, where $|C| < \kappa$.

2. Universal locally finite groups

In this section we prove our main theorems. We start with some construction of the countable universal locally finite group U which will play some special role in arguments below. In fact this is a slight modification of the original construction of Ph. Hall [2].

We build U as the union of the increasing chain of finite subgroups $A_0 < A_1 < \ldots < A_n < \ldots$ such that $A_0 = \langle c_0 \rangle$ is isomorphic to $(\mathbf{F}_2, +)$ and for any i any two isomorphic subgroups of A_i are conjugate in A_{i+1}. At step $i+1$ we define A_{i+1} as the symmetric group on $A_i \oplus \langle c_{i+1} \rangle$ ($|\langle c_{i+1} \rangle| = 2$). The group $A_i \oplus \langle c_{i+1} \rangle$ acts on itself by left multiplication. Then it can be considered as a subgroup of A_{i+1}. This defines the extension $A_i < A_{i+1}$. The fact that $\bigcup A_i$ is universal locally finite can be verified as in [2]. We now see that the group $C = \langle c_0, c_1, \ldots \rangle$ satisfies the following lemma.

Lemma 3. *Let U be the countable universal locally finite group. There exists a (residually finite) subgroup $C < U$ isomorphic to the vector space over $\mathbf{F}_2$ of dimension ω such that for any $g \in U$ the centralizer $C(g) \cap C$ is of finite index in C (thus C is inert).*

It is easy to see that any finite subgroup $F < U$ has a finite F-signalizer. By Proposition 2(1) there is an inert residually finite subgroup of U. Lemma 3 shows that C is an example of such a group.

We now state some further properties of the presentation $U = \bigcup A_i$. Let $C_i := \langle c_0, c_1, \ldots c_i \rangle$ $(= C \cap A_i)$, $i \in \omega$.

Lemma 4. *Let $i \in \omega$ and B be a finite group having A_i as a subgroup. Then there is j such that $B \oplus \langle c_{i+1}, \ldots, c_j \rangle$ embeds into A_j over $\langle A_i, C_j \rangle$.*

PROOF: First find k with $|B \oplus \langle c_{i+1}, \ldots, c_k \rangle| \leq |A_k|$. Then $B \oplus \langle c_{i+1}, \ldots, c_{k+1} \rangle$ is embeddable into A_{k+1}. Since any isomorphism between two subgroups of $A_{k+1} \oplus \langle c_{k+2} \rangle$ extends to an automorphism of A_{k+2}, the group $B \oplus \langle c_{i+1}, \ldots, c_{k+2} \rangle$ is embeddable into A_{k+2} over $A_i \oplus \langle c_{i+1}, \ldots, c_{k+2} \rangle$. We see that the statement of the lemma holds for $j = k+2$. □

Lemma 5. *For any locally finite extension of $B > C$ of finite index such that for any $g \in B$ the centralizer $C(g)$ has a cofinite intersection with the base $\{c_0, c_1, \ldots\}$ there exists a decomposition $B = B_0 \oplus \langle c_{n+1}, c_{n+2}, \ldots \rangle$ where B_0 is a finite subgroup of B.*

PROOF: Suppose that B is a locally finite extension of C satisfying the assumptions of the lemma. Then $B = \langle b_1, b_2, \ldots, b_m, C \rangle$, for some $b_1, b_2, \ldots, b_m$ and there is a natural number n such that

(i) for every $i > n$ and $j \leq m$, $c_i b_j = b_j c_i$.

It is easy to see that the number of non-trivial finite combinations $c_{n+1}^{\epsilon_{n+1}} c_{n+2}^{\epsilon_{n+2}} \dots c_{n+k}^{\epsilon_{n+k}}$, with $k \in \omega$ and $\epsilon_{n+j} \in \{0,1\}$, $1 \leq j \leq k$, belonging to $\langle b_1, b_2, \dots, b_m, c_0, c_1, \dots, c_n\rangle$ is finite. This implies that the number n can be enlarged so that

(ii) no non-trivial finite combination $c_{n+1}^{\epsilon_{n+1}} c_{n+2}^{\epsilon_{n+2}} \dots c_{n+k}^{\epsilon_{n+k}}$, where $k \in \omega$ and $\epsilon_{n+j} \in \{0,1\}$ for $1 \leq j \leq k$, belongs to $B_0 = \langle b_1, b_2, \dots, b_m, c_0, c_1, \dots, c_n\rangle$.

By (i) and (ii) we have $B = B_0 \oplus \langle c_{n+1}, c_{n+2}, \dots\rangle$. □

The following lemma was suggested by the referee. It has become basic to the main result of the section.

Lemma 6. *Let $B' > B \geq C$ be locally finite extensions of C of finite index such that for any $g \in B'$ the centralizer $C(g)$ has a cofinite intersection with the base $\{c_0, c_1, \dots\}$. Then every C-embedding of B into U extends to a C-embedding of B' into U.*

Proof: By Lemma 5 we find a natural number n such that B and B' are decomposed as follows:

$$B = B_0 \oplus \langle c_{n+1}, c_{n+2}, \dots\rangle \text{ and } B' = B'_0 \oplus \langle c_{n+1}, c_{n+2}, \dots\rangle,$$

where $B'_0 > B_0 > C_n$ are finite subgroups of $B' > B > C$ respectively.

We now see that any C_n-embedding of B'_0 into A_n can be naturally extended to an embedding of B' into U over C. By Lemma 4 we may assume that there is a C_n-embedding of B'_0 into A_n (enlarging n if necessary). Thus there exists an embedding $g : B' \to U$ over C.

Let $f : B \to U$ be a C-embedding. Find m such that $f(B_0) \subseteq A_m$. Then $f(\langle B_0, C_m\rangle) \subseteq A_m$. We may assume that $m \geq n$ where n is chosen as above. Then the natural isomorphism (induced by $f \cdot g^{-1}$)

$$g(\langle B_0, C_m\rangle) \oplus \langle c_{m+1}\rangle \to f(\langle B_0, C_m\rangle) \oplus \langle c_{m+1}\rangle$$

extends to an automorphism h of A_{m+1}. Define a C_{m+1}-embedding $\langle B'_0, C_{m+1}\rangle \to A_{m+1}$ by $f' := h \cdot g$. By the definition of h, the map f' extends f on $\langle B'_0, C_{m+1}\rangle$. Extending f' to a C-embedding we obtain the required embedding. □

Theorem 7. (1) *There is a universal locally finite group of cardinality ω_1 with a countable inert residually finite subgroup.*

(2) *For every uncountable cardinal κ there exists a universal locally finite group G of cardinality κ with the following properties:*
   (a) *there exists an inert residually finite subgroup $H < G$ of cardinality κ;*
   (b) *if κ is regular then there is no inert subgroup of cardinality $< \kappa$.*

Proof: (1) Let U be the countable universal locally finite group. Find a subgroup C as in Lemma 3. First we want to show that *there is a proper subgroup of U isomorphic to U over C.*

We again consider U as the union of a chain of finite subgroups $A_0 < A_1 < \ldots$ defined as in the construction before Lemma 3. We preserve all notation introduced there.

Let K be a non-trivial finite group. Define $B_i := \langle A_i, C\rangle \times K$ as a natural extension of $\langle A_i, C\rangle$. By Lemma 6 there is a sequence of C-embeddings $f_i : B_i \to U$ such that $f_i \subseteq f_{i+1}$ for all $i = 0, 1, \ldots$. Set $f = \bigcup f_i$. Then f is a C-embedding of $U \times K$ into U and $f(U)$ is a proper subgroup of U which is isomorphic to U over C. It is clear that C is inert in that subgroup.

We now build the group from the formulation of the theorem as the union of an increasing chain of universal locally finite groups containing C as an inert subgroup. At every step of the construction we use the embedding constructed above. If the chain is of length ω_1, then the union is a universal locally finite group satisfying the statement of the theorem.

(2) We construct G as the union $\bigcup G_\delta$ of an increasing chain. At every step δ we define an element $a_\delta \in G_\delta$. Then the group generated by all a_δ will serve as an inert residually finite subgroup. We start with a countable universal locally finite group G_0 and a non-trivial $a_0 \in G_0$. At step $\delta + 1$ extend $G_\delta \times G_\delta$ by an element c_δ of order 2 such that $(a, 1) \cdot c_\delta = c_\delta \cdot (1, a)$ for all $a \in G_\delta$. Here we identify G_δ with $(G_\delta, 1)$. Let $a_{\delta+1} \in (1, G_\delta)$ be non-trivial. It is easy to see that any finitely generated subgroup of the obtained group is finite. Thus it can be embedded into a universal locally finite group $G_{\delta+1}$ of the same cardinality. At the limit steps of the construction we take the unions.

The construction implies that $A = \langle\{a_\alpha : \alpha < \kappa\}\rangle$ is a direct sum of finite cyclic groups. Thus A is residually finite. Moreover for any $g \in G_\delta$ and any a_γ with $\delta < \gamma$ we have $g \cdot a_\gamma = a_\gamma \cdot g$. This guarantees that A is inert. If κ is regular and B is a subgroup with $|B| < |G|$ then there is $\gamma < \kappa$ such that $B \subseteq G_\gamma$. Then $B^{c_\gamma} \cap B = \{1\}$ and B is not inert in G. □

3. Frequency

Theorem 7 implies that there are several universal locally finite groups of cardinality ω_1 with inert residually finite subgroups. Macintyre and Shelah have shown in [6] that there exist 2^{ω_1} uncountable universal locally finite groups. Is it true that all of them have inert residually finite subgroups? We now analyse the method from [6]. It is based on some model theory for infinitary extensions of the first-order logic. The logic $L_{\omega_1,\omega}$ considers sets of formulas closed under quantification and countable conjunctions and disjunctions. For a structure M and a subset $A \subseteq M$ a *type over* A depending on variables $x_1, \ldots, x_n$ is a maximal set p of formulas $\phi(x_1, \ldots, x_n)$ with parameters from A such that the theory

$Th(M,a)_{a\in A} \cup p$ (in an appropriate logic) is consistent. A tuple $b_1,\ldots,b_n$ *realizes* p if every formula of p is satisfied by $b_1,\ldots,b_n$.

The following proposition develops Lemma 5 from [6].

Proposition 8. *Let H be an uncountable locally finite group and ϕ be an $L_{\omega_1,\omega}$-sentence which holds in H. Then the class K of all groups G such that $G \models \phi$ and G has an ω-inert residually finite subgroup, is not empty and K is a reduct of a class axiomatizable by a single sentence of $L_{\omega_1,\omega}$.*

Proof: Consider the following theory in the language of a group with a unary predicate P:

- $\{\phi\} \cup \{P$ defines an infinite subgroup $\}$;

$$(\forall x)(\bigvee_{n,m}(\exists y_1,\ldots,y_n \in P, z_1,\ldots,z_m \in P^x)(\forall y \in P, z \in P^x)$$
$$(\exists y', z' \in P \cap P^x) \bigvee_{i\le n; j\le m} (y = y_i \cdot y' \wedge z = z_j \cdot z'));$$

- P is a residually finite group.

By a theorem from [7] the latter property can be expressed in $L_{\omega_1,\omega}$ by a formula.

It remains to prove that for every uncountable locally finite group H, any theory of this form has a (countable) model. By the main result of [1] all locally finite groups having a finite subgroup without finite signalizers are countable. Thus for every finite subgroup $D < H$ there is a finite D-signalizer in H. This can be expressed by an infinite disjunction θ of first-order formulas of the following form

$$(\forall x_1,\ldots,x_n)(\bigwedge_{i,j}\bigvee_{k}(x_i x_j = x_k) \to (\exists y_1,\ldots,y_m)(\bigwedge_{i,j}\bigvee_{k}(y_i y_j = y_k) \wedge$$
$$(\{x_1,\ldots,x_n\} \cap \{y_1,\ldots,y_m\} = \{1\}) \wedge \bigwedge_{i,j}\bigvee_{k}(x_i^{-1} y_j x_i = y_k))).$$

By Downward Lowenheim-Skolem Theorem ([5, p. 69]) for any uncountable structure M and any sentence $\psi \in L_{\omega_1,\omega}$ with $M \models \psi$ there is a countable substructure satisfying ψ. Let H_0 be such a group chosen for H and $\phi \wedge \theta$. Then by the version of Proposition 2(1) for $\kappa = \aleph_0$ (originally proved in [1]) the group H_0 has an infinite inert residually finite subgroup. $\square$

By the $L_{\omega_1,\omega}$-version of Theorem 5.2 from [8] if in every uncountable κ there exist a model of the sentence constructed in the proof then for every infinite cardinality κ there is a group G as in the formulation and of cardinality κ with the additional property that for every countable $A \subset G$ the structure G realizes only countably many types over A. *The author does not know if in the case of universal locally finite groups the class K has a member in every cardinality.*

On the other hand the following constructions provides groups with opposite properties. Let α be an infinite cardinal $< \kappa$. Consider S_3^α where $S_3 = \langle g_1, g_2 : |g_1| = 2, |g_2| = 3, [g_1, g_2] \neq 1\rangle$ is the symmetric group on $\{0, 1, 2\}$. Let Π be a family of α^+ subsets of α. Let P_0 be the subgroup generated by all elements of the form $f_i \in \{1, g_1\}^\alpha$ with $f_i(\gamma) = g_1$ iff $\gamma = i$ $(i < \alpha)$. Let P be the subgroup generated by P_0 and all elements of the form $r_X \in \{1, g_2\}^\alpha$ with $r_X(\gamma) = 1$ iff $\gamma \in X$ $(X \in \Pi)$. Enumerate $P = \{p_\delta : \delta < \alpha^+\}$.

If $\alpha^+ < \kappa$, we construct a universal locally finite group $G(P)$ of cardinality κ as the union $\bigcup G_\delta$ of an increasing chain so that $P < G(P)$. At Step 0 let G_0 be a universal locally finite group of cardinality α^+ containing P. We now repeat the construction of the second part of Theorem 7. *The obtained group satisfies the statement of that theorem and P_0 is a subgroup of cardinality α such that α^+ types over P_0 are realized.* Each of these types consists of all formulas with parameters from P_0 realized by an appropriate r_X. Notice that for $X \neq X'$, say $i \in X \setminus X'$, the formula $[x, f_i] = 1$ is realized by r_X but not realized by $r_{X'}$.

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