Compact pospaces

Venu G. Menon

Abstract. Posets with property DINT which are compact pospaces with respect to the interval topologies are characterized.

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The purpose of this note is to characterize posets which are compact pospaces with respect to the interval topology. We restrict our attention to the posets which have, what Lawson [4] calls, property DINT. Lawson introduced this property as a generalization of the Property M studied by domain theorists. Among the quasicontinuous posets, these are precisely the ones with compact Lawson topology.

The topology generated by $\downarrow x$, $x \in P$ as subbasic closed sets is called *the upper topology*. The topology generated by $\uparrow x$, $x \in P$ as subbasic closed sets is called *the lower topology*. The *interval topology* is the supremum of the lower and upper topologies. A subset U of P is called *Scott-open* if (i) $U = \uparrow U$, and (ii) if D is a directed subset of P with $\sup D \in U$, then $U \cap D \neq \emptyset$. Scott-open sets form a topology, and is called the *Scott topology*. The *Lawson topology* is the common refinement of the lower and the Scott topologies. For subsets A and B of a poset P, we say A is way-below B, written $A \ll B$, if whenever D is a directed subset of P for which $\sup D$ exists and is in $\uparrow B$, then $D \cap \uparrow A \neq \emptyset$. An upcomplete poset is called *quasicontinuous* [3] if (i) for each $x \in P$, $\uparrow x = \bigcap\{\uparrow A : A \ll x, A \text{ is finite}\}$ and (ii) for finite subsets $F, G \ll x$, there exists a finite set H such that $H \ll x$, and $H \subseteq \uparrow F \cap \uparrow G$. A partially ordered set P is said to have *property DINT* (Lawson [4]) if every set closed in the lower topology is a directed intersection of finitely generated upper sets, that is sets of the form $\uparrow F$, where F is finite. In [4], Lawson gives several characterizations for quasicontinuous domains with property DINT.

Recall that a partially ordered set with a topology defined on it is called a pospace if the partial order is a closed subset of $X \times X$. In any pospace $\downarrow x$ and $\uparrow x$ are both closed, and hence the interval topology is contained in any pospace topology. In the following theorem we collect some well known results about pospaces. These results are proved in Section 1 of Chapter VI in [1].

Theorem 1.1. (a) *Let P be a poset with a topology defined on it. Then P is a pospace if and only if whenever $a \not\leq b$, there exist open sets U and V with $a \in U$ and $b \in V$ such that if $x \in U$ and $y \in V$, then $x \not\leq y$. Moreover, if P is compact, then U can be taken as an upper set and V can be taken as a lower set.*

(b) *If P is a compact pospace, then it has a basis of order convex compact neighborhoods at each point.*

(c) *If A is a compact subset of a pospace P, then $\downarrow A$ and $\uparrow A$ are closed subsets of P.*

The following theorem is due to M.E. Rudin [5].

Theorem 1.2. *If $\{\uparrow F_i : i \in I\}$ is a descending family of finitely generated upper sets in a partially ordered set P, then there exists a directed subset D of $\bigcup_{i\in I} F_i$ that intersects each F_i nontrivially.*

The following theorem is due to J.D. Lawson [4].

Theorem 1.3. *Let P be a partially ordered set satisfying property DINT. Let τ denote the lower topology, and suppose there exists a topology ν containing the upper topology with all open sets being upper sets such that P endowed with the join topology $\sigma := \tau \vee \nu$ is a compact Hausdorff space. Then the topology σ_* of σ-open lower sets is equal to τ.*

GCD lattices were introduced in Venugopalan [7] as a common generalization of completely distributive lattices and generalized continuous lattices, the latter being quasicontinuous posets which are also complete lattices [2]. Here we need a slightly general notion of GCD posets. The binary relation ρ defined below is a generalization of the relation in Raney [5].

Definition 1.4. Let P be a partially ordered set. We define a binary relation ρ on the set of subsets of P as follows: $A\rho B$ if and only if whenever S is a subset of P for which $\sup S$ exists and is in $\uparrow B$, then $S \cap \uparrow A \neq \emptyset$. Let $\rho(x) = \{A : A$ is finite and $A\rho x\}$. A quasicontinuous poset is called a *GCD poset* if and only if for all $x \in P$, $\uparrow x = \bigcap\{\uparrow A : A \in \rho(x)\}$. In [7] it was shown that GCD lattices have several of the pleasing properties of completely distributive complete lattices.

The following lemma, for the case of complete lattices, was proved in [7].

Lemma 1.5. *Let P be a GCD poset. For $x \in P$, $F \in \rho(x)$ implies that $\exists\, G \in \rho(x)$ such that $F\rho G$. If F and G are finite subsets of L such that $F\rho G$, then there exists a subset H of P such that $F\rho H$ and $H\rho G$.*

Proof: Let $x \in P$. Define $\Gamma = \{\uparrow A : A \in \rho(x)$ such that $A\rho B$ and $B \in \rho(x)\}$. First we shall show that Γ is nonempty. Let $B \in \rho(x)$. For each $b \in B$, pick a finite set F_b such that $F_b \in \rho(b)$. Let $A = \bigcup\{F_b : b \in B\}$. It is easy to see that $A\rho B\rho x$. Indeed, if $\sup S \in \uparrow B$, then $\sup S \geq b$ for some $b \in B$. This implies that $S \cap \uparrow F_b \neq \emptyset$; that is $S \cap \uparrow A \neq \emptyset$. Thus $A\rho B$. Next we show that $\bigcap \Gamma = \uparrow x$.

Since $A \in \rho(x)$ implies $x \in \uparrow A$, clearly $\uparrow x \subseteq \bigcap \Gamma$. To prove the reverse inclusion, let $y \in P \backslash \uparrow x$. Then, by the definition of a GCD poset, there exists $B \in \rho(x)$ such that $y \notin \uparrow B$. Similarly for each $b \in B$ pick $A_b \in \rho(b)$ such that $y \notin \uparrow A_b$. If $A = \bigcup\{A_b : b \in B\}$, then $A \rho B \rho x$. Therefore $\uparrow A \in \Gamma$ and $y \notin \uparrow A$. Hence $\bigcap \Gamma \subseteq \uparrow x$.

Let $F \in \rho(x)$. Suppose that there exists no $B \in \rho(x)$ such that $F \rho B \rho x$. For each $H \in \Gamma$, let $\operatorname{gen} H$ denote the set of minimal elements of H. Then the set $\operatorname{gen} H \backslash \uparrow F \neq \emptyset$. Let s_H be any element in $\operatorname{gen} H \backslash \uparrow F$. If $S = \{s_H : H \in \Gamma\}$, then by what was proved in the last paragraph, $\sup S \in \uparrow x$. Therefore there exists s_H such that $s_H \in \uparrow F$. This contradicts the choice of s_H. Therefore there exists $B \in \rho(x)$ such that $F \rho B \rho x$.

Suppose F and H are finite subsets of P such that $F \rho H$. For each $h \in H$, pick $G_h \in \rho(h)$ such that $F \rho G_h \rho h$. If $G = \bigcup\{G_h : h \in H\}$, then $F \rho G \rho H$. This completes the proof of the lemma. □

Theorem 1.6. *If P and P^{op} are posets satisfying property DINT, then the following statements are equivalent.*

(1) *P is a GCD poset.*
(2) *P is a compact pospace with respect to the interval topology.*
(3) *For $x, y \in P$ with $x \not\leq y$, there exist finite subsets F and G of P such that $x \notin \downarrow F$, $y \notin \uparrow G$ and $\downarrow F \bigcup \uparrow G = P$.*

Proof: **(1) $\Longrightarrow$ (2):** Let $x, y \in P$ with $x \not\leq y$. Then, since P is a GCD poset, there exists $F \in \rho(x)$ such that $y \notin \uparrow F$. Let $U = \{z : F \in \rho(z)\}$, and let $V = L \backslash \uparrow F$. Clearly V is open in the interval topology, and $U \cap V = \emptyset$. Also note that U is an upper set and V is a lower set. If we can also show that U is open, then it follows from Theorem 1.1(a) that P is a pospace.

Let $\sigma(P)$ denote the collection of all upper sets of P with the following property: For $S \subseteq P$, $\sup S \in U$ implies $S \cap U \neq \emptyset$. It is easy to verify that the topology generated by $\sigma(P)$ is precisely the upper topology. Therefore, it is enough to show that U has this property. Let $S \subseteq P$ such that $\sup S = s \in U$. Then $F \in \rho(s)$. Therefore, by Lemma 1.5, there exists $G \in \rho(s)$ such that $F \rho G$. Then there exists $t \in S$ such that $g \leq t$ for some $g \in G$. Since $F \in \rho(g)$, it follows that $F \in \rho(t)$; that is $t \in U$. This shows that U is open. This completes the proof that P is a pospace.

Since GCD posets are quasicontinuous, it follows from Lawson [4], that Lawson topology is compact, and since we have shown that the interval topology is Hausdorff, this means that the Lawson topology and the interval topology coincide. Thus P is a compact pospace with respect to the interval topology.

(2) $\Longrightarrow$ (3): Since P is a compact pospace, for any $x, y \in P$ with $x \not\leq y$, there exist an open upper set U and an open lower set V such that $U \cap V = \emptyset$. Since P and P^{op} have property DINT, it follows from Theorem 1.3 that U is open in the upper topology, and V is open in the lower topology. Therefore there exist finite sets F, and G such that $x \in P \backslash \downarrow F \subseteq U$ and $y \in P \backslash \uparrow G \subseteq V$. Since U and V are disjoint, $\downarrow F \cup \uparrow G = P$. Clearly $x \notin \downarrow F$, and $y \notin \uparrow G$.

(3) $\Longrightarrow$ (1): Let $x, y \in P$ such that $x \not\leq y$. Then there exist finite sets F and G such that $x \notin \downarrow F$, $y \notin \uparrow G$, and $\downarrow F \cup \uparrow G = P$. We shall show that $G \rho x$. Suppose S is a subset of P such that $\sup S \in \uparrow x$. That is $\bigcap_{s \in S} \uparrow s \in \uparrow x$. Since $\bigcap_{s \in S} \uparrow s$ is a closed subset in the lower topology, by property DINT, $\bigcap_{s \in S} \uparrow s = \bigcap_{i \in I} \uparrow F_i \in \uparrow x$, where I is directed. Then, by Rudin's Theorem, there exists a directed set D such that $D \cap F_i \neq \emptyset$, and $\sup D \in \uparrow x$. Now, since $x \notin \downarrow F$, $\sup D \notin \downarrow F$. But since D is directed and F is finite this means that $D \not\subseteq \downarrow F$ which implies that $D \cap \downarrow G \neq \emptyset$. This show that $G \rho x$. $\square$

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Department of Mathematics, University of Connecticut, Stamford, Connecticut 06901, USA

E-mail: venu.menon@uconn.edu

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