Non-existence of some canonical constructions on connections

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Abstract. For a vector bundle functor $H : \mathcal{M}f \to \mathcal{VB}$ with the point property we prove that H is product preserving if and only if for any m and n there is an $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$. For a bundle functor $E : \mathcal{FM}_{m,n} \to \mathcal{FM}$ with some weak conditions we prove non-existence of $\mathcal{FM}_{m,n}$-natural operators D transforming connections Γ on (m,n)-dimensional fibered manifolds $Y \to M$ into connections $D(\Gamma)$ on $EY \to M$.

Keywords: (general) connection, natural operator

Classification: 58A20

0. Introduction

We recall that a (general) connection on a fibered manifold $p : Y \to M$ is a smooth section $\Gamma : Y \to J^1Y$ of the first jet prolongation of Y, which can be also interpreted as the lifting map (denoted by the same symbol)

$$\Gamma : Y \times_M TM \to TY\,.$$

Let H be a bundle functor on the category of smooth manifolds and all smooth maps and let $\Gamma : Y \to J^1Y$ be a connection on the fibered manifold $p : Y \to M$. It is well known that if H preserves products, then Γ induces a connection $\mathcal{H}\Gamma$ on $Hp : HY \to HM$. More precisely, there is the canonical flow equivalence $THM = HTM$ and the lifting map of $\mathcal{H}\Gamma$ is of the form

$$\mathcal{H}\Gamma : HY \times_{HM} THM \to THY\,.$$

We recall that the connection $\mathcal{H}\Gamma$ has been constructed by I. Kolář [2] in the case of higher order velocities functors and then by J. Slovák [6] in the general case.

In the present paper we study the non-existence of natural operators D lifting connections Γ on $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$ for non-product preserving vector bundle functors $H : \mathcal{M}f \to \mathcal{VB}$ with the point property $H(pt) = pt$ (pt is a one-point manifold). If H is without the point property, then such D can exist, see [1].

In Section 1, we prove that a vector bundle functor $H : \mathcal{M}f \to \mathcal{VB}$ with the point property is product preserving if and only if for any m and n there is an $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.

In particular, if $H = T^{(2)} = (J^2(.,\mathbb{R})_0)^*$ is the second order vector tangent bundle functor, we get negative answer to the question (formulated by I. Kolář) about the existence of natural operators D transforming connections Γ on fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $T^{(2)}p : T^{(2)}Y \to T^{(2)}M$.

In next sections, for a bundle functor $E : \mathcal{FM}_{m,n} \to \mathcal{FM}$ with some weak condition we prove the non-existence of $\mathcal{FM}_{m,n}$-natural operators D transforming connections Γ on (m,n)-dimensional fibered manifolds $Y \to M$ into connections $D(\Gamma)$ on $EY \to M$. This is a generalization of the result of [3, Proposition 45.9].

Unless otherwise specified, we use the terminology and notation from the book [3]. All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$.

1. The case $HY \to HM$

Let $H : \mathcal{M}f \rightsquigarrow \mathcal{VB}$ be a vector bundle functor with the point property. Let m, n be natural numbers.

Define a natural bundle $F : \mathcal{M}f_m \to \mathcal{FM}$ by

$$FM = H(M \times \mathbb{R}^n) \ \text{ and } \ F\varphi = H(\varphi \times \mathrm{id}_{\mathbb{R}^n})$$

for an $\mathcal{M}f_m$-object M and an $\mathcal{M}f_m$-morphism φ.

If $p_M : M \times \mathbb{R}^n \to M$ is the obvious projection, then p_M is a surjective submersion, so is $H(p_M)$ ([3]) and hence $GM = \ker H(p_M)$ is a regular submanifold. Define a natural bundle $G : \mathcal{M}f_m \to \mathcal{FM}$ by

$$GM = \ker H(p_M) \text{ and } G\varphi = \text{ the restriction of } \ H(\varphi \times \mathrm{id}_{\mathbb{R}^n})$$

for an $\mathcal{M}f_m$-object M and an $\mathcal{M}f_m$-morphism φ.

We have an $\mathcal{M}f_m$-natural equivalence of natural bundles $GM \times_M HM \tilde{=} FM$ given by

$$\Phi(\omega, \tilde{\omega}) = \omega + H(i^y_M)(\tilde{\omega}),$$

where $\omega \in H_{(x,y)}(M \times \mathbb{R}^n) \cap G_x M$, $\tilde{\omega} \in H_x M$, $(x,y) \in M \times \mathbb{R}^n$, $+$ is the sum in the vector space $H_{(x,y)}(M \times \mathbb{R}^n)$ and $i^y_M = (\mathrm{id}_M, y) : M \to M \times \mathbb{R}^n$. The inverse isomorphism is given by $\Phi^{-1}(\omega) = (\omega - H(i^y_M \circ p_M)(\omega), H(p_M)(\omega))$, where $\omega \in H_{(x,y)}(M \times \mathbb{R}^n)$, $(x,y) \in M \times \mathbb{R}^n$.

Proposition 1. *The natural bundle G is of order 0 if and only if $H(\mathbb{R}^{m+n}) = H(\mathbb{R}^m) \times H(\mathbb{R}^n)$ modulo a diffeomorphism, i.e. iff H preserves product in dimension m and n.*

Proof: If the equality holds, then $G_0\mathbb{R}^m = H(\mathbb{R}^n)$ and then G is of order 0. If G is of order 0, then $G_0(\mathbb{R}^m) = H(t\,\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n})(G_0(\mathbb{R}^m))$ for all $t \neq 0$. Putting

$t \to 0$ we obtain $G_0(\mathbb{R}^m) = H(\{0\}\times\mathbb{R}^n) = H(\mathbb{R}^n)$. Then $\dim(H_{(0,0)}(\mathbb{R}^m\times\mathbb{R}^n)) = \dim(H_0(\mathbb{R}^m)) + \dim(H_0(\mathbb{R}^n))$, and Proposition 38.14 in [3] completes the proof. □

Proposition 2. *If G is not of order 0, then there is no $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.*

Proof: Suppose that we have an $\mathcal{FM}_{m,n}$-natural operator D lifting connections Γ on $p : Y \to M$ into connections $D(\Gamma) : HY \times_{HM} THM \to THY$ on $Hp : HY \to HM$. Then we can define a natural operator $A : T_{\mathcal{M}f_m} \rightsquigarrow TG$ by

$$A(X)_\omega = T\,\mathrm{pr}_1(D(\Gamma_M)(\omega, \mathcal{H}X_{0_x})),$$

where $\omega \in G_xM$, $x \in M$, $0_x = 0 \in H_xM$, $\mathcal{H}X$ is the flow lifting of X to HM, $\mathrm{pr}_1 : FM \tilde{=} GM \times_M HM \to GM$ is the obvious projection and Γ_M is the trivial connection on the trivial bundle $p_M : M \times \mathbb{R}^n \to M$.

Since $\mathcal{H}X_{0_x}$ depends only on X_x, A is of order 0.

Since $D(\Gamma_M)$ is a lifting transformation, $A(X)$ covers X. Hence

$$A(X) = \mathcal{G}X + \mathcal{V}(X),$$

where $\mathcal{G}X$ is the flow lifting of X to GM and $\mathcal{V}(X)$ is a vertical type operator $T_{\mathcal{M}f_m} \rightsquigarrow TG$. Clearly, $\mathcal{G}$ is of order $\mathrm{ord}(G) \geq 1$ and not of order $\mathrm{ord}(G) - 1$ and $\mathcal{V}$ is of order $\mathrm{ord}(G) - 1$, see Lemma 1 in [5] (or Appendix of the present paper). So, A is not of order 0, which is a contradiction. □

Thus we have proved the following general fact.

Theorem 1. *A vector bundle functor $H : \mathcal{M}f \to \mathcal{VB}$ with the point property is product preserving if and only if for any m and n there is an $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.*

Any product-preserving vector bundle functor $H : \mathcal{M}f \to \mathcal{VB}$ is equivalent to some vector bundle functor $T^{[s]} : \mathcal{M}f \to \mathcal{VB}$, $T^{[s]}M = TM \otimes \mathbb{R}^s$, $T^{[s]}f = Tf \otimes \mathrm{id}_{\mathbb{R}^s}$, see [3]. So, we have the following classification theorem.

Theorem 1'. *Up to natural equivalence the $T^{[s]}$ for $s = 0, 1, 2, \ldots$ are all vector bundle functors $H : \mathcal{M}f \to \mathcal{VB}$ with the point property such that for any m and n there is an $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.*

Open problem: Our conjecture is that a bundle functor $H : \mathcal{M}f \to \mathcal{FM}$ with the point property is product preserving if and only if for any m and n there is an $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.

2. The case $EY \to M$

Theorem 2. *Let $E : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{E} : \mathcal{M}f_m \to \mathcal{FM}$, $\tilde{E}M = E(M \times \mathbb{R}^n)$, $\tilde{E}\varphi = E(\varphi \times \mathrm{id}_{\mathbb{R}^n})$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $Y \to M$ into connections $D(\Gamma)$ on $EY \to M$.*

Proof: Suppose we have such an $\mathcal{FM}_{m,n}$-natural operator $D(\Gamma)$. Then we can define a natural operator $A : T_{\mathcal{M}f_m} \rightsquigarrow T\tilde{E}$ by

$$A(X)_\omega = D(\Gamma_M)(\omega, X_x),$$

where $\omega \in \tilde{E}_x M$, $x \in M$, X is a vector field on M and Γ_M is the trivial connection on the trivial bundle $p_M : M \times \mathbb{R}^n \to M$.

Then A is of order 0 and $A(X)$ covers X.

This is a contradiction by the same arguments as at the end of the proof of Proposition 2. $\square$

For $E = J^1$ we reobtain Proposition 45.9 from [3] without the order assumption.

Remark 2. The existence of a connection $\mathcal{V}^F\Gamma$ on a vertical bundle $V^F Y \to M$ canonically depending on a connection Γ on $Y \to M$ ([4]) shows that the assumption of Theorem 2 is essential.

3. The case $EY \to Y$

Theorem 3. *Let $E : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{E} : \mathcal{M}f_m \to \mathcal{FM}$, $\tilde{E}M = E(M \times \mathbb{R}^n)$, $\tilde{E}\varphi = E(\varphi \times \mathrm{id}_{\mathbb{R}^n})$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $Y \to M$ into connections $D(\Gamma)$ on $EY \to Y$.*

Proof: Suppose that such $D(\Gamma)$ exists. Composing $D(\Gamma)$ with Γ we obtain a connection on $EY \to M$ canonically dependent on Γ. This contradicts Theorem 2. $\square$

We remark that in [5] we proved the following theorem.

Theorem 4 ([5])**.** *Let $E : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\overline{E} : \mathcal{M}f_n \to \mathcal{FM}$, $\overline{E}N = E(\mathbb{R}^m \times N)$, $\overline{E}\varphi = E(\mathrm{id}_{\mathbb{R}^m} \times \varphi)$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-natural operator D transforming connections Γ on (m,n)-dimensional fibered manifolds $Y \to M$ into connections $D(\Gamma)$ on $EY \to Y$.*

4. Appendix

Because Lemma 1 from [5] is essential in the proof of Proposition 2, we cite this lemma with the proof here for the reader's convenience.

Lemma 1 ([5])**.** *Let $G : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle of order $r \geq 1$. Then any natural operator $\mathcal{V} : T_{\mathcal{M}f_n} \rightsquigarrow TG$ of vertical type is of order $r - 1$.*

PROOF: ([5]) Let $X_1, X_2 \in \mathcal{X}(N)$ be two vector fields with $j^{r-1}_x(X_1) = j^{r-1}_x(X_2)$, $x \in N$. Let $w \in G_x N$. Because of the regularity of $\mathcal{V}$ we can assume that $X_1(x) \neq 0$. There is an x-preserving local diffeomorphism $\varphi : N \to N$ such that $j^r_x \varphi = \mathrm{id}$ and $\varphi_* X_1 = X_2$ near x, see [3]. Then $\mathcal{V}(X_2)(w) = \mathcal{V}(\varphi_* X_1)(w) = TG_x(\varphi) \circ \mathcal{V}(X_1) \circ G_x(\varphi^{-1})(w) = \mathcal{V}(X_1)(w)$ since $G_x(\varphi) = \mathrm{id}$ as G is of order r and $j^r_x \varphi = \mathrm{id}$. □

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(*Received* March 3, 2003, *revised* June 18, 2003)