Best approximations and porous sets

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Abstract. Let D be a nonempty compact subset of a Banach space X and denote by $S(X)$ the family of all nonempty bounded closed convex subsets of X. We endow $S(X)$ with the Hausdorff metric and show that there exists a set $\mathcal{F} \subset S(X)$ such that its complement $S(X) \setminus \mathcal{F}$ is σ-porous and such that for each $A \in \mathcal{F}$ and each $\tilde{x} \in D$, the set of solutions of the best approximation problem $\|\tilde{x} - z\| \to \min$, $z \in A$, is nonempty and compact, and each minimizing sequence has a convergent subsequence.

Keywords: Banach space, complete metric space, generic property, Hausdorff metric, nearest point, porous set

Classification: 41A50, 41A52, 41A65, 54E35, 54E50, 54E52

Given a nonempty closed convex subset A of a Banach space $(X, \|\cdot\|)$ and a point $x \in X$, we consider the minimization problem

$$\min\{\|x - z\| : \ z \in A\}. \tag{P}$$

It is well known that if X is reflexive, then problem (P) always has at least one solution. This solution is unique when X is strictly convex. In this connection we recall that the minimization problem (P) is said to be well posed if it has a unique solution, say a_0, and every minimizing sequence of (P) converges to a_0. It is said to be well posed in the generalized sense of [6] if the set of minimizers is nonempty and compact, and each minimizing sequence has a convergent subsequence.

When X is not reflexive, several generic existence results for best approximation problem are still available [3], [4], [9]. In these papers the authors considered a framework the main feature of which is that the set A in problem (P) can vary. For example, in their recent paper [2], De Blasi, Georgiev and Myjak fix a point $x \in X$ and show, inter alia, that for most bounded closed convex subsets of X, problem (P) is well posed.

In order to formulate their result more precisely, we first set, for each $x \in X$ and each $A \subset X$,

$$\rho(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

We denote by $S(X)$ the family of all nonempty bounded closed convex subsets of X. For each $A, B \in S(X)$, define the Hausdorff distance between A and B by

$$H(A, B) = \max\{\sup\{\rho(x, B) : \ x \in A\}, \ \sup\{\rho(y, A) : \ y \in B\}\}.$$

It is not difficult to see that H is a metric on $S(X)$ and that the metric space $(S(X), H)$ is complete.

We also need to recall the notion of porosity [1], [2], [5], [8], [9], [12]–[14].

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous (with respect to the metric d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called σ-porous (with respect to d) if it is a countable union of porous subsets of Y.

Remark 1. It is known that in the above definition of porosity, the point y can be assumed to belong to E.

Other notions of porosity have been used in the literature [1], [7], [11]–[13]. We use the rather strong notion which appears, for example, in [2], [5], [8], [9], [14].

Fixing $x \in X$ and denoting by F the set of all elements $A \in S(X)$ for which the minimization problem (P) is well posed, De Blasi, Georgiev and Myjak [2] have proved that the complement $S(X) \setminus F$ is σ-porous in $S(X)$. More applications of porosity concepts to best approximation theory can be found in [5], [7], [10], [11] and in the references mentioned there.

In the present paper, instead of $x \in X$ we fix a set $D \subset X$. If D is countable, then the De Blasi-Georgiev-Myjak result implies that the collection of all sets $A \in S(X)$ for which problem (P) is well posed for any $x \in D$, has a σ-porous complement. When the set D is uncountable, the problem is more difficult and less understood. Nevertheless, we are able to prove a partial positive result when D is compact in the norm topology. More precisely, we establish the existence of a set $\mathcal{F} \subset S(X)$ which has a σ-porous complement such that for any $A \in \mathcal{F}$ and any $x \in D$, problem (P) is well posed in the generalized sense of [6].

We are now ready to state and prove our result.

Theorem. *Given a nonempty compact subset D of a Banach space $(X, \|\cdot\|)$, there exists a set $\mathcal{F} \subset S(X)$ such that its complement $S(X) \setminus \mathcal{F}$ is σ-porous in $(S(X), H)$ and such that for each $A \in \mathcal{F}$ and each $\tilde{x} \in D$, the following property holds:*

(C) *The set $\{y \in A : \|\tilde{x} - y\| = \rho(\tilde{x}, A)\}$ is nonempty and compact and each sequence $\{y_i\}_{i=1}^{\infty} \subset A$ which satisfies $\lim_{i\to\infty} \|\tilde{x} - y_i\| = \rho(\tilde{x}, A)$ has a convergent subsequence.*

Proof: Let $(X, \|\cdot\|)$ be a Banach space. Denote by $\mathrm{co}(A)$ and $\mathrm{clco}(A)$ the convex hull and the closed convex hull of A, respectively.

For each $A \subset X$, set

$$\mathrm{rad}(A) = \sup\{\|x\| : x \in A\}. \tag{1}$$

Let D be a nonempty compact subset of X. For each natural number n, denote by $\mathcal{F}_n$ the set of all $A \in S(X)$ which have the following property:

(C1) There exist a nonempty compact set $E \subset A$ and a positive number δ such that for each $u \in D$ and each $z \in A$ satisfying

$$\|u - z\| \leq \rho(u, A) + \delta, \tag{2}$$

the inequality

$$\rho(z, E) \leq 1/n \tag{3}$$

is valid.

Set

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n. \tag{4}$$

Let $A \in \mathcal{F}$. By (4) and property (C1), for each natural number n, there exist a nonempty finite set $E_n \subset A$ and a positive number δ_n such that the following property holds:

(C2) For each $u \in D$ and each $z \in A$ satisfying $\|u - z\| \leq \rho(u, A) + \delta_n$, the inequality $\rho(z, E_n) < 2/n$ holds.

Let $u \in D$. Consider a sequence $\{x_i\}_{i=1}^{\infty} \subset A$ such that $\lim_{i\to\infty} \|u - x_i\| = \rho(u, A)$. By (C2), for each integer $n \geq 1$, there exists a subsequence $\{x_i^{(n)}\}_{i=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that the following two properties hold:

$\{x_i^{(n+1)}\}_{i=1}^{\infty}$ is a subsequence of $\{x_i^{(n)}\}_{i=1}^{\infty}$ for all integers $n \geq 1$;

for any integer $n \geq 1$, $\|x_j^{(n)} - x_k^{(n)}\| < 4/n$ for all integers $j, k \geq 1$.

These properties imply that there exists a subsequence $\{\bar{x}_i\}_{i=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ which is a Cauchy sequence. Therefore $\{\bar{x}_i\}_{i=1}^{\infty}$ converges to a point $\bar{x} \in A$ which satisfies

$$\|u - \bar{x}\| = \lim_{i\to\infty} \|u - x_i\| = \rho(u, A).$$

In other words, for each $A \in \mathcal{F}$ and each $\tilde{x} \in D$, property (C) holds. In order to complete the proof of our theorem, it is sufficient to show that $S(X) \setminus \mathcal{F}$ is σ-porous in $(S(X), H)$. Since

$$S(X) \setminus \mathcal{F} = \bigcup_{n=1}^{\infty} S(X) \setminus \mathcal{F}_n,$$

it suffices to show that $S(X) \setminus \mathcal{F}_n$ is σ-porous in $(S(X), H)$ for each natural number n.

To this end, let n be a natural number. For each natural number m, set

$$\Omega_m = \{A \in S(X) : \ \mathrm{rad}(A) \le m\}. \tag{5}$$

Since

$$S(X) \setminus \mathcal{F}_n = \bigcup_{m=1}^{\infty} \Omega_m \setminus \mathcal{F}_n,$$

it is sufficient to show that for each natural number m, the set $\Omega_m \setminus \mathcal{F}_n$ is porous in $(S(X), H)$.

Let m be a natural number. Choose a positive number α satisfying

$$\alpha < (4 \cdot 48n(2m+1))^{-1}. \tag{6}$$

Let

$$A \in \Omega_m \setminus \mathcal{F}_n \text{ and } r \in (0, 1]. \tag{7}$$

Since D is compact, there are a natural number q and a finite subset $\{\xi_1, \dots, \xi_q\}$ of D such that

$$D \subset \bigcup_{i=1}^{q} \{z \in X : \ \|z - \xi_i\| < \alpha r\}. \tag{8}$$

For each $i \in \{1, \dots, q\}$, we pick $\tilde{\xi}_i \in X$ satisfying

$$\rho(\tilde{\xi}_i, A) \le r/8, \ i = 1, \dots, q, \tag{9}$$

in the following way:

Let $i \in \{1, \dots, q\}$. There are two cases: 1) $\rho(\xi_i, A) \le r/8$; 2) $\rho(\xi_i, A) > r/8$. In the first case we set

$$\tilde{\xi}_i = \xi_i. \tag{10}$$

In the second case, we first choose $x_i \in A$ for which

$$\|\xi_i - x_i\| \le \rho(\xi_i, A) + (64n(2m+1))^{-1} r, \tag{11}$$

and then choose

$$\tilde{\xi}_i \in \{\gamma x_i + (1-\gamma)\xi_i : \ \gamma \in (0, 1)\} \tag{12}$$

such that

$$\|\tilde{\xi}_i - x_i\| = r/8 \ \text{ and } \ \|\tilde{\xi}_i - \xi_i\| = \|x_i - \xi_i\| - r/8. \tag{13}$$

If the second case holds, then by (11) and (13),

$$\|\xi_i - \tilde{\xi}_i\| = \|x_i - \xi_i\| - r/8 \le -r/8 + \rho(\xi_i, A) + r(64n(2m+1))^{-1}. \tag{14}$$

Clearly, (9) holds in both cases.

Let

$$\tilde{A} := \mathrm{clco}(A \cup \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}) \tag{15}$$

and

$$E := \mathrm{co}\{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}. \tag{16}$$

By (9) and (15),

$$H(A, \tilde{A}) \le r/8. \tag{17}$$

We will now show that the following property holds:

(C3) If $u \in D$, $z \in \tilde{A}$, and $\|u - z\| \le \rho(u, \tilde{A}) + 4\alpha r$, then $\rho(z, E) \le n^{-1}$.

Indeed, assume that

$$u \in D,\ z \in \tilde{A}, \quad \text{and} \quad \|u - z\| \le \rho(u, \tilde{A}) + 4\alpha r. \tag{18}$$

We claim that $\rho(z, E) \le n^{-1}$. Assume the contrary. Then

$$\rho(z, E) > n^{-1}. \tag{19}$$

By (8) and (18), there exists $j \in \{1, \dots, q\}$ such that

$$\|u - \xi_j\| < \alpha r. \tag{20}$$

The relations (20) and (18) imply that

$$\begin{aligned}\|\xi_j - z\| &\le \|u - z\| + \|\xi_j - u\| < \alpha r + \rho(u, \tilde{A}) + 4\alpha r \\ &\le 5\alpha r + \rho(\xi_j, \tilde{A}) + \|u - \xi_j\| < 6\alpha r + \rho(\xi_j, \tilde{A}).\end{aligned}$$

Thus

$$\|\xi_j - z\| < 6\alpha r + \rho(\xi_j, \tilde{A}). \tag{21}$$

If $\rho(\xi_j, A) \le r/8$, then by the definition of $\tilde{\xi}_j$ (see (10)), $\xi_j = \tilde{\xi}_j$; now by (21), the inclusion $\tilde{\xi}_j \in \tilde{A}$ (see (15)), and (6),

$$\|\xi_j - z\| < 6\alpha r + \rho(\xi_j, \tilde{A}) = 6\alpha r < n^{-1},$$

and in view of (16),

$$\rho(z, E) \le \|z - \xi_j\| < n^{-1},$$

a contradiction (see (19)). Therefore we have

$$\rho(\xi_j, A) > r/8. \tag{22}$$

Since $z \in \tilde{A}$, it follows from (15) that there exists a sequence

$$\{z_i\}_{i=1}^{\infty} \subset \mathrm{co}(A \cup \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}) \tag{23}$$

such that

$$\|z - z_i\| \to 0 \text{ as } i \to \infty. \tag{24}$$

The relations (23), (16) and the convexity of A imply that for each natural number i, there exist

$$\lambda_i \in [0, 1], \ z_{i1} \in A, \quad \text{and} \quad z_{i2} \in E \tag{25}$$

such that

$$z_i = \lambda_i z_{i1} + (1 - \lambda_i) z_{i2}. \tag{26}$$

It follows from (25), (19), (26), (7), (5), (16), (1) and (9) that for each natural number i,

$$\begin{aligned} n^{-1} < \|z - z_{i2}\| &\le \|z - z_i\| + \|z_i - z_{i2}\| = \|z - z_i\| + \lambda_i \|z_{i1} - z_{i2}\| \\ &\le \|z - z_i\| + \lambda_i (\|z_{i1}\| + \|z_{i2}\|) \\ &\le \|z - z_i\| + \lambda_i (2m + 1), \end{aligned}$$

and by (24),

$$n^{-1} \le \liminf_{i \to \infty} (\|z - z_i\| + \lambda_i (2m + 1)) = (2m + 1) \liminf_{i \to \infty} \lambda_i.$$

Thus

$$\liminf_{i \to \infty} \lambda_i \ge ((2m + 1)n)^{-1}. \tag{27}$$

Let Δ be any positive number. By (25), (16) and (9), for each natural number i, there exists $\tilde{z}_{i2} \in X$ such that

$$\tilde{z}_{i2} \in A \text{ and } \|z_{i2} - \tilde{z}_{i2}\| \le r/8 + \Delta. \tag{28}$$

It follows from (25), (26) and (28) that for each natural number i,

$$\lambda_i z_{i1} + (1-\lambda_i)\tilde{z}_{i2} \in A$$

and

$$\begin{aligned}\rho(\xi_j, A) &\le \|\xi_j - (\lambda_i z_{i1} + (1-\lambda_i)\tilde{z}_{i2})\| \\ &\le \|\xi_j - (\lambda_i z_{i1} + (1-\lambda_i) z_{i2})\| + \|(1-\lambda_i)(z_{i2} - \tilde{z}_{i2})\| \\ &= \|\xi_j - z_i\| + (1-\lambda_i)\|z_{i2} - \tilde{z}_{i2}\| \le \|\xi_j - z_i\| + (1-\lambda_i)(r/8 + \Delta).\end{aligned}$$

By these relations, (24) and (27),

$$\begin{aligned}\rho(\xi_j, A) &\le \liminf_{i\to\infty}(\|\xi_j - z_i\| + (1-\lambda_i)(r/8+\Delta)) = \|\xi_j - z\| \\ &+ \liminf_{i\to\infty}(1-\lambda_i)(r/8+\Delta) \le \|\xi_j - z\| + (r/8+\Delta)(1-((2m+1)n)^{-1}).\end{aligned} \tag{29}$$

By (29), (21), (15), (22), the definition of $\tilde{\xi}_j$ (see (12), (13)) and (14),

$$\begin{aligned}\rho(\xi_j, A) &\le \|\xi_j - z\| + (8^{-1}r + \Delta)(1-((2m+1)n)^{-1}) \\ &< (8^{-1}r + \Delta)(1-((2m+1)n)^{-1}) + 6\alpha r + \rho(\xi_j, \tilde{A}) \\ &\le (8^{-1}r + \Delta)(1-((2m+1)n)^{-1}) + 6\alpha r + \|\xi_j - \tilde{\xi}_j\| \\ &\le (8^{-1}r + \Delta)(1-((2m+1)n)^{-1}) \\ &\quad + 6\alpha r - r/8 + \rho(\xi_j, A) + r((2m+1)64n)^{-1}.\end{aligned}$$

Since Δ is an arbitrary positive number, we conclude that

$$\begin{aligned}0 &\le (1-((2m+1)n)^{-1}) + 48\alpha - 1 + ((2m+1)8n)^{-1} \\ &= 48\alpha - ((2m+1)n)^{-1} + ((2m+1)8n)^{-1} \le 48\alpha - ((2m+1)2n)^{-1},\end{aligned}$$

a contradiction (see (6)). The contradiction we have reached proves that $\rho(z, E) \le n^{-1}$, as claimed. Thus property (C3) holds true.

Now assume that

$$B \in S(X),\ H(B, \tilde{A}) \le \alpha r, \tag{30}$$

and

$$u \in D,\ z \in B,\ \|u - z\| \le \rho(u, B) + \alpha r. \tag{31}$$

By (30) and (31),

$$\rho(u, B) \le \rho(u, \tilde{A}) + H(B, \tilde{A}) \le \rho(u, \tilde{A}) + \alpha r \tag{32}$$

and there is $z_1 \in X$ such that

$$z_1 \in \tilde{A} \text{ and } \|z_1 - z\| \le 2\alpha r. \tag{33}$$

The relations (31), (33) and (32) imply that

$$\|u - z_1\| \le \|u - z\| + \|z - z_1\| \le \rho(u, B) + \alpha r + 2\alpha r \le \rho(u, \tilde{A}) + 4\alpha r.$$

Thus

$$\|u - z_1\| \le \rho(u, \tilde{A}) + 4\alpha r. \tag{34}$$

It now follows from (31), (33), (34) and property (C3) that $\rho(z, E) \le n^{-1}$. Hence $B \in \mathcal{F}_n$. The inequalities (30), (17) and (6) imply that

$$H(A, B) \le H(B, \tilde{A}) + H(\tilde{A}, A) \le \alpha r + r/8 \le r.$$

In other words, we have shown that each B satisfying (30) belongs to $\mathcal{F}_n$ and satisfies $H(A, B) \le r$. Therefore

$$\{B \in S(X) : \ H(B, \tilde{A}) \le \alpha r\} \subset \{B \in S(X) : \ H(B, A) \le r\} \cap \mathcal{F}_n$$

and the set $\Omega_m \setminus \mathcal{F}_n$ is porous in $(S(X), H)$. This completes the proof of our theorem. $\square$

Since every σ-porous set is of the first category, the following corollary is an immediate consequence of our theorem.

Corollary. *Given a nonempty compact subset D of a Banach space $(X, \|\cdot\|)$, there exists a residual set $\mathcal{F} \subset S(X)$ such that for each $A \in \mathcal{F}$ and each $\tilde{x} \in D$, the following property holds:*

The set $\{y \in A : \|\tilde{x} - y\| = \rho(\tilde{x}, A)\}$ is nonempty and compact and each sequence $\{y_i\}_{i=1}^{\infty} \subset A$ which satisfies $\lim_{i\to\infty} \|\tilde{x} - y_i\| = \rho(\tilde{x}, A)$ has a convergent subsequence.

A natural question which can be raised at this point is whether analogous results hold in a nonconvex setting. A theorem in this direction can indeed be found in [10]. However, the setting there is different; it is concerned with complete hyperbolic spaces, closed (not necessarily bounded) subsets, and with the notion of porosity with respect to a pair of metrics. At first sight, the result of [10] seems to be more general. Indeed, here we have considered only Banach spaces (a special case of complete hyperbolic spaces), porosity with respect to only one metric, and the subclass of bounded closed convex subsets. But sometimes it is more difficult to establish a generic result for a subspace than for the whole space. This is because one of the most crucial points in the proofs of generic results is the construction of a small perturbation of a given element. Thus in [10], when we perturb a given set, we need only to make sure that the perturbed set is closed, while in the present paper we need to ensure that it is also convex. It may also be of interest to mention that we do not know if the theorem of the present paper can be extended to closed convex, not necessarily bounded, subsets. Also, it seems that its proof cannot be modified so as to apply to all complete hyperbolic spaces.

Acknowledgments. The work of the first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund. Both authors are grateful to the referee for providing them with a detailed report and several helpful comments.

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(*Received* September 10, 2002, *revised* July 9, 2003)