On the uniqueness of loops $M(G, 2)$

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Abstract. Let G be a finite group and C_2 the cyclic group of order 2. Consider the 8 multiplicative operations $(x, y) \mapsto (x^i y^j)^k$, where $i, j, k \in \{-1, 1\}$. Define a new multiplication on $G \times C_2$ by assigning one of the above 8 multiplications to each quarter $(G \times \{i\}) \times (G \times \{j\})$, for $i, j \in C_2$. If the resulting quasigroup is a Bol loop, it is Moufang. When G is nonabelian then exactly four assignments yield Moufang loops that are not associative; all (anti)isomorphic, known as loops $M(G, 2)$.

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1. Introduction

Because of the specialized topic of this paper, we assume that the reader is familiar with the theory of Bol and Moufang loops (cf. [6)]).

Chein introduced the following construction in [1] to obtain Moufang loops from groups: Let G be a finite group and let $\overline{G} = \{\overline{x};\ x \in G\}$ be a set of new elements. Define multiplication $*$ on $G \cup \overline{G}$ by

$$x * y = xy, \quad x * \overline{y} = \overline{yx}, \quad \overline{x} * y = \overline{xy^{-1}}, \quad \overline{x} * \overline{y} = y^{-1}x, \tag{1}$$

where $x, y \in G$. The resulting Moufang loop $M(G, 2)$ is associative if and only if G is abelian, according to [1].

Loops $M(G, 2)$ play an important role among Moufang loops of small order (cf. [1], [5]). Recently, it was found that all Moufang loops of order $n \in \{16, 24, 32\}$ can be obtained by modifying one quarter of the multiplication tables of loops $M(G, 2)$ in a certain way [4]. The smallest nonassociative Moufang loop is isomorphic to $M(S_3, 2)$, where S_3 is the symmetric group on 3 points (cf. [3], [7]).

We are going to study a generalization of Chein's construction (1). Given a group G, consider the 8 multiplicative operations on G: $(x, y) \mapsto (x^i y^j)^k$, where $i, j, k \in \{-1, 1\}$. Let C_2 be the cyclic group of order 2. Define a new multiplication on $G \times C_2$ by assigning one of the above 8 multiplications to each quarter $(G \times \{i\}) \times (G \times \{j\})$, for $i, j \in C_2$. Let M be the resulting quasigroup.

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In this note, we characterize when M is a loop (Lemma 1); we show that if M is a Bol loop, it is Moufang (Lemma 2); and we prove that when G is nonabelian then there are exactly 4 assignments that yield nonassociative Moufang loops, all (anti)isomorphic to the loop $M(G,2)$. See Theorem 6 for details.

Chein's construction (1) is therefore unique, in a sense.

2. Notation

Let us introduce a notation that will better serve our purposes. Consider the permutations ι, σ, τ of $G\times G$ defined by $(x,y)\iota=(x,y)$, $(x,y)\sigma=(y,x)$, and $(x,y)\tau=(y^{-1},x)$. Since $\sigma^2=\tau^4=\iota$ and $\sigma\tau\sigma=\tau^{-1}$, the group A generated by σ and τ is isomorphic to Q_8, the quaternion group of order 8. The elements ψ of A are described by

$$\begin{array}{c|cccccccc} \psi & \iota & \sigma & \tau & \tau^2 & \tau^3 & \sigma\tau & \sigma\tau^2 & \sigma\tau^3 \\ (x,y)\psi & (x,y) & (y,x) & (y^{-1},x) & (x^{-1},y^{-1}) & (y,x^{-1}) & (x^{-1},y) & (y^{-1},x^{-1}) & (x,y^{-1}) \end{array}.$$

We like to think of these elements as multiplications in G, and often identify $\psi\in A$ with the map $\psi\Delta: G\times G\to G$, where $(x,y)\Delta=xy$. For instance, the permutation $\sigma\tau$ determines the multiplication $x*y=x^{-1}y$. Note that $\sigma\Delta=\iota\Delta$ when G is abelian, and that $A\Delta=\iota\Delta$ when G is an elementary abelian 2-group.

To avoid trivialities, we assume throughout the paper that G *is not an elementary abelian* 2*-group, and that* $|G|>1$.

It is natural to split the multiplication table of $M(G,2)$ into four quarters $G\times G$, $G\times\overline{G}$, $\overline{G}\times G$ and $\overline{G}\times\overline{G}$, as in

$$\begin{array}{c|cc} * & G & \overline{G} \\ \hline G & & \\ \overline{G} & & \end{array}.$$

Then Chein's construction (1) can be represented by the matrix

$$M_c=\begin{pmatrix} \iota & \sigma \\ \sigma\tau^3 & \tau \end{pmatrix}. \tag{2}$$

For example, we can see from M_c that $\overline{x}*y=\overline{(x,y)\sigma\tau^3}=\overline{xy^{-1}}$, for $x,y\in G$.

3. Main result

When we look at Chein's construction (1) via (2), it appears to be somewhat arbitrary. Let us therefore investigate all multiplications

$$M=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{3}$$

where $\alpha, \beta, \gamma, \delta \in A$. We will no more distinguish between the matrix M and the groupoid it defines.

We note in passing that every M is a quasigroup. The next lemma characterizes all loops M. In the course of the proof we encounter several identities of the form $w_1 = w_2$, where w_i is a word in some symbols $x_1, \dots, x_m \in G$. When w_1, w_2 reduce to the same word in the free group on $x_1, \dots, x_m$, then $w_1 = w_2$ surely holds in G. Conversely, since we assumed that G is not an elementary abelian 2-group and $|G| > 1$, there are many identities that do not hold in G, no matter what G is. For instance, $x \neq x^{-1}$, $y \neq xy^{-1}x^{-1}$ (set $x = y$), and so on.

Lemma 1. *M is a loop if and only if $\alpha \in \{\iota, \sigma\}$, $\beta \in \{\iota, \sigma, \tau^3, \sigma\tau\}$ and $\gamma \in \{\iota, \sigma, \tau, \sigma\tau^3\}$. When M is a loop, its neutral element coincides with the neutral element of G.*

Proof: We first show that if M is a loop, its neutral element e coincides with the neutral element 1 of G. This is clear, as for some $\varepsilon \in A$ we have $1 = 1 * e = (1,\, e)\varepsilon \in \{e,\, e^{-1}\}$, and thus $1 = e$.

The equation $y = 1 * y$ holds for every $y \in G$ if and only if $y = (1, y)\alpha$, which happens if and only if $\alpha \in \{\iota, \sigma, \tau^3, \sigma\tau\}$. Similarly, the equation $y = y * 1$ holds for every $y \in G$ if and only if $\alpha \in \{\iota, \sigma, \tau, \sigma\tau^3\}$. Altogether, $y = y * 1 = 1 * y$ holds for every $y \in G$ if and only if $\alpha \in \{\iota, \sigma\}$.

Following the same strategy, $\overline{y} = 1 * \overline{y}$ holds for every $y \in G$ if and only if $\beta \in \{\iota, \sigma, \tau^3, \sigma\tau\}$, and $\overline{y} = \overline{y} * 1$ holds for every $y \in G$ if and only if $\gamma \in \{\iota, \sigma, \tau, \sigma\tau^3\}$. $\square$

Once M is a loop, it must have two-sided inverses:

Lemma 2. *If M is a loop then it is an inverse property loop. In particular, if M happens to be a Bol loop, it must be Moufang.*

Proof: Assume that $x * y = 1$ for some $x, y \in G \cup \overline{G}$. Then both x, y belong to G, or both belong to $\overline{G}$, by Lemma 1. We therefore want to show that $(x, y)\varepsilon = 1$ implies $(y, x)\varepsilon = 1$ for every $\varepsilon \in A$ and $x, y \in G$.

Pick $\varepsilon \in A$. Then $(x, y)\varepsilon = (x^i y^j)^k$ for some $i, j, k \in \{-1,\ 1\}$. Assume that $(x, y)\varepsilon = 1$. Then $x^i y^j = 1$ and $y^j x^i = 1$. If $i = j$, we conclude from the latter equality that $y^i x^j = 1$, and thus $(y, x)\varepsilon = 1$. The inverse of the former equality yields $y^{-j}x^{-i} = 1$. If $i = -j$, we immediately have $y^i x^j = 1$, and thus $(y, x)\varepsilon = 1$.

Hence M is an inverse property loop. It is well-known that a Bol loop is Moufang if and only if it is an inverse property loop (cf. [2]). $\square$

Given M as in (3), let

$$M^{\mathrm{op}} = \begin{pmatrix} \sigma\alpha & \sigma\gamma \\ \sigma\beta & \sigma\delta \end{pmatrix}.$$

Lemma 3. *The quasigroup M^{op} is opposite to M.*

Proof: Denote by $\circ$ the multiplication in M^{op}. Then

$$\begin{aligned} x\circ y &= (x,y)\sigma\alpha = (y,x)\alpha = y*x,\\ x\circ \overline{y} &= \overline{(x,y)\sigma\gamma} = \overline{(y,x)\gamma} = \overline{y}*x,\\ \overline{x}\circ y &= \overline{(x,y)\sigma\beta} = \overline{(y,x)\beta} = y*\overline{x},\\ \overline{x}\circ \overline{y} &= (x,y)\sigma\delta = (y,x)\delta = \overline{y}*\overline{x}, \end{aligned}$$

for every $x,y\in G$. □

Let us assume from now on that G *is nonabelian*. Then the identity $xy=yx$ and any other identity that reduces to $xy=yx$ do not hold in G, of course. We will come across the identity $xxy=yxx$. Note that this identity holds in G if and only if the center of G is of index 2 in G.

We would like to know when M is a Bol (and hence Moufang) loop. Assume from now on that M is a loop.

Recall that the opposite of a Moufang loop is again Moufang. We can therefore combine Lemmas 1, 3 and assume that the loop M satisfies $\alpha=\iota$. Since every Moufang loop is diassociative, we are going to have a look at such loops:

Lemma 4. *If G is nonabelian and M is a diassociative loop with $\alpha=\iota$ then (β,γ,δ) is one of the eight triples*

$$\begin{array}{llll} (\iota,\iota,\iota), & (\tau^3,\iota,\sigma\tau), & (\sigma,\sigma,\sigma), & (\sigma\tau,\sigma,\tau^3),\\ (\tau^3,\tau,\tau^2), & (\iota,\tau,\sigma\tau^3), & (\sigma,\sigma\tau^3,\tau), & (\sigma\tau,\sigma\tau^3,\sigma\tau^2)\ . \end{array} \tag{4}$$

Proof: The identities $(\overline{x}*\overline{x})*y=\overline{x}*(\overline{x}*y)$, $\overline{x}*(y*\overline{x})=(\overline{x}*y)*\overline{x}$ hold in M, for every $x,y\in G$. They translate into

$$(x,x)\delta y = (x,(x,y)\gamma)\delta, \tag{5}$$

$$(x,(y,x)\beta)\delta = ((x,y)\gamma,x)\delta, \tag{6}$$

respectively. We are first going to check which pairs (γ,δ) satisfy (5).

Assume that $\gamma=\iota$. Then (5) becomes $(x,x)\delta y=(x,xy)\delta$. Denote this identity by $I(\delta)$. Then $I(\iota)$ is $xxy=xxy$ (true), $I(\sigma)$ is $xxy=xyx$ (false), $I(\tau)$ is $y=y^{-1}$ (false), $I(\tau^2)$ is $x^{-2}y=x^{-1}y^{-1}x^{-1}$ (false), $I(\tau^3)$ is $y=xyx^{-1}$ (false), $I(\sigma\tau)$ is $y=y$ (true), $I(\sigma\tau^2)$ is $x^{-2}y=y^{-1}x^{-1}x^{-1}$ (false), and $I(\sigma\tau^3)$ is $y=xy^{-1}x^{-1}$ (false).

Assume that $\gamma=\sigma$. Then (5) becomes $(x,x)\delta y=(x,yx)\delta$. Verify that this identity holds only if $\delta=\sigma$ or $\delta=\tau^3$. (The case $\delta=\sigma$ leads to the identity $xxy=yxx$ mentioned before this lemma.)

When $\gamma=\tau$, (5) holds only if $\delta=\tau^2$ or $\delta=\sigma\tau^3$.

When $\gamma=\sigma\tau^3$, (5) holds only if $\delta=\tau$ or $\delta=\sigma\tau^2$.

Altogether, (5) can be satisfied only when (γ,δ) is one of the 8 pairs (ι,ι), $(\iota,\sigma\tau)$, (σ,σ), (σ,τ^3), (τ,τ^2), $(\tau,\sigma\tau^3)$, $(\sigma\tau^3,\tau)$, $(\sigma\tau^3,\sigma\tau^2)$. All these pairs will now be tested on (6).

Straightforward calculation shows that (6) can be satisfied only when (β,γ,δ) is one of the 8 triples listed in (4). □

The *Moufang identity* $((xy)x)z=x(y(xz))$ will help us eliminate 4 out of the 8 possibilities in (4). We have $((x*\overline{y})*x)*z=x*(\overline{y}*(x*z))$ in M, and thus

$$(((x,y)\beta,x)\gamma,z)\gamma=(x,(y,xz)\gamma)\beta. \tag{7}$$

The pairs $(\beta,\gamma)=(\sigma,\sigma)$, (τ^3,ι), (ι,τ), $(\sigma\tau,\sigma\tau^3)$ do not satisfy (7). For instance, $(\beta,\gamma)=(\sigma,\sigma)$ turns (7) into $zxyx=xzyx$, i.e., $zx=xz$.

The four remaining triples from (4) yield Moufang loops, as we are going to show.

The quadruple $(\alpha,\beta,\gamma,\delta)=(\iota,\iota,\iota,\iota)=G_\iota$ corresponds to the direct product of G and the two-element cyclic group. The quadruple $(\iota,\sigma,\sigma\tau^3,\tau)=M_c$ is the Chein Moufang loop $M(G,2)$ that is associative if and only if G is abelian, by [1]. (We can also verify this directly.)

Set $G_\tau=(\iota,\tau^3,\tau,\tau^2)$ and $M_\sigma=(\iota,\sigma\tau,\sigma,\tau^3)$. We claim that G_ι is isomorphic to G_τ, and M_c is isomorphic to M_σ.

Lemma 5. *Define* $T:A^4\to A^4$ *by*

$$M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\mapsto\begin{pmatrix}\alpha & \tau^3\beta\\ \gamma\tau & \tau^2\delta\end{pmatrix}=MT.$$

If $((x,y)\beta\Delta)^{-1}=(y^{-1},x^{-1})\beta\Delta$ *and* $((x,y)\gamma\Delta)^{-1}=(x^{-1},y)\gamma\tau\Delta$ *then* M *is isomorphic to* MT.

Proof: Consider the permutation f of $G\cup\overline{G}$ defined by $f(x)=x$, $f(\overline{x})=\overline{x^{-1}}$, for $x\in G$. Let $*$ be the multiplication in M and $\circ$ the multiplication in MT. We show that $(x*y)f=xf\circ yf$ for every $x,y\in G\cup\overline{G}$. With $x,y\in G$, we have

$$\begin{aligned}(x*y)f&=(x,y)\alpha\Delta f=(x,y)\alpha\Delta=x\circ y=xf\circ yf,\\ (\overline{x}*\overline{y})f&=(x,y)\delta\Delta f\ =(x,y)\delta\Delta\ =(x^{-1},y^{-1})\tau^2\delta\Delta=\overline{x}f\circ\overline{y}f.\end{aligned}$$

Using the assumption on β and γ, we also have

$$(x*\overline{y})f=\overline{(x,y)\beta\Delta}f=\overline{((x,y)\beta\Delta)^{-1}}=\overline{(y^{-1},x^{-1})\beta\Delta}=\overline{(x,y^{-1})\tau^3\beta\Delta}=xf\circ\overline{y}f,$$

and

$$(\overline{x} * y)f = \overline{(x,y)\gamma\Delta} f = \overline{((x,y)\gamma\Delta)^{-1}} = \overline{(x^{-1},y)\gamma\tau\Delta} = \overline{x}f \circ yf. \qquad \square$$

Note that $G_\iota T = G_\tau$ and $M_c T = M_\sigma$. Now, $\beta \in \{\iota, \sigma\}$ satisfies $((x,y)\beta\Delta)^{-1} = (y^{-1}, x^{-1})\beta\Delta$, and $\gamma \in \{\iota, \sigma\tau^3\}$ satisfies $((x,y)\gamma\Delta)^{-1} = (x^{-1}, y)\gamma\tau\Delta$. By Lemma 5, G_ι is isomorphic to G_τ, and M_c is isomorphic to M_σ.

We have proved:

Theorem 6. *Let G with $|G| > 1$ be a finite group that is not an elementary abelian 2-group. With the above conventions, let*

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

specify the multiplication in $L = G \cup \overline{G}$, where $\alpha, \beta, \gamma, \delta \in A = \langle \sigma, \tau \rangle$, and $(x,y)\sigma = (y,x)$, $(x,y)\tau = (y^{-1}, x)$. If L is a Bol loop then it is Moufang.

When G is nonabelian, then L is a Bol loop if and only if M is equal to one of the following matrices:

$$G_\iota = \begin{pmatrix} \iota & \iota \\ \iota & \iota \end{pmatrix}, \qquad G_\iota^{\mathrm{op}} = \begin{pmatrix} \sigma & \sigma \\ \sigma & \sigma \end{pmatrix},$$

$$G_\tau = \begin{pmatrix} \iota & \tau^3 \\ \tau & \tau^2 \end{pmatrix}, \qquad G_\tau^{\mathrm{op}} = \begin{pmatrix} \sigma & \sigma\tau \\ \sigma\tau^3 & \sigma\tau^2 \end{pmatrix},$$

$$M_c = \begin{pmatrix} \iota & \sigma \\ \sigma\tau^3 & \tau \end{pmatrix}, \quad M_c^{\mathrm{op}} = \begin{pmatrix} \sigma & \tau^3 \\ \iota & \sigma\tau \end{pmatrix},$$

$$M_\sigma = \begin{pmatrix} \iota & \sigma\tau \\ \sigma & \tau^3 \end{pmatrix}, \quad M_\sigma^{\mathrm{op}} = \begin{pmatrix} \sigma & \iota \\ \tau & \sigma\tau^3 \end{pmatrix}.$$

The loops X^{op} are opposite to the loops X. The isomorphic loops G_ι, G_τ and their opposites are groups. The isomorphic loops M_c, M_σ and their opposites are Moufang loops that are not associative.

Even when G is abelian, it turns out there are no additional matrices M besides those listed in Theorem 6 that yield Bol loops. The proof of this observation is not long and can be found in [8].

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