Cardinal characteristics of the ideal of Haar null sets

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Abstract. We calculate the cardinal characteristics of the σ-ideal $\mathcal{HN}(G)$ of Haar null subsets of a Polish non-locally compact group G with invariant metric and show that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathfrak{b} \le \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \le \mathrm{non}(\mathcal{HN}(G)) \le \mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$. If $G = \prod_{n\ge 0} G_n$ is the product of abelian locally compact groups G_n, then $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$, $\mathrm{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ and $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{N})$, where $\mathcal{N}$ is the ideal of Lebesgue null subsets on the real line. Martin Axiom implies that $\mathrm{cof}(\mathcal{HN}(G)) > 2^{\aleph_0}$ and hence G contains a Haar null subset that cannot be enlarged to a Borel or projective Haar null subset of G. This gives a negative (consistent) answer to a question of S. Solecki. To obtain these estimates we show that for a Polish non-locally compact group G with invariant metric the ideal $\mathcal{HN}(G)$ contains all o-bounded subsets (equivalently, subsets with the small ball property) of G.

Keywords: Polish group, Haar null set, Martin Axion, cardinal characteristics of an ideal, o-bounded set, the small ball property

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A subset N of a topological group G is called *Haar null* if it is contained in a universally measurable set $B \subset G$ for which there exists a σ-additive Borel probability measure μ on G such that $\mu(gBh) = 0$ for all $g, h \in G$ (a subset B of a topological space X is *universally measurable* if it is measurable with respect to any Borel σ-additive probability measure on X). The family $\mathcal{HN}(G)$ of Haar null subsets of a Polish group G is closed under translations, taking subsets and countable unions, see [THJ, 2.4.5]. The notion of Haar null sets is a natural extension of the notion of sets of Haar measure zero: if G happens to be locally compact, then Haar null sets are precisely the sets of Haar measure zero. Since the publication of Christensen's paper [C] who introduced this new notion, Haar null sets have found many applications, see [BL], [PZ].

In this paper we estimate the principal cardinal characteristics of the σ-ideal $\mathcal{HN}(G)$ of Haar null subsets of a Polish group G. There is nothing surprising about $\mathcal{HN}(G)$ if the group G is locally compact and non-discrete. In this case the ideal $\mathcal{HN}(G)$ is isomorphic to the σ-ideal $\mathcal{N}$ of Lebesgue null subsets of the real line $\mathbb{R}$ in the sense that there is a Borel isomorphism $h : G \to \mathbb{R}$ such that a subset $A \subset G$ belongs to $\mathcal{HN}(G)$ if and only if $h(A) \in \mathcal{N}$ (this follows from the classical

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theorem on isomorphism of Borel measure spaces, see [Ke, 17.41]). Consequently, for a non-discrete locally compact Polish group G the σ-ideals $\mathcal{HN}(G)$ and $\mathcal{N}$ have the same cardinal characteristics. Let us remind their definitions, see [V].

Given a σ-ideal $\mathcal{I}$ of subsets of a set X let

$$\begin{aligned}
&\mathrm{add}(\mathcal{I}) = \min\big\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \bigcup\mathcal{J} \notin \mathcal{I}\big\};\\
&\mathrm{cov}(\mathcal{I}) = \min\big\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \bigcup\mathcal{J} = X\big\};\\
&\mathrm{non}(\mathcal{I}) = \min\big\{|A| : A \subset X \text{ and } A \notin \mathcal{I}\big\};\\
&\mathrm{cof}(\mathcal{I}) = \min\big\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \mathcal{I} = \{A \subset X : \exists\, E \in \mathcal{J} \text{ with } A \subset E\}\big\}.
\end{aligned}$$

It is easy to see that these cardinals are related as follows:

$$\aleph_1 \le \mathrm{add}(\mathcal{I}) \le \min\{\mathrm{non}(\mathcal{I}), \mathrm{cov}(\mathcal{I})\} \le \max\{\mathrm{non}(\mathcal{I}), \mathrm{cov}(\mathcal{I})\} \le \mathrm{cof}(\mathcal{I}).$$

It follows from the famous Cichoń diagram (see [V], [BS]) that $\aleph_1 \le \mathrm{add}(\mathcal{N}) \le \mathfrak{b} \le \mathfrak{d} \le \mathrm{cof}(\mathcal{N})$, where $\mathfrak{b}$ and $\mathfrak{d}$ are two well-known small cardinals introduced by E. van Douwen in his seminal paper [vD]. Since for any non-discrete locally compact Polish group G the cardinal characteristics of the ideals $\mathcal{HN}(G)$ and $\mathcal{N}$ coincide, we get

$$\aleph_1 \le \mathrm{add}(\mathcal{HN}(G)) \le \mathfrak{b} \le \mathfrak{d} \le \mathrm{cof}(\mathcal{HN}(G)).$$

In [S$_2$, 3.4] S. Solecki proved that the same estimates hold also for any non-locally compact Polish group G with invariant metric. There is however one crucial difference between locally compact and non-locally compact cases: for a Polish non-locally compact group the cardinal $\mathrm{cof}(\mathcal{HN}(G))$ always exceeds $\aleph_1$. Moreover, under Martin Axiom, it exceeds the size of continuum. Thus for a non-locally compact Polish group G the σ-ideal $\mathcal{HN}(G)$ differs substantially from other classical ideals whose cardinal characteristics lie between $\aleph_1$ and $\mathfrak{c}$ (and thus fall into the category of so-called small cardinals). Unlike to the cofinality $\mathrm{cof}(\mathcal{HN}(G))$, the other cardinal characteristics of the σ-ideal $\mathcal{HN}(G)$ behave not so wildly and for some special groups (like $\mathbb{R}^\omega$ or $\mathbb{Z}^\omega$) they can be expressed via known small cardinals $\mathfrak{b}$, $\mathfrak{d}$, $\mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{N})$, and $\mathrm{non}(\mathcal{N})$.

To calculate the cardinal characteristics of the ideal $\mathcal{HN}(G)$ for a non-locally compact Polish group G with invariant metric we shall prove that for such a group G the ideal $\mathcal{HN}(G)$ contains the σ-ideal $o\mathcal{B}(G)$ of o-bounded subsets of G. Following O. Okunev and M. Tkachenko [Tk, 3.9] we define a subset B of a topological group G to be *o-bounded* if for any sequence $(U_n)_{n\ge0}$ of neighborhoods of the neutral element of G there is a sequence $(F_n)_{n\ge0}$ of finite subsets of G such that $B \subset \bigcup_{n\ge0} F_nU_n$ (this is equivalent to saying that there is a sequence $(F_n)_{n\ge0}$ of finite subsets of G with $B \subset \bigcap_{k\ge0}\bigcup_{n\ge k} F_nU_n$, see [HRT, 2.7]). Recently o-bounded sets attracted a lot of attention, see [Tk2], [HRT], [Her], [Ba1], [Ba2],

[BNS], [Ts]. It should be mentioned that in Banach space theory they are known as sets with the small ball property, i.e., sets which can be covered by a sequence of small balls whose radii tend to zero, see [BK]. It is easy to see that the family $o\mathcal{B}(G)$ of all o-bounded subsets of a topological group G forms a σ-ideal containing all compact subsets of G.

Our main instrument in estimation of cardinal characteristics of the ideal $\mathcal{HN}(G)$ is

Theorem 1. *Let G be a non-locally compact Polish group.*

1. *If G admits an invariant metric, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$;*
2. *If $G = \prod_{n\geq 0} G_n$ is the countable product of locally compact groups, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$;*
3. *For a continuous homomorphism $h : G \to H$ onto a non-discrete (locally compact) Polish group H, a subset $A \subset H$ is Haar null (if and) only if its preimage $h^{-1}(H)$ is Haar null in G, which implies that $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(H))$ and $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(H))$.*

The first statement of Theorem 1 generalizes the result of Dougherty [D] who proved that for a Polish non-locally compact group with invariant metric the ideal $\mathcal{HN}(G)$ contains all compact subsets of G (for abelian G this fact was proven by Christensen [C]). Theorem 1 will help us to make the following estimations of the cardinal characteristics of the ideal $\mathcal{HN}(G)$.

Theorem 2. *Suppose G is a non-discrete Polish group.*

1. *If G is locally compact, then $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{HN}(G)) = \mathrm{non}(\mathcal{N})$, and $\mathrm{cof}(\mathcal{HN}(G)) = \mathrm{cof}(\mathcal{N})$.*
2. *If G is not locally compact and has an invariant metric, then $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathfrak{b}$, $\mathrm{non}(\mathcal{HN}(G)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ and $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.*
3. *If G contains a closed normal subgroup H such that either H or G/H is locally compact and not discrete, then $\mathrm{add}(\mathcal{HN}(G)) \leq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{N})$.*
4. *If the center $Z = \{g \in G : \forall x \in G\ gx = xg\}$ of G is not locally compact, then $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(Z)) \leq \mathfrak{b}$, and $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(Z)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.*
5. *If G admits a surjective continuous homomorphism onto a non-locally compact group H with invariant metric, then $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(H)) \leq \mathfrak{b}$, and $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(H)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.*

For linear complete metric spaces, Theorem 2(2),(3) implies

Corollary 1. *If X is an infinite-dimensional linear complete metric space, then*

1. $\mathrm{add}(\mathcal{HN}(X)) \le \mathrm{add}(\mathcal{N})$;
2. $\mathrm{cov}(\mathcal{HN}(X)) \le \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$;
3. $\mathrm{non}(\mathcal{HN}(X)) \ge \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$;
4. $\mathrm{cof}(\mathcal{N}) \le \mathrm{cof}(\mathcal{HN}(X)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

For groups G which are countable products of locally compact amenable groups the first three inequalities of Corollary 1 can be reversed. Haar null subsets in such groups were characterized by S. Solecki [S]. We remind that a locally compact group G is *amenable* if it admits a left invariant mean on the space $L^\infty(G)$ of all essentially bounded complex functions measurable with respect to the Haar measure. It is well-known [Pa, 4.10] that a locally compact group G endowed with a left-invariant Haar measure μ is amenable if and only if it satisfies the Følner condition: for any $\varepsilon > 0$ and any compact subset $C \subset G$ there is a compact subset $K \subset G$ such that $\mu(xK \triangle K) < \varepsilon\mu(K)$ for all $x \in C$. The class of amenable locally compact groups contains all abelian (and even exponentially bounded) locally compact groups, see [Pa, Chapter 6].

Another class containing all abelian groups is the class of groups admitting a finitely supported kaleidoscopical measure. A probability measure λ on a topological group G is called *kaleidoscopical* if there is a partition $G = A_1 \cup \cdots \cup A_n$ of G into $n > 1$ λ-measurable pieces such that $\mu(xA_iy) = \frac{1}{n}$ for every $i \le n$ and all $x, y \in G$. Groups admitting a kaleidoscopical finitely supported measure will be called *kaleidoscopical*, cf. [BP, §8]. We shall say that a group G is *almost kaleidoscopical* if for any $\varepsilon > 0$ there is a finitely supported probability measure μ on G and a partition $G = A_1 \cup \cdots \cup A_n$, $n > 1$, such that $\left|\mu(xA_iy) - \frac{1}{n}\right| < \frac{\varepsilon}{n}$ for all $x, y \in G$ and $i \le n$.

The following result proved in [BP, §8] shows that the class of (almost) kaleidoscopical groups is quite large. We recall that a topological group G is a *SIN-group* (abbreviated from "Small Invariant Neighborhoods") if it has a neighborhood base $\mathcal{B}$ at the unit such that $gUg^{-1} = U$ for any $U \in \mathcal{B}$ and $g \in G$. It is well-known that each first countable SIN-group admits an invariant metric and that a topological group is a SIN-group if it is *totally bounded* in the sense that for any neighborhood $U \subset G$ of the unit there is a finite subset $F \subset G$ with $G = UF = FU$.

Proposition 1 ([BP, §8]). 1. *A group admitting a homomorphism onto an* (*almost*) *kaleidoscopical group is* (*almost*) *kaleidoscopical.*

2. *A group G is kaleidoscopical provided G admits a homomorphism onto a group containing a finite non-trivial normal subgroup.*
3. *A group G is almost kaleidoscopical provided G admits a homomorphism onto a topological SIN-group containing a totally bounded non-trivial normal subgroup.*

Question 1. *Is every (amenable) group almost kaleidoscopical?*

Now we can give some estimates of cardinal characteristics of the ideal $\mathcal{HN}(G)$ for Polish groups which are products of locally compact groups.

Theorem 3. *Suppose that a Polish non-locally compact group $G = \prod_{n\geq 0} G_n$ is the countable product of locally compact groups G_n. Then*

1. $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathfrak{b}$, $\mathrm{non}(\mathcal{HN}(G)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$, *and* $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$;
2. *if all but finitely many groups G_n are amenable, then* $\mathrm{add}(\mathcal{HN}(G)) \geq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \geq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$ *and* $\mathrm{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$;
3. *if some group G_n is non-discrete or infinitely many of the groups G_n are almost kaleidoscopical, then* $\mathrm{add}(\mathcal{HN}(G)) \leq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{N})$, *and* $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{N})$;
4. *if G is abelian, then* $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$, $\mathrm{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$, *and* $\mathrm{cof}(\mathcal{N}) \leq \mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

The strict inequality $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ together with Martin Axiom has very strange consequences displaying a striking difference between properties of the σ-ideal $\mathcal{HN}(G)$ in the locally compact and non-locally compact cases.

It is well-known that any subset of zero Haar measure in a locally compact Polish group can be enlarged to a G_δ-set of zero Haar measure. In [S, p. 208] S. Solecki asked if the same is true for Haar null subsets in non-locally compact groups. We shall show that the answer to this question is negative under Martin Axiom. More precisely, in each non-locally compact Polish group we shall find a universally null subset that cannot be enlarged to a σ-projective Haar null (more generally, 2-Zorn) set.

Following [Ke, 39.15] we call a subset A of a Polish space X *σ-projective* if it belongs to the smallest σ-algebra $\sigma\mathbf{P}(X)$ containing X and such that the image $f(A)$ of any set $A \in \sigma\mathbf{P}(X)$ under a continuous map $f : A \to X$ belongs to $\sigma\mathbf{P}(X)$. The σ-algebra $\sigma\mathbf{P}(X)$ contains all analytic and consequently all Borel subsets of X.

Generalizing the notion of a Zorn set [PZ] let us call a subset Z of a group G a *κ-Zorn set*, where κ is a cardinal, if $G \neq F \cdot Z$ for any subset $F \subset G$ of size $|F| \leq \kappa$. It is clear that each Haar null subset of a Polish group G is κ-Zorn for any $\kappa < \mathrm{cov}(\mathcal{HN}(G))$. The family of all κ-Zorn subsets of a topological G group will be denoted by $\mathcal{Z}_\kappa(G)$. By $\mathcal{UN}(G)$ we denote the ideal of all universally null subsets of G (a subset $N \subset G$ is universally null if it has zero measure with respect to any Borel non-atomic measure on G). Denote by $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G))$ the smallest size $|\mathcal{Z}|$ of a family $\mathcal{Z} \subset \mathcal{Z}_2(G)$ of 2-Zorn subsets of G such that each universally null subset of G lies in some set $Z \in \mathcal{Z}$. Since $\mathcal{UN}(G) \subset \mathcal{HN}(G) \subset \mathcal{Z}_2(G)$ we get $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) \le \mathrm{cof}(\mathcal{HN}(G))$.

It is well-known that Martin Axiom implies $\mathfrak{b} = \mathfrak{d} = \mathrm{add}(\mathcal{N}) = \mathfrak{c}$, where $\mathfrak{c}$ is the size of continuum.

Theorem 4. *Let G be a Polish non-locally compact group.*

1. *If $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$, then $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d}$.*
2. *If $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ (which holds under Martin Axiom), then the group G contains a universally null (and thus Haar null) subset that cannot be enlarged to a σ-projective Haar null (more generally, 2-Zorn) subset of G.*

It should be mentioned that the strict inequality $\mathfrak{c} < \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G))$ from Theorem 4 cannot be proven in ZFC. According to [La] there is a model of ZFC in which $2^{\aleph_1} = \mathfrak{c}$ and each universally null set has size $\le \aleph_1$. In this model $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) \le \mathrm{cof}(\mathcal{UN}(G)) \le \mathfrak{c}^{\aleph_1} = \mathfrak{c}$.

Problem 1. *Is the inequality $\mathrm{cof}(\mathcal{HN}(G)) \le \mathfrak{c}$ consistent with ZFC for some Polish non-locally compact group G?*

Assuming Martin Axiom we get $\mathrm{add}(\mathcal{N}) = \mathfrak{b} = \mathrm{non}(\mathcal{N}) = \mathfrak{c}$ and thus $\mathrm{non}(\mathcal{HN}(G)) = \mathfrak{c} < \mathrm{cof}(\mathcal{HN}(G))$ for any Polish non-locally compact group G.

Problem 2. *Let G be a nondiscrete Polish group (with invariant metric). Is $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{cov}(\mathcal{HN}(G)) = \mathfrak{c}$ under MA or PFA?*

Two topological groups G, H are called *Haar null isomorphic* if there is a bijection $h : G \to H$ such that a subset $N \subset G$ is Haar null in G if and only if $h(N)$ is Haar null in H. It follows from Isomorphism Theorem for non-atomic measure spaces [Ke, 17.41] that any two non-discrete locally compact Polish groups are Haar null isomorphic. On the other hand, the failure of the countable chain condition for the ideal $\mathcal{HN}(G)$ in the non-locally compact case [S_1] implies that a Polish locally compact group cannot be Haar null isomorphic to a Polish non-locally compact group with invariant metric.

Problem 3. *Are there two Polish non-locally compact groups (with invariant metric) that fail to be Haar null isomorphic? In particular, is the Hilbert space ℓ^2 Haar null isomorphic to $\mathbb{R}^\omega$ or $\mathbb{Z}^\omega$? Have the ideals $\mathcal{HN}(\ell^2)$ and $\mathcal{HN}(\mathbb{R}^\omega)$ the same cardinal characteristics?*

Cardinal characteristics of the ideal $o\mathcal{B}(G)$

In this section we shall estimate the cardinal characteristics of the ideal $o\mathcal{B}(G)$ of o-bounded sets in a Polish non-locally compact group G with invariant metric. First we remind the definition of the small cardinals $\mathfrak{b}$ and $\mathfrak{d}$. For two functions $f, g \in \mathbb{N}^\omega$ we write $f \le^* g$ if $f(n) \le g(n)$ for all sufficiently large n. A subset $B \subset \mathbb{N}^\omega$ is called

- *bounded* in $\mathbb{N}^\omega$ if there is $f \in \mathbb{N}^\omega$ such that $g \le^* f$ for all $g \in B$;

- *dominating* if for any $f \in \mathbb{N}^\omega$ there is $g \in B$ with $f \le^* g$.

By definition, $\mathfrak{b}$ is the smallest size of an unbounded subset of $\mathbb{N}^\omega$ while $\mathfrak{d}$ is the smallest size of a dominating subset of $\mathbb{N}^\omega$, see [vD] or [V].

It is well-known (and easily seen) that the family $\mathcal{B}$ (resp. $\mathcal{ND}$) of bounded (resp. non-dominating) subsets of $\mathbb{N}^\omega$ forms a σ-ideal. As we shall see, the ideal $\mathcal{ND}$ is closely related to the ideal $o\mathcal{B}(G)$ while $\mathcal{B}$ is related to the σ-ideal $\mathcal{B}(G)$ generated by compact subsets of G.

Lemma 1. *If G is a Polish non-locally compact group, then* $\mathrm{cov}(o\mathcal{B}(G)) \le \mathrm{cov}(\mathcal{ND}) = \mathfrak{b} \le \mathfrak{d} = \mathrm{non}(\mathcal{ND}) \le \mathrm{non}(o\mathcal{B}(G))$.

Proof: To prove the lemma we shall construct a function $\psi : G \to \mathbb{N}^\omega$ such that for any non-dominating subset $D \subset \mathbb{N}^\omega$ the set $\psi^{-1}(D)$ is o-bounded in G. Fix a decreasing neighborhood base $(U_n)_{n\ge 0}$ at the unit of the group G and a countable dense subset $\{a_k\}_{k\in\omega}$ of G. Define a function $\psi : G \to \mathbb{N}^\omega$ assigning to each $x \in G$ the function $y \in \mathbb{N}^\omega$ such that $y(n)$ is the smallest number with $x \in a_{y(n)}U_n$. We claim that the map $\psi : G \to \mathbb{N}^\omega$ satisfies our requirements.

Fix any non-dominating subset $D \subset \mathbb{N}^\omega$ and consider the preimage $\psi^{-1}(D) \subset G$. To show that $\psi^{-1}(D)$ is o-bounded in G, fix any sequence $(W_n)_{n\ge 0}$ of neighborhoods of the origin of G. By induction construct an increasing function $f : \omega \to \omega$ such that $U_{f(n)} \subset W_n$ for all $n \in \omega$. Since D is not dominating, there is an increasing function $y \in \mathbb{N}^\omega$ such that $y \not\le^* z$ for all $z \in D$. Take any function $g \in \mathbb{N}^\omega$ such that $\min\{g(i) : f(k) \le i < f(k+1)\} \ge y(f(k+1))$ for every $k \ge 0$.

For every $n \ge 0$ let $F_n = \{a_k : k \le g(n)\}$. We claim that $\psi^{-1}(D) \subset \bigcup_{n\ge 0} F_nU_n$. Assuming the converse find a point $x \in \psi^{-1}(D) \setminus \bigcup_{n\ge 0} F_nU_n$. Consider the function $z = \psi(x) \in D$. It follows from the definition of ψ that $z(i) > g(i)$ for all $i \ge 0$. Let us show that $z(i) \ge y(i)$ for all $i \ge f(0)$. Indeed, given such an i, find $k \ge 0$ with $f(k) \le i < f(k+1)$ and observe that $z(i) \ge g(i) \ge y(f(k+1)) \ge y(i)$. Thus $y \le^* z \in D$ which contradicts the choice of y.

It follows from the property of the function ψ that $\mathrm{cov}(o\mathcal{B}(G)) \le \mathrm{cov}(\mathcal{ND})$ and $\mathrm{non}(o\mathcal{B}(G)) \ge \mathrm{non}(\mathcal{ND})$. To complete the proof it rests to note that $\mathrm{cov}(\mathcal{ND}) = \mathfrak{b}$ and $\mathrm{non}(\mathcal{ND}) = \mathfrak{d}$. To establish these equalities observe that a subset $D \subset \mathbb{N}^\omega$ is not dominating if and only if $D \subset \{x \in \mathbb{N}^\omega : f \not\le^* x\}$ for some $f \in \mathbb{N}^\omega$. $\square$

In the sequel we shall also need some information concerning cardinal characteristics of the σ-ideal $\mathcal{B}(G)$ generated by compact subsets of a topological group G.

Lemma 2. *Suppose G is a Polish non-locally compact group. Then* $\mathrm{add}(\mathcal{B}(G)) = \mathrm{add}(\mathcal{B}) = \mathfrak{b} = \mathrm{non}(\mathcal{B}) = \mathrm{non}(\mathcal{B}(G))$ *and* $\mathrm{cov}(\mathcal{B}(G)) = \mathrm{cov}(\mathcal{B}) = \mathfrak{d} = \mathrm{cof}(\mathcal{B}) = \mathrm{cof}(\mathcal{B}(G))$.

Proof: Let $\bar{G}$ be any metrizable compactification of G and $f : K \to \bar{G}$ be a continuous surjective map from a zero-dimensional compact space. Consider the preimage $f^{-1}(G)$ and by Zorn Lemma find a minimal closed subset $Z \subset f^{-1}(G)$ with $f(Z) = G$. Then Z, being Polish, zero-dimensional and nowhere locally compact, is homeomorphic to $\mathbb{N}^\omega$ according to the Aleksandrov-Urysohn Theorem [Ke, 7.7]. Since the map $f|Z$ is proper (that is the preimages of compact subsets are compact) we get that the space G is the image of the space $\mathbb{N}^\omega$ under a continuous proper map $\pi : \mathbb{N}^\omega \to G$.

Call a subset of G *σ-bounded* if it lies in a σ-compact subset of G. Observe that a subset $B \subset \mathbb{N}^\omega$ lies in a σ-compact subset of $\mathbb{N}^\omega$ if and only if it is bounded in the sense of the pre-order $\leq^*$. Consequently, for any bounded subset A of $(\mathbb{N}^\omega, \leq^*)$ the image $\pi(A)$ is σ-bounded in G and for any σ-bounded subset $B \subset G$ the preimage $\pi^{-1}(B)$ is bounded in $(\mathbb{N}^\omega, \leq^*)$. This observation together with known equalities $\mathrm{add}(\mathcal{B}) = \mathfrak{b} = \mathrm{non}(\mathcal{B})$ and $\mathrm{cov}(\mathcal{B}) = \mathfrak{d} = \mathrm{cof}(\mathcal{B})$ allow us to conclude that $\mathrm{add}(\mathcal{B}(G)) = \mathrm{add}(\mathcal{B}) = \mathfrak{b} = \mathrm{non}(\mathcal{B}) = \mathrm{non}(\mathcal{B}(G))$ and $\mathrm{cov}(\mathcal{B}(G)) = \mathrm{cov}(\mathcal{B}) = \mathfrak{d} = \mathrm{cof}(\mathcal{B}) = \mathrm{cof}(\mathcal{B}(G))$. □

Finally let us prove another useful lemma which probably belongs to the mathematical folklore.

Lemma 3. *Let $\mathcal{F}$ be a family of universally measurable subsets of a Polish space X. If $|\mathcal{F}| < \mathrm{add}(\mathcal{N})$, then the union $\bigcup \mathcal{F}$ is universally measurable in X.*

Proof: Fix any finite Borel measure μ. We have to show that the union $\bigcup \mathcal{F}$ is μ-measurable. Let $C = \{x \in X : \mu(\{x\}) > 0\}$. It is clear that the set C is at most countable and thus Borel. Consider the discrete measure $\nu = \sum_{x \in C} \mu(\{x\})\delta_x$ where δ_x is the Dirac measure concentrated at x. Then $\eta = \mu - \nu$ is a non-atomic measure. Since each subset of X is ν-measurable, it suffices to show that the set $\bigcup \mathcal{F}$ is η-measurable. That is so if $\eta = 0$. So we consider the case of non-trivial measure η. Multiplying η by a suitable constant we may assume that η is a probability measure. Then by Isomorphism Theorem for non-atomic probability measures [Ke, 17.41] the measure η is equivalent to the Lebesgue measure λ on $[0, 1]$. Hence we may assume that $X = [0, 1]$ and $\eta = \lambda$. Let $\lambda_*(\bigcup \mathcal{F}) = \sup\{\lambda(S) : S \subset \cup \mathcal{F} \text{ is } \sigma\text{-compact}\}$ and find a σ-compact subset $S \subset \bigcup \mathcal{F}$ with $\lambda(S) = \lambda_*(\bigcup \mathcal{F})$. Then $\lambda(B) = 0$ for any measurable subset $B \subset \bigcup \mathcal{F} \setminus S$. It follows that $\lambda(F \setminus S) = 0$ for each $F \in \mathcal{F}$. Since $|\mathcal{F}| < \mathrm{add}(\mathcal{N})$ we conclude $\lambda(\bigcup \mathcal{F} \setminus S) = 0$ which implies that $\bigcup \mathcal{F} = S \cup (\bigcup \mathcal{F} \setminus S)$ is λ-measurable. □

Proof of Theorem 1

We divide the proof of Theorem 1 into three lemmas.

Lemma 4. *If G is a Polish non-locally compact group with invariant metric, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$.*

Proof: Fix any complete invariant metric d on the group G. Since G is not locally compact, no non-empty open subset of G is totally bounded. Using this observation we can inductively construct a sequence $(\varepsilon_n)_{n\geq 0} \subset (0,1]$ of positive reals such that for every $n \geq 0$ the ε_n-ball $B(\varepsilon_n) = \{x \in G : d(x,0) < \varepsilon_n\}$ around the origin of G fails to have a finite $6\varepsilon_{n+1}$-net. Then $\varepsilon_{n+1} \leq \frac{1}{6}\varepsilon_n \leq \frac{1}{6^n}$ for all n. By the invariance of the metric d we get $F \cdot B(\varepsilon_n) = B(\varepsilon_n) \cdot F$ for any finite subset $F \subset G$.

To show that $o\mathcal{B}(G) \subset \mathcal{HN}(G)$, fix any o-bounded subset $B \subset G$ and find a sequence $(F_n)_{n\geq 0}$ of finite subsets of G such that $B \subset \bigcap_{k\geq 0} \bigcup_{n\geq k} F_n B(\varepsilon_n)$. Observe that the set $M = \bigcap_{k\geq 0} \bigcup_{n\geq k} F_n B(\varepsilon_n)$ is Borel in G.

Using the fact that the ε_n-ball $B(\varepsilon_n)$ admits no finite $6\varepsilon_{n+1}$-net, for every $n \geq 0$ fix a finite subset $D_n \subset B(\varepsilon_n)$ of size $|D_n| = 2^{n+1}|F_{n+2}|$ which is $6\varepsilon_{n+1}$-separated in the sense that $d(x,y) \geq 6\varepsilon_{n+1}$ for any distinct points $x, y \in D_n$.

Let $D = \bigcup_{n\geq 0} \prod_{k\leq n} D_k$ and let $\bar{D} = \prod_{k\geq 0} D_k$ be the infinite product endowed with the Tychonov product topology. Consider the map $\psi : D \to G$ assigning to each finite sequence $(x_0, \dots, x_n) \in D$ the product $x_0 \cdots x_n$ in G. Also let $\varphi : \bar{D} \to G$ be the continuous map assigning to each infinite sequence $(x_n)_{n\geq 0}$ the limit $\lim_{n\to\infty} x_0 \cdots x_n$ of the sequence $(x_0 \cdots x_n)_{n\geq 0}$. It can be shown that for any distinct sequences $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$ in $\bar{D}$ with $k = \min\{n \in \omega : x_n \neq y_n\}$ we get $d(\varphi(x), \varphi(y)) \geq 6\varepsilon_{k+1} - 2\sum_{i=k+1}^{\infty} \varepsilon_i > 2\varepsilon_{k+1}$. This implies that for any $g, h \in G$ and any $k \geq 1$ the preimage $\varphi^{-1}(gB(\varepsilon_{k+1})h)$ is small in the sense that there is a finite sequence $(x_0, \dots, x_{k-1}) \in D$ such that for any $y \in \varphi^{-1}(gB(\varepsilon_{k+1})h)$ we get $y_i = x_i$ for all $i < k$.

Let $\lambda = \bigotimes_{n\geq 0} \lambda_n$ be the tensor product of probability counting measures λ_n on D_n (i.e., $\lambda_n(A) = |A|/|D_n|$ for $A \subset D_n$) and μ be the image of the measure λ under the map φ (i.e., $\mu(A) = \lambda(\varphi^{-1}(A))$ for a Borel subset $A \subset G$).

We claim that $\mu(gMh) = 0$ for each $g, h \in G$. For this we note that for any $g, h \in G$ and $k \geq 1$ we get $\mu(gB(\varepsilon_{k+1})h) = \lambda\big(\varphi^{-1}(gB(\varepsilon_{k+1})h)\big) \leq \big(\prod_{i<k} |D_i|\big)^{-1}$. Consequently, $\mu(gF_{k+1}B(\varepsilon_{k+1})h) \leq |F_{k+1}|\big(\prod_{i<k} |D_i|\big)^{-1} \leq \frac{|F_{k+1}|}{|D_{k-1}|} = \frac{1}{2^k}$ and

$$\mu(gMh) \leq \mu\Big(\bigcup_{i>k} gF_iB(\varepsilon_i)h\Big) \leq \sum_{i>k} \frac{1}{2^{i-1}} = \frac{1}{2^{k-1}}\,.$$

Sending k to ∞ we get $\mu(gMh) = 0$, which means that B lies in the Haar null G_δ-subset M of G. □

Lemma 5. *If $\pi : G \to H$ is a continuous surjective homomorphism from a Polish group G onto a non-discrete (locally compact) Polish group H, then a subset $A \subset H$ is Haar null (if and) only if its preimage $\pi^{-1}(A)$ is Haar null in G, which implies* $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(H))$ *and* $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(H))$.

Proof: To prove the "only if" part assume that a subset A is Haar null in H. Without loss of generality, A is universally measurable in H. Then its preimage

$\pi^{-1}(A)$ is universally measurable in G. Fix any probability measure μ on H with $\mu(xAy) = 0$ for all $x, y \in H$ and find any probability measure η on G that maps onto μ by the homomorphism π (the existence of such a measure η follows from the Jankov, von Neumann Uniformization Theorem [Ke, 18.1]). Then for any $x, y \in G$ we get $\eta(x\pi^{-1}(A)y) = \eta\big(\pi^{-1}(\pi(x)A\pi(y))\big) = \mu(\pi(x)A\pi(y)) = 0$, which means that $\pi^{-1}(A)$ is Haar null.

To prove that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(H))$ take any cover $\mathcal{C}$ of H by Haar null sets with $|\mathcal{C}| = \mathrm{cov}(\mathcal{HN}(H))$ and observe that $\pi^{-1}(\mathcal{C}) = \{\pi^{-1}(C) : C \in \mathcal{C}\}$ is a cover of G by Haar null sets, which yields $\mathrm{cov}(\mathcal{HN}(G)) \le |\pi^{-1}(\mathcal{C})| \le \mathrm{cov}(\mathcal{HN}(H))$.

To prove that $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H))$ take any subset $A \subset G$ of size $|A| < \mathrm{non}(\mathcal{HN}(H))$. Then $|\pi(A)| < \mathrm{non}(\mathcal{HN}(H))$ and hence $\pi(A)$ is Haar null in H while $\pi^{-1}(\pi(A)) \supset A$ is Haar null in G. Thus $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H))$.

To prove the "if" part, suppose that the group H is locally compact and $A \subset H$ is such that $\pi^{-1}(A)$ is Haar null in G. Let λ denote a left invariant Haar measure on H. We should show that $\lambda(A) = 0$. The set $\pi^{-1}(A)$, being Haar null, is contained in a universally measurable subset $M \subset G$ for which there exists a probability measure μ with compact support on G such that $\mu(xMy) = 0$ for all $x, y \in G$. Find a locally finite Borel measure η on G that maps onto the Haar measure λ by the homomorphism π. Consider the convolution $\eta * \mu$ of the measures η and μ, i.e., a measure assigning to a continuous function $f : G \to \mathbb{R}$ the integral $\int_\eta \int_\mu f(xy)\,dx\,dy$. It follows from the Fubini Theorem that $\eta * \mu(M) = 0$, see [THJ, 2.4.4].

Let us show that the homomorphism π maps the measure $\eta * \mu$ onto the Haar measure λ. Indeed, given a Borel subset $B \subset H$ denote by $\chi_B : H \to \{0, 1\}$ the characteristic function of the set B and applying the Fubini Theorem conclude that

$$
\begin{aligned}
\eta * \mu(\pi^{-1}(B)) &= \int_\eta \int_\mu \chi_B \circ \pi(xy)\,dx\,dy \\
&= \int_\mu \int_\eta \chi_B \circ \pi(xy)\,dy\,dx = \int_\mu \eta\big(\pi^{-1}(\pi(x^{-1})B)\big)\,dx \\
&= \int_\mu \lambda\big(\pi(x^{-1})B\big)\,dx = \int_\mu \lambda(B)\,dx = \lambda(B).
\end{aligned}
$$

Since $\eta * \mu(M) = 0$ there is a σ-compact set $S \subset G \setminus M$ such that $\eta * \mu(G \setminus S) = 0$. Then $\lambda(H \setminus \pi(S)) = 0$ and hence $\lambda(A) = 0$ since $A \cap \pi(S) = \emptyset$. $\square$

Lemma 6. *If a non-locally compact Polish group $G = \prod_{n \ge 0} G_n$ is the product of locally compact groups, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$.*

Proof: We remind that the *modular function* on a locally compact group H endowed with a left invariant Haar measure λ is a unique homomorphism $\triangle$:

$H \to \mathbb{R}_+$ into the multiplicative group of positive real numbers such that $\lambda(Bx) = \triangle(x)\lambda(B)$ for any $x \in H$ and a Borel subset $B \subset H$, see [He, §1.2] or [Za, §4]. A locally compact group H is *unimodular* if its modular function is constant (this is equivalent to saying that any left invariant Haar measure on H is right invariant).

To prove that $o\mathcal{B}(G) \subset \mathcal{HN}(G)$ fix any o-bounded subset $B \subset G = \prod_{n\ge 0} G_n$. Without loss of generality, we can assume that all the groups G_n are not compact.

If infinitely many groups G_n fail to be unimodular, then the Polish abelian group $H = \prod_{n\ge 0} G_n/\operatorname{Ker}(\triangle_n)$ is not locally compact and consequently, the group G admits a continuous homomorphism $\pi : G \to H$ onto the Polish non-locally compact abelian group H. By [Tk, 3.10] the set $\pi(B)$ is o-bounded in H. Since the abelian group H has invariant metric we may apply Lemma 4 to conclude that $o\mathcal{B}(H) \subset \mathcal{HN}(H)$ and thus the set $\pi(B)$ is Haar null in H. Applying Lemma 5 we get that the preimage $\pi^{-1}(\pi(B)) \supset B$ is Haar null in G.

Now consider the case when almost all the groups G_n are unimodular. Without loss of generality, we can assume that the groups G_n are unimodular for all $n \ge 1$. For every $n \ge 0$ fix a left invariant Haar measure λ_n on the locally compact group G_n and a neighborhood $W_n \subset G_n$ of the unit, having compact closure in G_n. Let $U_n = \{(x_i)_{i\ge 0} \in G : x_i \in W_i \text{ for } i \le n\}$, $n \ge 1$. Using the o-boundedness of the set B find a sequence $(F_n)_{n\ge 1}$ of finite subsets of the group G such that $B \subset \bigcap_{k\ge 1}\bigcup_{n\ge k} F_nU_n$. Note that the set $M = \bigcap_{k\ge 1}\bigcup_{n\ge k} F_nU_n$ is Borel and hence universally measurable. We claim that it is Haar null in G.

To find a suitable measure μ on G, for every $n \ge 0$ fix a compact subset $K_n \subset G_n$ with $\lambda_n(K_n) \ge 2^n|F_n|\lambda_n(W_n)$ (such a set K_n exists since G_n is not compact and the measure λ_n is unbounded). Next, consider the probability measure μ_n on G_n defined by $\mu_n(B) = \frac{\lambda_n(B\cap K_n)}{\lambda_n(K_n)}$ for a Borel subset $B \subset G_n$. Finally consider the tensor product $\mu = \bigotimes_{n\ge 0}\mu_n$ of the measures μ_n.

We claim that $\mu(xMy) = 0$ for any $x, y \in G$. For this notice that by the invariance of the measures λ_n, for every $n \ge 1$ we get

$$\mu(xF_nU_ny) \le |F_n|\frac{\lambda_n(W_n)}{\lambda_n(K_n)} \le \frac{1}{2^n}\,.$$

Consequently, for every $k \ge 0$

$$\mu(xMy) \le \mu\Big(\bigcup_{n\ge k} xF_nU_ny\Big) \le \sum_{n\ge k}\frac{1}{2^n} = \frac{1}{2^{k-1}}\,.$$

Sending k to ∞ we get $\mu(xMy) = 0$ which means that $M \supset B$ is Haar null in G. □

Proof of Theorem 2

Suppose that G is a non-discrete Polish group.

1. If G is locally compact, then a subset $A \subset G$ is Haar null if and only if A has zero measure with respect to a left-invariant Haar measure λ on G, see [THJ, p. 374]. Replace the Haar measure λ by a Borel probability measure μ equivalent to λ in the sense that $\mu(B) = 0$ for a Borel subset $B \subset G$ if and only if $\lambda(B) = 0$. By Theorem [Ke, 17.41] on the isomorphism of measure spaces, there is a Borel isomorphism $f : G \to [0,1]$ such that for any Borel subset $B \subset G$ $\mu(B) = \tau(f(B))$ where τ is the Lebesgue measure on $[0,1]$. This shows that the ideal $\mathcal{HN}(G)$ is isomorphic to the ideal $\mathcal{N}$ of Lebesgue null subsets of $[0,1]$ and consequently these ideals have the same cardinal characteristics, i.e., $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{HN}(G)) = \mathrm{non}(\mathcal{N})$, and $\mathrm{cof}(\mathcal{HN}(G)) = \mathrm{cof}(\mathcal{N})$.

2. Suppose that G is not locally compact and admits an invariant metric. The inclusion $o\mathcal{B}(G) \subset \mathcal{HN}(G)$ and estimates $\mathrm{cov}(o\mathcal{B}(G)) \le \mathfrak{b}$, $\mathrm{non}(o\mathcal{B}(G)) \ge \mathfrak{d}$ proved in Lemmas 4 and 1 imply $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(o\mathcal{B}(G)) \le \mathfrak{b}$ and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(o\mathcal{B}(G)) \ge \mathfrak{d}$. The estimate $\mathrm{non}(\mathcal{N}) \le \mathrm{non}(\mathcal{HN}(G))$ follows from the inclusion $\mathcal{UN}(G) \subset \mathcal{HN}(G)$ and the well-known equality $\mathrm{non}(\mathcal{UN}) = \mathrm{non}(\mathcal{N})$ (holding because of the Isomorphism Theorem for non-atomic measure spaces [Ke, 17.41]). Therefore $\mathrm{cov}(\mathcal{HN}(G)) \le \mathfrak{b} \le \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \le \mathrm{non}(\mathcal{HN}(G)) \le \mathrm{cof}(\mathcal{HN}(G))$.

To show that $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ we first prove that $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$ implies $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d}$ (this will be used for the proof of Theorem 4).

Assuming that $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$ and $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) \le \mathfrak{d}$, fix a family $\{Z_\alpha\}_{\alpha<\mathfrak{d}}$ of 2-Zorn subsets of G such that each universally null subsets of G lies in Z_α for some ordinal $\alpha < \mathfrak{d}$ (as usual, we identify cardinals with initial ordinals). Since $\mathrm{cof}(\mathcal{B}(G)) = \mathrm{cof}(\mathcal{B}) = \mathfrak{d}$, we can also fix a family $\{C_\alpha\}_{\alpha<\mathfrak{d}}$ of σ-compact subsets of G such that each σ-compact sets C lies in some C_α.

Let us show that for any ordinal $\alpha < \mathfrak{d}$ we get $G \ne Z_\alpha \cup \big(\bigcup_{\beta\le\alpha} C_\alpha\big)$. Let $S_\alpha = \bigcup_{\beta\le\alpha} C_\beta$ and consider the set $S_\alpha \cdot S_\alpha^{-1} = \bigcup_{\beta,\gamma\le\alpha} C_\beta \cdot C_\gamma^{-1}$ which is the union of $< \mathfrak{d}$ compact subsets of G. Since $\mathrm{cov}(\mathcal{B}(G)) = \mathrm{cov}(\mathcal{B}) = \mathfrak{d}$, there is an element $g \in G\setminus(S_\alpha\cdot S_\alpha^{-1})$. For this element g we get $S_\alpha \cap gS_\alpha = \emptyset$. Assuming that $G = Z_\alpha \cup S_\alpha$ we would get $gS_\alpha \subset Z_\alpha$ and $S_\alpha \subset g^{-1}Z_\alpha$. Then $G = Z_\alpha \cup g^{-1}Z_\alpha$ which is not possible as Z_α is 2-Zorn. Consequently, $G \ne Z_\alpha \cup S_\alpha$ and we can pick a point $x_\alpha \in G \setminus (Z_\alpha \cup S_\alpha)$.

We claim that the subset $X = \{x_\alpha : \alpha < \mathfrak{d}\}$ is universally null. Fix any probability non-atomic measure μ on G and find a σ-compact subset $C \subset G$ with $\mu(G \setminus C) = 0$, see [Ke, 17.11]. By the choice of the family $\{C_\alpha\}$, there is an ordinal $\alpha < \mathfrak{d}$ with $C \subset C_\alpha$. It follows from the construction of X that $X \cap C_\alpha \subset \{x_\beta : \beta \le \alpha\}$ and $|X \cap C_\alpha| < \mathfrak{d}$. Since $\mathfrak{d} \le \mathrm{non}(\mathcal{N}) = \mathrm{non}(\mathcal{UN}(G))$, the set $X \cap C_\alpha$ is universally null. Consequently, $\mu(X) \le \mu(X \cap C_\alpha) + \mu(G \setminus C_\alpha) = 0$, i.e., X is universally null and hence Haar null. By the choice of the family $\{Z_\alpha\}$, there is an ordinal $\alpha < \mathfrak{d}$ with $X \subset Z_\alpha$. On the other hand $X \setminus Z_\alpha \ni x_\alpha$, which is a contradiction.

Thus $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d} \geq \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ under $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}$. If $\mathrm{non}(\mathcal{N}) < \mathfrak{d}$, then again $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(G)) \geq \mathfrak{d} > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

3. Assume that H is a non-discrete locally compact group and either H is a closed normal subgroup of G or else H is a quotient of G. In both cases we shall construct a map $p : G \to H$ such that a subset $N \subset H$ is Haar null in H if and only if $p^{-1}(N)$ is Haar null in G. If H is a quotient group of G, then let $p : G \to H$ be the quotient homomorphism and apply Lemma 5.

So now consider the case when H is a closed normal subgroup of G. According to [Ke, 12.17] the quotient homomorphism $\pi : G \to G/H$ admits a Borel section $s : G/H \to G$. The set $T = s(G/H)$, being the image of the Polish space G/H under an injective Borel map, is Borel in G, see [Ke, 15.1].

Consider the map $p : G \to H$ assigning to a point $x \in G$ the point $p(x) = (s \circ \pi(x))^{-1}x$. We claim that $p^{-1}(B) = TB$ for any subset $B \subset H$. Indeed, for any $t \in T$ and $b \in B$

$$p(tb) = (s \circ \pi(tb))^{-1}tb = (s \circ \pi(t))^{-1}tb = t^{-1}tb = b \in B.$$

On the other hand, if $p(x) = b \in B$, then $b = p(x) = (s \circ \pi(x))^{-1}x$ and thus $x = (s \circ \pi(x))b \in TB$.

We claim that a subset $N \subset H$ is Naar null in H if and only if TN is Haar null in G. Fix a left-invariant Haar measure λ on H.

Suppose that N is Haar null in H. Then $\lambda(N) = 0$ and we can assume that N is Borel in H. The product $TN = p^{-1}(N)$, being the image of the Borel space $T \times N$ under an injective continuous map, is Borel and thus universally measurable in G, see [Ke, 15.1]. We claim that $\lambda(xTNy) = 0$ for all $x, y \in G$. Given points $x, y \in G$ let $t = s(x^{-1}y^{-1})$ and observe that $xTNy \cap H = xtNy$ and hence $\lambda(xTNy) = \lambda(xtNy) = \lambda(Ny) = \triangle(y)\lambda(N) = 0$ which means that TN is Haar null in G.

Now assume conversely, that TN is Haar null in G. To show that N is Haar null in H it suffices to verify that $\lambda(N) = 0$. Fix a universally measurable subset $M \supset TN$ of G and a probability measure μ on G such that $\mu(xMy) = 0$ for all $x, y \in G$. Consider the convolution $\lambda * \mu$ assigning to a Borel function $f : G \to \mathbb{R}$ the integral $\int_\lambda \int_\mu f(xy)\,dx\,dy$. It is standard to show that $\lambda*\mu(M) = 0$, see [THJ, 2.4.4]. Denote by $\chi_M : G \to \{0, 1\}$ the characteristic function of the set M and applying Fubini Theorem conclude that

$$0 = \lambda * \mu(M) = \int_\lambda \int_\mu \chi_M(xy)\,dx\,dy = \int_\mu \int_\lambda \chi_M(xy)\,dy\,dx = \int_\mu \lambda(x^{-1}M)\,dx.$$

Then $\lambda(x^{-1}M) = 0$ for some $x \in G$. Since $M \supset TN$, we get $0 = \lambda(x^{-1}TN) = \lambda(x^{-1}(s \circ \pi(x))N) = \lambda(N)$.

Therefore a subset $N \subset H$ is Haar zero if and only if $p^{-1}(N)$ is Haar null in G. Using this observation it is easy to show that $\mathrm{add}(\mathcal{HN}(G)) \le \mathrm{add}(\mathcal{HN}(H)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(H)) = \mathrm{cov}(\mathcal{N})$, and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H)) = \mathrm{non}(\mathcal{N})$.

To show that $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{HN}(H)) = \mathrm{cof}(\mathcal{N})$ fix any family $\mathcal{F} \subset \mathcal{HN}(G)$ of size $|\mathcal{F}| = \mathrm{cof}(\mathcal{HN}(G))$ such that each Haar null subset of G lies in some $F \in \mathcal{F}$. For each set $F \in \mathcal{F}$ consider the subset $F' = H \setminus p(G \setminus F)$ of H which is Haar null in H since $p^{-1}(F') \subset F$. We claim that the family $\{F' : F \in \mathcal{F}\}$ is cofinal in $\mathcal{HN}(H)$. Indeed, for any Haar null set $N \subset H$ the set $p^{-1}(N)$ is Haar null in G. Then $p^{-1}(N) \subset F$ for some $F \in \mathcal{F}$ and hence $N \subset F'$. Therefore $\mathrm{cof}(\mathcal{N}) = \mathrm{cof}(\mathcal{HN}(H)) \le |\{F' : F \in \mathcal{F}\}| \le |\mathcal{F}| = \mathrm{cof}(\mathcal{HN}(G))$.

4. Assume that the center $Z = \{g \in G : \forall x \in G\ xg = gx\}$ of G is not locally compact. Then $\mathrm{cov}(\mathcal{HN}(Z)) \le \mathfrak{b} \le \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \le \mathrm{non}(\mathcal{HN}(Z))$ by the second statement of this theorem. So it rests to verify that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(Z))$ and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(Z))$.

According to [Ke, 12.17] the quotient homomorphism $\pi : G \to G/Z$ admits a Borel section $s : G/Z \to G$. Let $T = s(G/Z)$ and consider the map $p : G \to Z$ defined by $p(x) = (s \circ \pi(x))^{-1}x$ for $x \in G$. In the preceding item we have shown that p is a Borel map with $p^{-1}(N) = TN$ for any subset $N \subset H$.

We claim that for any universally measurable Haar null set $N \subset Z$ the set TN is Haar null in G. First we note that the set $TN = p^{-1}(N)$, being the preimage of the universally measurable set N under the Borel map p, is universally measurable.

Since the set N is Haar null in Z, there is a Borel measure μ on Z such that $\mu(xNy) = 0$ for all $x, y \in H$. We claim that $\mu(xTNy) = 0$ for all $x, y \in G$. Given any points $x, y \in G$ let $t = s \circ \pi(x^{-1}y^{-1})$ and note that $xty \in Z$ and $xTNy \cap Z = xtNy = xtyN$. Then $\mu(xTNy) = \mu(xtyN) = 0$ which means that $TN = p^{-1}(N)$ is a Haar null subset of G.

Therefore for any Haar null subset $N \subset Z$ the preimage $p^{-1}(N) = TN$ is Haar null in G. Using this fact it is trivial to show that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(Z))$ and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(Z))$.

5. If G admits a surjective continuous homomorphism onto a non-locally Polish compact group H with invariant metric, then Lemma 5 and the second item of this theorem imply that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(H)) \le \mathfrak{b}$, and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H)) \ge \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

Proof of Theorem 3

Suppose that a non-locally compact Polish group $G = \prod_{n \ge 0} G_n$ is the product of locally compact Polish groups G_n. Without loss of generality we can assume that the groups G_n are not trivial.

1. The estimates $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(o\mathcal{B}(G)) \le \mathfrak{b}$ and

$$\begin{aligned}\max\{\mathrm{non}(\mathcal{N}),\mathfrak{d}\} &\le \max\{\mathrm{non}(\mathcal{UN}(G)),\mathrm{non}(o\mathcal{B}(G))\}\\ &\le \mathrm{non}(\mathcal{HN}(G)) \le \mathrm{cof}(\mathcal{HN}(G))\end{aligned}$$

follow from the inclusion $\mathcal{UN}(G)\cup o\mathcal{B}(G) \subset \mathcal{HN}(G)$ (see Theorem 1) and Lemma 1. To prove that $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d},\mathrm{non}(\mathcal{N})\}$ we consider separately two cases.

If $\mathrm{non}(\mathcal{N}) < \mathfrak{d}$, then $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathfrak{d} > \min\{\mathfrak{d},\mathrm{non}(\mathcal{N})\}$. If $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$, then $\mathrm{cof}(\mathcal{HN}(G)) > \mathfrak{d} \ge \min\{\mathfrak{d},\mathrm{non}(\mathcal{N})\}$ according to (the proof of) Theorem 2(2).

2. Suppose that all but finitely many groups G_n are amenable. Without loss of generality we can assume that the groups G_n are amenable for $n\ge 1$. For every $n \ge 0$ fix a left-invariant Haar measure λ_n on G_n. Each group G_n, $n\ge1$, being amenable, satisfies the Følner condition. Using this condition, for every $n\ge 1$ we can construct an increasing sequence $(K_{n,m})_{m\ge0}$ of compact subsets of the group G_n such that $\bigcup_{m\ge0} K_{n,m} = G_n$, each $K_{n,m}$ lies in the interior of $K_{n,m+1}$ and $\lambda_n(xK_{n,m+1}\triangle K_{n,m+1}) < 2^{-m}\lambda(K_{n,m+1})$ for any $x\in K_{n,m}$.

For every $n\ge 0$ fix a probability measure $\tilde\lambda_n$ on G_n equivalent to the Haar measure λ_n (in the sense that they have the same null sets) and let $\lambda_{0,m} = \tilde\lambda_0$ for all $m\in\mathbb{N}$. For every $n,m\in\mathbb{N}$ define a probability measure $\lambda_{n,m}$ on the group G_n letting

$$\lambda_{n,m}(B) = \Big(1-\frac{1}{2^m}\Big)\frac{\lambda_n(B\cap K_{n,m})}{\lambda_n(K_{n,m})} + \frac{1}{2^m}\tilde\lambda_n(B)$$

for any Borel subset $B\subset G_n$. For any function $f\in\mathbb{N}^\omega$, denote by μ_f the tensor product $\mu_f = \bigotimes_{n\ge0}\lambda_{n,f(n)}$ which is a probability measure on G. In (the proof of) Theorem 4.1 [S$_2$] S. Solecki has shown that a universally measurable subset $N\subset G$ is Haar null in G if and only if there is a function $f\in\mathbb{N}^\omega$ such that $\mu_f(xNy)=0$ for any $x,y\in G$ if and only if there is a function $f\in\mathbb{N}^\omega$ such that $\mu_g(xNy)=0$ for any $x,y\in G$ and any $g\in\mathbb{N}^\omega$ with $f\le^* g$.

To estimate the cardinals $\mathrm{add}(\mathcal{HN}(G))$ and $\mathrm{cov}(\mathcal{HN}(G))$ fix any family $\mathcal{S}\subset\mathcal{HN}(G)$ of universally measurable Haar null subsets of G with $|\mathcal{S}|<\mathfrak{b}$. Using the mentioned result of S. Solecki, for any $S\in\mathcal{S}$ find a function $f_S\in\mathbb{N}^\omega$ such that $\mu_g(xSy)=0$ for any $x,y\in G$ and any $g\in\mathbb{N}^\omega$ with $f_S\le^* g$. Since $|\mathcal{S}|<\mathfrak{b}$, the set $\{f_S : S\in\mathcal{S}\}$ is bounded in $(\mathbb{N}^\omega,\le^*)$. Consequently, there is a function $f\in\mathbb{N}^\omega$ such that $f_S\le^* f$ for all $S\in\mathcal{S}$. For this function f we get $\mu_f(xSy)=0$ for all $x,y\in G$ and $S\in\mathcal{S}$. Now consider the union $\bigcup\mathcal{S}$. If $|\mathcal{S}|<\mathrm{add}(\mathcal{N})$, then $\bigcup\mathcal{S}$ is universally measurable by Lemma 3 and $\mu_f(x(\bigcup\mathcal{S})y)=0$ for all $x,y\in G$. Applying Solecki's Theorem 4.1 [S$_2$] we conclude that the union $\bigcup\mathcal{S}$ is Haar null in G and hence $\mathrm{add}(\mathcal{HN}(G)) \ge \min\{\mathfrak{b},\mathrm{add}(\mathcal{N})\} = \mathrm{add}(\mathcal{N})$. If $|\mathcal{S}|<\mathrm{cov}(\mathcal{N})$, then $\bigcup S\ne G$ (being the union of $<\mathrm{cov}(\mathcal{N})$ many μ_f-zero sets) and thus $\mathrm{cov}(\mathcal{HN}(G)) \le \min\{\mathfrak{b},\mathrm{cov}(\mathcal{N})\}$.

To prove that $\mathrm{non}(\mathcal{HN}(G)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$, fix any dominating subset $D \subset \mathbb{N}^\omega$ of size $|D| = \mathfrak{d}$. For any $f \in D$ find a subset $N_f \subset G$ of size $|N_f| = \mathrm{non}(\mathcal{N})$ such that $\mu_f(N_f) \neq 0$. Then the union $N = \bigcup_{f\in D} N_f$ has size $|N| \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ and is not Haar null. Otherwise, using the Solecki's result we would find a function $f \in D$ such that $\mu_f(N) = 0$ which is not possible since $\mu_f(N_f) \neq 0$ and $N_f \subset N$.

3. If one of the groups G_n is non-discrete, then we may apply Theorem 2(3) to conclude that $\mathrm{add}(\mathcal{HN}(G)) \leq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{N})$, and $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{N})$. So assume that all groups G_n are discrete and infinitely many of them are almost kaleidoscopical. Without loss of generality we can assume that each group G_{2n}, $n \geq 0$, is almost kaleidoscopical.

Fix any sequence $(\varepsilon_n)_{n\geq 0}$ of positive real numbers such that

$$\frac{1}{2} < \prod_{n\geq 0}(1-\varepsilon_n) < \prod_{n\geq 0}(1+\varepsilon_n) < 2.$$

Since the groups G_{2n} are almost kaleidoscopical, for every $n \geq 0$ we can find a finitely supported probability measure μ_{2n} on G_{2n} and a nontrivial finite partition $\mathcal{P}_n$ of G_{2n} such that $|\mu_{2n}(xPy) - \frac{1}{|\mathcal{P}_n|}| < \frac{\varepsilon_n}{|\mathcal{P}|}$ for each $x, y \in G$ and $P \in \mathcal{P}_n$. Endow each set $\mathcal{P}_n$ with the discrete topology and consider the map $p_n : G_{2n} \to \mathcal{P}_n$ assigning to a point $g \in G$ a unique element $P \in \mathcal{P}_n$ containing g. Now consider the continuous map $\pi : \prod_{n\geq 0} G_n \to \prod_{n\geq 0}\mathcal{P}_n$ assigning to a sequence $(x_n)_{n\geq 0} \in \prod_{n\geq 0} G_n$ the sequence $(p_n(x_{2n}))_{n\geq 0}$ in $\prod_{n\geq 0}\mathcal{P}_n$. For every $n \geq 0$ fix any probability measure μ_{2n+1} on the group G_{2n+1} and endow the group $G = \prod_{n\geq 0} G_n$ with the measure μ equal to the tensor product $\bigotimes_{n\geq 0}\mu_n$ of the measures μ_n.

On the product $\mathbb{P} = \prod_{n\geq 0}\mathcal{P}_n$ consider the measure λ equal to the tensor product $\bigotimes_{n\geq 0}\lambda_n$ of uniformly distributed measures of $\mathcal{P}_n$'s. For every $m \geq 1$ let $\mathrm{pr}_m : \prod_{n\geq 0}\mathcal{P}_n \to \prod_{0\leq n<m}\mathcal{P}_n$ be the projection onto the first m coordinates. Let us call a subset C of $\mathbb{P}$ *cylindrical* if $C = \mathrm{pr}_m^{-1}(A)$ for some $m \geq 1$ and some set $A \subset \prod_{0\leq n<m}\mathcal{P}_n$. It follows from the choice of the measures μ_i that for any point $y \in \prod_{0\leq n<m}\mathcal{P}_n$ the preimage $P = (\mathrm{pr}_m \circ \pi)^{-1}(y)$ has μ-measure satisfying

$$\prod_{0\leq n<m}\frac{1-\varepsilon_n}{|\mathcal{P}_n|} \leq \mu(P) \leq \prod_{0\leq n<m}\frac{1+\varepsilon_n}{|\mathcal{P}_n|}$$

and the same estimate is true for any shift xPy of P. Consequently, for any $x, y \in G$ and any cylindrical set $C = \mathrm{pr}_m^{-1}(A)$ we have

$$\lambda(C) = |A|\prod_{0\leq n<m}\frac{1}{|\mathcal{P}_n|} \tag{1}$$

and for its preimage $\pi^{-1}(C)$ we get

$$|A| \prod_{0\le n<m} \frac{1-\varepsilon_n}{|\mathcal{P}_n|} \le \mu(x\pi^{-1}(C)y) \le |A| \prod_{0\le n<m} \frac{1+\varepsilon_n}{|\mathcal{P}_n|}. \tag{2}$$

Dividing (2) by (1) we get

$$\frac{1}{2} \le \prod_{0\le n<m} (1-\varepsilon_n) \le \frac{\mu(x\pi^{-1}(C)y)}{\lambda(C)} \le \prod_{0\le n<m} (1+\varepsilon_n) \le 2. \tag{3}$$

Next, we show that the same estimate

$$\frac{1}{2}\lambda(M) \le \mu(x\pi^{-1}(M)y) \le 2\lambda(M) \tag{4}$$

holds for any universally measurable subset M of $\mathbb{P}$. Assuming that $\mu(x\pi^{-1}(M)y) > 2\lambda(M)$ for some universally measurable set $M \subset \mathbb{P}$ and points $x, y \in G$, find a compact subset $K \subset \pi^{-1}(M)$ with $\mu(xKy) > 2\lambda(M)$. Express the compact set $\pi(K)$ as a countable intersection $\pi(K) = \bigcap_{n\ge 0} C_n$ of a decreasing sequence of cylindrical subsets of $\mathbb{P}$. Since $\mu(x\pi^{-1}(\pi(K))y) \ge \mu(xKy) > 2\lambda(M) \ge 2\lambda(\pi(K))$, the countable additivity of the measures μ and λ imply that $\mu(x\pi^{-1}(C_n)y) > 2\lambda(C_n)$ for a sufficiently large n, which is not possible because of (3). By a similar argument we can show that $\mu(x\pi^{-1}(M)y) \ge \frac{1}{2}\lambda(M)$ for any universally measurable set $M \subset \mathbb{P}$ and any points $x, y \in G$ and thus finish the proof of (4).

This estimate implies that for any universally measurable set $M \subset \mathbb{P}$ with $\lambda(M) = 0$ we get $\mu(x\pi^{-1}(M)y) = 0$ for all $x, y \in G$, which means that $\pi^{-1}(M)$ is Haar null in G.

Now we prove that the converse is also true, that is a subset $N \subset \mathbb{P}$ has zero λ-measure if its preimage $\pi^{-1}(N)$ is Haar null in G. Assuming that the set $\pi^{-1}(N)$ is Haar null in G, fix a universally measurable set $M \supset \pi^{-1}(N)$ of G and a probability measure ν on G such that $\nu(xMy) = 0$ for all $x, y \in G$. Now consider the convolution $\mu * \nu$ assigning to a bounded continuous function $f : G \to \mathbb{R}$ the integral $\int_\mu \int_\nu f(xy)\,dx\,dy$. It follows from the Fubini Theorem that $\mu * \nu(M) = 0$. Consequently, there is a σ-compact set $S \subset G \setminus M$ with $\mu * \nu(S) = 1$. Now consider the image $\pi(S) \subset \mathbb{P}$ and note that it is a σ-compact set disjoint with N. Let $Q = \pi^{-1}(\mathbb{P} \setminus \pi(S))$ and note that $S \cap Q = \emptyset$ and hence $\mu * \nu(Q) = 0$. Denote by $\chi_Q : G \to \{0, 1\}$ the characteristic function of the set Q. Using the Fubini Theorem and the estimate (4) we get

$$\begin{aligned}
0 = \mu * \nu(Q) &= \int_\mu \int_\nu \chi_Q(xy)\,dx\,dy = \int_\nu \int_\mu \chi_Q(xy)\,dy\,dx \\
&= \int_\nu \mu(x^{-1}Q)\,dx \ge \frac{1}{2}\int_\nu \lambda(\mathbb{P} \setminus \pi(S))\,dx = \frac{1}{2}\lambda(\mathbb{P} \setminus \pi(S)).
\end{aligned}$$

Hence $\lambda(\mathbb{P} \setminus \pi(S)) = 0$ and $\lambda(N) = 0$ since $N \cap \pi(S) = \emptyset$. Therefore we have shown that a subset $N \subset \mathbb{P}$ has measure $\lambda(N) = 0$ if and only if its preimage $\pi^{-1}(N)$ is Haar null in G.

Now the estimations $\mathrm{add}(\mathcal{HN}(G)) \le \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N}) \le \mathrm{cof}(\mathcal{HN}(G))$ can be derived by analogy with the proof of Theorem 2(3).

4. Finally assume that the group G is abelian. Then every group G_n, being abelian is amenable and kaleidoscopical, see Proposition 1. Applying Theorem 3(1),(2),(3) we get the required estimations.

Proof of Theorem 4

Let G be a Polish non-locally compact group. The estimate $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d}$ under $\mathrm{non}(\mathbb{N}) \ge \mathfrak{d}$ was proven in (the proof of) Theorem 2(2). Then under $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ we get $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{c}$. Since the σ-algebra $\sigma\mathbf{P}$ has size $|\sigma\mathbf{P}| = \mathfrak{c}$, we conclude that there is a universally null subset of G, contained in no σ-projective 2-Zorn subset of G.

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