A sufficient condition for maximal resolvability of topological spaces

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Abstract. We show a new theorem which is a sufficient condition for maximal resolvability of a topological space. We also discuss some relationships between various theorems about maximal resolvability.

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At the beginning of XX century the problem of resolvability of a topological space became a matter of intense research and the subject of various publications. The first results were achieved by W. Sierpiński in [S]. He proved that if in a metric space X each non-void open set contains at least $m \geq \aleph_0$ points, then X is the union of m disjoint sets every of which contains at least m points of each non-void open set in X. In 1943 E. Hewitt considered the problem of determining the largest number of possible pairwise disjoint dense subsets in a topological space (including locally compact Hausdorff spaces and metric spaces). In 1964 J.G. Ceder generalized some of these results in the work [C]. In [CGF] W.W. Comfort and S. Garcia-Ferreira gave a brief introduction to the theory of spaces which are resolvable in the Hewitt sense. We will make use of some definitions and theorems introduced in [C] and [CGF]. The main aim of this paper is to give some theorems on resolvability in the language of bases at points.

Throughout the paper X will denote a topological space which is dense-in-itself, i.e. no point of X is isolated in X. Let $w(X)$ stand for the weight of X, that means

$$w(X) = \min\{|B| : B \text{ is a base in } X\}.$$

A dispersion character of X is a cardinal given by

$$\Delta(X) = \min\{|U| : U \text{ is a non-void open subset of } X\}.$$

Let κ be an arbitrary cardinal greater than 1. A space X is called κ-resolvable if there is a family of κ-many pairwise disjoint and dense subsets of X. If X is

κ-resolvable it becomes apparent that $\kappa \le \Delta(X)$. The space X is called maximally resolvable if it is $\Delta(X)$-resolvable. We say that a space X is cardinality-homogeneous (in short card-homogeneous) if every non-empty open subset V of X satisfies condition $|V| = |X|$.

W.W. Comfort and S. Garcia-Ferreira in [CGF] presented the following theorem:

Theorem 1. *If a topological space X is card-homogeneous satisfying condition $w(X) \le |X|$, then it is maximally resolvable.*

We can formulate a similar theorem using bases at points.

Theorem 2. *Let X be a card-homogeneous topological space and X_0 be a dense subset of X. If for every $x \in X_0$ there exists a point base $\mathcal{B}(x)$ such that*

$$|\mathcal{B}(x)| \le |X|,$$

then X is maximally resolvable.

From [CGF, Lemma 3.5 and Remark 3.6] (see also [B, Proposition 2]), by the use the notions of π-base and π-weight of a topological space we obtain the following

Theorem 3. *If X is a card-homogeneous topological space such that $\pi w(X) \le |X|$, then X is maximally resolvable.*

Remark 4. The following implications take place:
assumption of Th. 1 $\Rightarrow$ assumption of Th. 2 $\Rightarrow$ assumption of Th. 3.

(Hence Theorem 1 results from Theorem 2 and Theorem 2 results from Theorem 3.) Indeed, assume that $w(X) \le |X|$. Let $\mathcal{B}$ be a base of X such that $|\mathcal{B}| = w(X)$. For $X_0 = X$ and a point $x \in X$ we put

$$\mathcal{B}(x) = \{U \in \mathcal{B} : x \in \mathcal{B}\}.$$

Then $\mathcal{B}(x) \subset \mathcal{B}$, so $|\mathcal{B}(x)| \le |\mathcal{B}| = w(X) \le |X|$.

Now, assume that for every point x belonging to a dense X_0 subset of X there exists a point base $\mathcal{B}(x)$ such that $|\mathcal{B}(x)| \le |X|$. Put $\mathcal{B} = \bigcup_{x \in X_0} \mathcal{B}(x)$. Then $\mathcal{B}$ is a π-base of X such that $|\mathcal{B}| \le |X|$, so $\pi w(X) \le |\mathcal{B}| \le |X|$. $\square$

Observe that for metric spaces, the assumptions of Theorems 1, 2 and 3 are equivalent. We do not know whether this holds in general.

The main aim of next theorems is to omit the assumption that X is card-homogeneous. By a dispersion character of a space X at a point $x \in X$ we mean a cardinal number

$$\Delta(X, x) = \min\{|U| : U \text{ is an open neighbourhood of } x \text{ in } X\}.$$

Lemma 5. *If X is a dense-in-itself topological space of cardinality κ, then there exist pairwise disjoint, open and card-homogeneous sets G_α, $\alpha < \kappa$, such that*

$$X = \overline{\bigcup_{\alpha<\kappa} G_\alpha}.$$

Proof: Consider a relation $\prec$ on X defined as follows. Let $\Gamma = \{\Delta(X,x) : x \in X\}$. For each $\gamma \in \Gamma$ let $\prec_\gamma$ be a well ordering of the set $K_\gamma = \{x \in X : \Delta(X,x) = \gamma\}$. For any $x, y \in X$ we say that $x \prec y$ if either $\Delta(X,x) < \Delta(X,y)$ or x, y are in K_γ for some $\gamma \in \Gamma$ and $x \prec_\gamma y$. Then X is well ordered by $\prec$. Thus $\prec$ is isomorphic to the set of ordinals less than an ordinal ξ of cardinality $|\xi| = \kappa$, with the usual ordering (this set is usually identified with ξ). Hence we can arrange points of the set X as x_α, $\alpha < \xi$, and we have $x_\alpha \prec x_\beta$ iff $\alpha < \beta$.

For the point x_0, pick its neighbourhood U such that $|U| = \Delta(X, x_0)$. Put $G_0 = U$. Thus G_0 is card-homogeneous since if $V \neq \emptyset$ is an open subset of G_0 then for any $y \in V$ we have $y = x_\beta$ for some $\beta \geq 0$, so

$$|V| \leq |G_0| = \Delta(X, x_0) \leq \Delta(X, x_\beta) \leq |V|.$$

Assume that $0 < \alpha < \xi$ and that the sets G_γ, $\gamma < \alpha$ have been chosen. If $X = \overline{\bigcup_{\gamma<\alpha} G_\gamma}$, we put $G_\gamma = \emptyset$ for $\alpha \leq \gamma < \xi$. Otherwise, pick the smallest $\zeta < \xi$ such that $x_\xi \in X \setminus \overline{\bigcup_{\gamma<\alpha} G_\gamma}$. Take a neighbourhood V of x_ζ such that $|V| = \Delta(X, x_\zeta)$ and put

$$G_\alpha = V \setminus \overline{\bigcup_{\gamma<\alpha} G_\gamma}.$$

Then $|G_\alpha| = \Delta(X, x_\zeta)$ and similarly as for G_0 we show that G_α is card-homogeneous. This finishes the construction. Since $|\xi| = \kappa$, we can renumber sets G_α by indices $\alpha < \kappa$. Thus $X = \overline{\bigcup_{\gamma<\kappa} G_\gamma}$. □

Theorem 6. *Let X be an arbitrary dense-in-itself topological space and let X_0 be a dense subset of X. If for every point $x \in X_0$ there exists a local base $\mathcal{B}(x)$ at x such that*

$$|\mathcal{B}(x)| \leq \Delta(X),$$

then X is maximally resolvable.

Proof: By Lemma 5 there exists a disjoint family of open and card-homogeneous sets G_α, $\alpha < |X|$, such that $X = \overline{\bigcup_{\alpha} G_\alpha}$. We will use Theorem 2 for sets G_α and their dense subsets $G_\alpha \cap X_0$. Fix $\alpha < |X|$. For every point $x \in G_\alpha \cap X_0$ pick a local base $\mathcal{B}(x)$ satisfying condition $|\mathcal{B}(x)| \leq \Delta(X) \leq |G_\alpha|$. For any α, each set

G_α is $|G_\alpha|$-resolvable and according to the inequality $\Delta(X) \le |G_\alpha|$ it is $\Delta(X)$-resolvable. Hence we obtain $\Delta(X)$-many pairwise disjoint and dense sets $S_\gamma^{(\alpha)}$, $\gamma < \Delta(X)$. Denote $X_\gamma = \bigcup_\alpha S_\gamma^{(\alpha)}$, $\gamma < \Delta(X)$. Then, the sets X_γ are pairwise disjoint. We shall prove that they are dense in X. For a fixed α, every set $S_\gamma^{(\alpha)}$ is dense in G_α, so

$$\overline{S_\gamma^{(\alpha)}} \supset G_\alpha \quad \Rightarrow \quad \bigcup_\alpha \overline{S_\gamma^{(\alpha)}} \supset \bigcup_\alpha G_\alpha .$$

Hence

$$\overline{X_\gamma} = \overline{\bigcup_\alpha S_\gamma^{(\alpha)}} = \overline{\bigcup_\alpha \overline{S_\gamma^{(\alpha)}}} \supset \overline{\bigcup_\alpha G_\alpha} = X.$$

The above inclusions imply that X_γ are dense in X and the space X is maximally resolvable. $\square$

In [C] J.G. Ceder obtained a similar theorem

Theorem 7. *If X is dense-in-itself topological space satisfying condition $w(X) \le \Delta(X)$, then X is maximally resolvable.*

A general sufficient condition for the maximal resolvability of a topological space is the following theorem proved by A. Bella in [B] (see also [CGF]).

Theorem 8. *If X is dense-in-itself topological space satisfying condition $\pi w(X) \le \Delta(X)$, then X is maximally resolvable.*

Remark 9. We claim that if a space X satisfies the assumption of Theorem 7, then it satisfies the assumptions of Theorems 6 and 8.

First we show that the assumption of Theorem 6 drives from inequality $w(X) \le \Delta(X)$. Put $X_0 = X$. We can take a base $\mathcal{B}$ such that $|\mathcal{B}| = w(X)$ and for $x \in X$ we put $\mathcal{B}(x) = \{U \in \mathcal{B} : x \in \mathcal{B}\}$. Then $\mathcal{B}(x) \subset \mathcal{B}$, so $|\mathcal{B}(x)| \le |\mathcal{B}| = w(X) \le \Delta(X)$.

The assumption of Theorem 8 drives immediately from inequality $\pi w(X) \le w(X) \le \Delta(X)$.

The following examples witness that the above assumptions are not equivalent.

Example 10. We shall find a space which fulfils the assumption of Theorem 6 but does not these of Theorems 7 and 8.

Let X_1 be a discrete topological space of cardinality $\mathfrak{c} = |\mathbb{R}|$ and $\mathbb{Q}_+$ be the set of nonnegative rationals. Put $X = X_1 \times \mathbb{Q}_+$. In X we introduce a topology in the following way: if $p = (x, 0)$, then a neighbourhood of p is of the form $U(p, r) = \{x\} \times ([0, r) \cap \mathbb{Q}_+)$ where $r > 0$; if $p = (x, q)$, $q \ne 0$ then a neighbourhood of p is of the form $U(p, r) = \{x\} \times ((x - r, x + r) \cap \mathbb{Q}_+)$, where $0 < r < |q|$. In this topology every open set is countable, so $\Delta(X) = \aleph_0$. The space X has a countable base at every point p, which can be taken as the family of open sets

$U(p, \frac{1}{n})$, $n \in \mathbb{N}$. Hence X fulfils the assumption of Theorem 6, so is maximally resolvable.

The space X does not fulfil the assumption of Theorem 8 (so it does not fulfil the assumption of Theorem 7). Observe that $d(X) = \mathfrak{c}$, where $d(X)$ stands for the smallest cardinality of a dense subset of X. Indeed, $\{U(p, 1) : p \in X_1\}$ is a disjoint family of cardinality $\mathfrak{c}$, it consists of open sets, and each member of its family contains a point of a fixed dense set. As for the space X the following inequalities $w(X) \geq \pi w(X) \geq d(X) > \Delta(X) = \aleph_0$ come true, the space X neither has a countable π-base nor a countable base.

Example 11. This example introduces a space which fulfils the assumption of Theorem 8 but does not fulfil the assumption of Theorem 7 (does not have a countable base).

Let X_1 be a space of negative rationals equipped with the natural topology. Let X_2 be the space of positive real numbers equipped with Hashimoto topology generated by the ideal of nowhere dense sets. We put $X = X_1 \cup X_2$. As open sets in X_1 are countable, $\Delta(X) = \aleph_0$. Obviously X_1 has a countable base which is a countable π-base of this space. As every open set in X_2 contains an interval, an Euclidean base of X_2 is a π-base of X_2. Hence we obtain $\pi w(X) = \aleph_0$.

The space X does not have a countable base, because X_2 does not have such a base. This follows from the fact that the number of open sets in X_2 is $\leq 2^{w(X_2)}$ but in X_2 we have $2^{\mathfrak{c}}$ open sets (every set of the form $(0, 1) \setminus A$, where A is a subset of the Cantor set, is open in X_2).

Example 12. There exists a space X which fulfils the assumption of Theorem 8 but does not fulfil the assumption of Theorem 6 (does not have a countable base in any point).

Let $X = (\mathbb{R}, \mathcal{T})$ be the space introduced in Example 11 (it does have a countable π-base). Let us fix a point $x \in \mathbb{R}$. Suppose that the family $\{B_\alpha : \alpha < \omega\}$ is a countable base at the point x. Each of the sets B_α is of the form $B_\alpha = U_\alpha \setminus I_\alpha$ (U_α is an interval, I_α is nowhere dense in natural topology) and $x \in B_\alpha$ for every $\alpha < \omega$. The set $I = \bigcup_{\alpha<\omega} I_\alpha$ is of first category. Let $C \subset \mathbb{R} \setminus I$ be a Cantor set such that $x \in C$. The set $U = (\mathbb{R} \setminus C) \cup \{x\}$ is an open neighbourhood of x in $\mathcal{T}$ and does not include any set B_α.

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