Comment.Math.Univ.Carolin. 45,1 (2004)145–151

In search for Lindelöf C_p's

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Abstract. It is shown that if X is a first-countable countably compact subspace of ordinals then $C_p(X)$ is Lindelöf. This result is used to construct an example of a countably compact space X such that the extent of $C_p(X)$ is less than the Lindelöf number of $C_p(X)$. This example answers negatively Reznichenko's question whether Baturov's theorem holds for countably compact spaces.

Keywords: $C_p(X)$, space of ordinals, Lindelöf space

Classification: 54C35, 54D20, 54F05

1. Introduction

We prove that $C_p(X)$ is Lindelöf for every first-countable countably compact subspace of ordinals. Thus, we widen the class of all spaces X for which it is known that $C_p(X)$ is Lindelöf. This result gives some possible directions where one might find other spaces with Lindelöf C_p's (see questions in Section 3). Using the main result we construct an example of a countably compact space X such that $l(C_p(X)) \neq e(C_p(X))$. In the above equality $l(Y)$ stands for *Lindelöf number*, that is, the smallest infinite cardinal τ such that every open covering of Y contains a subcovering of cardinality $\leq \tau$. And $e(Y)$ is the *extent* of Y defined as the supremum of cardinalities of closed discrete subsets. This example answers Reznichenko's question whether Baturov's theorem [BAT] holds for countably compact spaces. Recall that Baturov's theorem states that $l(Y) = e(Y)$ for every $Y \subset C_p(X)$, where X is a Σ-Lindelöf space. A counterexample to Reznichenko's question also answers negatively the question posed in [BUZ] whether $C_p(X)$ is a D-space if X is countably-compact. The notion of D-space was introduced by Eric van Douwen [DOU].

A *neighborhood assignment* for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for any $x \in X$. A space X is a *D-space*, if for any neighborhood assignment φ for X there exists a closed discrete subset D of X such that $X = \bigcup_{d\in D} \varphi(d)$.

Throughout the paper, all spaces are assumed to be Tychonov. By R we denote the space of all real numbers endowed with standard topology. In notation and terminology we will follow [ARH] and [ENG].

Research supported by PSC-CUNY grant 64457-00 33.

2. Main result

Let $\tau_\omega = \{\alpha \leq \tau : cf(\alpha) \leq \omega\}$. Since in this section we deal only with τ_ω's and their function spaces, let us agree that for any $\alpha, \beta \in (\tau+1)$, by the interval $[\alpha, \beta]$ we mean the set $\{\gamma \in \tau_\omega : \alpha \leq \gamma \leq \beta\}$ (the same concerns open and half-open intervals). This agreement significantly simplifies our notation but is valid only within this section. If U is a standard open set of $C_p(X)$ we say that U depends on a finite set $\{x_1, \ldots, x_n\} \subset X$ if there exist $B_1, \ldots, B_n$ open in R such that $U = \{f \in C_p(X) : f(x_i) \in B_i \text{ for } i \leq n\}$.

Definition 2.1. *Let $A \subset \tau_\omega$. We say that B is an ω-support of A if B is countable and the following conditions are satisfied:*

(1) *$0 \in B$;*
(2) *$A \subset B$;*
(3) *if $b \in B$ is non-isolated in τ_ω then b is an accumulation point for B.*

Lemma 2.2. *If $A \subset \tau_\omega$ is countable, then there exists an ω-support B of A.*

PROOF: For each $a \in A$ non-isolated in τ_ω, fix a countable strictly increasing sequence X_a of isolated ordinals converging to a. Let $B = A \cup \{0\} \cup (\bigcup_{a\in A} X_a)$.

The set B is countable as a countable union of countable sets. Conditions (1) and (2) are met by definition. Let us verify (3). Take any $b \in B$ non-isolated in τ_ω. Since all X_a's consist of isolated ordinals, we have $b \in A$. Therefore, b is an accumulation point for $X_b \subset B$ and, as a consequence, for B as well. $\square$

Notice that if $A_n \subset \tau_\omega$ is an ω-support of itself for each n, then $\bigcup_n A_n$ is an ω-support of itself as well.

Definition 2.3. *Let $A \subset \tau_\omega$ be countable and an ω-support of itself. Let $f \in C_p(\tau_\omega)$. Define $c_{f,A}$ as follows: $c_{f,A}(x) = f(a_x)$, where $a_x = \sup(\{a \in \bar{A} : a \leq x\})$.*

First notice that the set $\{a \in \bar{A} : a \leq x\}$ is not empty for every x because $0 \in A$ (see the definition of ω-support). Since $\bar{A}$ is countable and τ_ω contains all ordinals not exceeding τ of countable cofinality, a_x exists for each x. And since the supremum is unique, $c_{f,A}$ is a well-defined function of τ_ω to R. Also, notice that $c_{f,A}$ coincides with f on $\bar{A}$ as $a_x = x$ for each $x \in \bar{A}$.

Lemma 2.4. *Let $A \subset \tau_\omega$ be countable and an ω-support of itself. Let $f \in C_p(\tau_\omega)$. Then $c_{f,A} \in C_p(\tau_\omega)$.*

PROOF: To show continuity of $c_{f,A}$ it is enough to show that for each $x_n \to x$ in τ_ω one can find a subsequence $\{x_m\} \subset \{x_n\}$ such that $c_{f,A}(x_m) \to c_{f,A}(x)$. If $x_n \in \bar{A}$ for infinitely many of n's then we are done since $c_{f,A} = f$ on $\bar{A}$.

Otherwise, we can assume that all x_n's are not in $\bar{A}$ and are distinct. For each $y \in \tau_\omega$, put $b_y = \tau$ if $(y, \tau] \cap \bar{A} = \emptyset$ and $b_y = \inf\{b \in A : b > y\}$ otherwise. For each x_n, consider $[a_{x_n}, b_{x_n})$, where a_{x_n} is from the definition of $c_{f,A}$. Notice that either $b_y = \tau$ or b_y is an isolated ordinal. Indeed, if $b_y \neq \tau$ then $b_y = \inf\{b \in A : b > y\} \in A$. And since A is an ω-support of itself, b_y is an isolated ordinal (see condition (3) in Definition 2.1).

The intervals $[a_{x_n}, b_{x_n})$ are either disjoint or coincide. Assume they coincide for infinitely many of m's with $[a_{x_3}, b_{x_3})$. If b_{x_3} is isolated then $x \in [a_{x_3}, b_{x_3})$ and $c_{f,A}([a_{x_3}, b_{x_3}))$ is a singleton. Therefore, $c_{f,A}(x_m) \to c_{f,A}(x)$. Otherwise b_{x_3} is not isolated and equal to τ. In this case $(a_{x_3}, \tau] \cap \bar{A} = \emptyset$ and $c_{f,A}([a_{x_3}, b_{x_3}])$ is a singleton again.

If the intervals are mutually disjoint then $a_{x_n} \to x \in \bar{A}$. And now use the facts that $f = c_{f,A}$ on $\bar{A}$ and $c_{f,A}(x_n) = f(a_{x_n})$. □

Lemma 2.5. *Let $A \subset \tau_\omega$ be countable and an ω-support of itself and $\mathcal{B}$ be a base of R. Let $f \in C_p(\tau_\omega)$. Let $U \subset C_p(\tau_\omega)$ be open and contain $c_{f,A}$. Then there exist sequences $\{[a_1, b_1], \dots, [a_n, b_n]\}$ and $\{B_1, \dots, B_n\}$ with the following properties:*

1. *$a_i \in A$;*
2. *$b_i \in A$ for $i < n$ and $b_n = \tau$;*
3. *$B_i \in \mathcal{B}$;*
4. *$c_{f,A} \in \{g \in C_p(\tau_\omega) : g([a_i, b_i)) \subset B_i$ if $a_i \neq b_i$ and $g(a_i) \in B_i$ if $a_i = b_i\} \subset U$.*

Proof: Without loss of generality, there exist $c_1 < .. < c_l \in \tau_\omega$ and $V_1, .., V_l \in \mathcal{B}$ such that $U = \{g \in C_p(\tau_\omega) : g(c_i) \in V_i\}$. We may assume that $c_l \geq \sup(\bar{A})$.

Step 1.

Let $m = \min\{i : c_i \geq \sup(\bar{A})\}$. Find $B_1 \in \mathcal{B}$ such that $c_{f,A}(c_m) \in B_1 \subset V_m \cap V_{m+1} \cap \dots \cap V_l$. Note that such a B_1 exists since $c_{f,A}$ is constant starting from $\sup(\bar{A})$. Find $a_1 \in A$ such that $c_{f,A}([a_1, \tau]) \subset B_1$ and $a_1 > c_i$ for all $i < m$. Due to continuity of $c_{f,A}$, such an a_1 can be found somewhere close to $\sup(\bar{A})$ (if $\sup(\bar{A}) \in A$, it can serve as a_1). Put $b_1 = \tau$.

Step k $\leq$ l.

If $c_i \geq a_{k-1}$ for all i, stop construction. Let $m = \max\{i : c_i < a_{k-1}\}$. Let $a'_k = \sup(\{a \in A : a \leq c_m\})$ and $b_k = \inf(\{a \in A : c_m \leq a\})$. Obviously $b_k \in A$. If $b_k = c_m = a'_k$ put $a_k = c_m$ and $B_k = V_m$. Otherwise, find $B_k \in \mathcal{B}$ such that $c_{f,A}([a'_k, b_k)) \subset B_k \subset V_m$. Such a B_k exists because $c_{f,A}([a'_k, b_k)) = f(a'_k) = c_{f,A}(c_m)$. If $a'_k = c_{m-1}$ we also require that $B_k \subset V_m \cap V_{m-1}$. If $a'_k \in A$ put $a_k = a'_k$. Otherwise a'_k is an accumulation point for A. And, due to continuity, we can find an $a_k \in A$ such that $[a_k, a'_k)$ contains no c_i's and $c_{f,A}([a_k, b_k)) \subset B_k$.

Re-enumerate $B_1, \dots, B_n$ and corresponding intervals in reverse order. Properties (1)–(4) hold by our construction. □

Theorem 2.6. *$C_p(\tau_\omega)$ is Lindelöf for any τ.*

PROOF: Let $\mathcal{B}$ be a countable base of R. Let $\mathcal{U}$ be an arbitrary open covering of $C_p(\tau_\omega)$. We will choose a countable subcovering $\{U_n\}$ inductively. From Step 2, we will follow our induction using elements in $\mathcal{S}_1$ defined at Step 1. However, at each Step n we might need to enlarge our inductive set by new elements. To ensure that every old element keeps the old tag we agree to enumerate $\mathcal{S}_1$ by prime numbers while new elements added at Step n by numbers p^{n+1}, where p is any prime.

Step 1.

Take any $U_1 \in \mathcal{U}$. The set U_1 depends on finite X_1. Let A_1 be an ω-support of X_1. Let $\mathcal{S}_1$ consist of all pairs $(\{[a_1, b_1], \dots, [a_k, b_k]\}, \{B_1, \dots, B_k\})$, where $B_i \in \mathcal{B}$, $a_i \in A_1$, $b_i \in A_1$ for $i < k$, $b_k = \tau$, and k is any natural number. Enumerate $\mathcal{S}_1$ by prime numbers.

Step n.

If $U_1 \cup \dots \cup U_{n-1}$ covers $C_p(\tau_\omega)$ stop induction. Otherwise, take the first $S = (\{[a_1, b_1], \dots, [a_k, b_k]\}, \{B_1, \dots, B_k\}) \in \mathcal{S}_{n-1}$ such that there exist f and $U_n \in \mathcal{U}$ containing f and the following property is satisfied.

Property. $f \in \{g \in C_p(\tau_\omega) : g([a_i, b_i)) \subset B_i$ if $a_i \neq b_i$ and $g(a_i) \in B_i$ if $a_i = b_i\} \subset U_n$.

If no such an S exists, just take any $U_n \in \mathcal{U}$ such that $U_n \setminus \bigcup_{i<n} U_i \neq \emptyset$.

The set U_n depends on X_n. Let A_n be an ω-support of $A_{n-1} \cup X_n$. Let $\mathcal{S}_n$ be the set of all pairs $(\{[a_1, b_1], \dots, [a_k, b_k]\}, \{B_1, \dots, B_k\})$, where $B_i \in \mathcal{B}$, $a_i \in A_n$, $b_i \in A_n$ for $i < k$, $b_k = \tau$, and k is any natural number. Enumerate $\mathcal{S}_n \setminus \mathcal{S}_{n-1}$ by numbers p^{n+1}, where p is any prime number. Enumeration on $\mathcal{S}_{n-1}$ is left unchanged.

Let us show that $\bigcup_n U_n$ covers $C_p(\tau_\omega)$. Take any $f \in C_p(\tau_\omega)$. Let $A = \bigcup_n A_n$. The set A is an ω-support of itself. Consider the function $c_{f,A}$. Since $\mathcal{U}$ covers $C_p(\tau_\omega)$ there exists $U \in \mathcal{U}$ that contains $c_{f,A}$.

By Lemma 2.5, there exists a pair $S = (\{[a_1, b_1], \dots, [a_k, b_k]\}, \{B_1, \dots, B_k\})$ with the following properties:

(1) $a_i \in A$;
(2) $b_i \in A$ for $i < k$ and $b_k = \tau$;
(3) $B_i \in \mathcal{B}$;
(4) $c_{f,A} \in \{g \in C_p(\tau_\omega) : g([a_i, b_i)) \subset B_i$ if $a_i \neq b_i$ and $g(a_i) \in B_i$ if $a_i = b_i\} \subset U$.

That is, $S \in \mathcal{S}_n$ for some n. Therefore, starting from some Step p^{n+1}, S must satisfy the *Property* and eventually it will be the first such. Therefore, $c_{f,A}$ must be covered by some U_m chosen at Step m. However, U_m depends on $X_m \subset A_m \subset A$ while $c_{f,A}$ coincides with f on $\bar{A}$. Therefore, U_m covers f. $\square$

Since any first-countable countably compact subspace of ordinals is homeomorphic to τ_ω for some τ we can restate our result as follows.

Theorem 2.7. *Let X be a first-countable countably compact subspace of ordinals. Then $C_p(X)$ is Lindelöf.*

3. Corollaries and related questions

Many papers are devoted to finding classes of spaces with Lindelöf C_p's. How good a space should be to have such a nice covering property as Lindelöfness in its function space? It is known that even a linearly orderable first countable compactum is not such unless it is metrizable. This fact follows from the theorem of Nahmanson in [NAH] (a detailed proof is in [ARH]). His theorem states that if X is a linearly ordered compactum then the Lindelöf number of $C_p(X)$ equals the weight of X. Even first-countable compacta with metrizable closures of countable sets do not have to have Lindelöf C_p's. Again this follows from the Nahmanson theorem and existence of non-metrizable first countable linearly ordered compacta in which closures of countable sets are metrizable (an example of such a compactum is Aronszajn continuum).

However, what happens if we strengthen the requirement of metrizable closures to countable closures? Notice that spaces in our main result (Theorem 2.7) are first-countable countably compact and, the closures of countable sets are countable. Therefore, the following questions might be of interest.

Question 3.1. *Let X be countably compact and first countable. Assume also that the closure of any countable set is countable in X. Is then $C_p(X)$ Lindelöf?*

Question 3.2. *Let X be first-countable and countably compact. Assume also that the closure of any countable set is countable in X. Is then $C_p(X)^\omega$ Lindelöf?*

Question 3.3. *Let $X = X_1 \oplus \cdots \oplus X_n \oplus \ldots$, where each X_n is first-countable and countably compact. Assume also that the closure of any countable set is countable in X_n. Is then $C_p(X)$ Lindelöf?*

Notice that spaces in Question 3.3 can be obtained from spaces in Question 3.1 by removing a point of countable character. Therefore the following question might worth consideration.

Question 3.4. *Suppose that $C_p(X)$ is Lindelöf for a space X. Let $x \in X$ have countable character in X. Is $C_p(X \setminus \{x\})$ Lindelöf? What if X is first countable (countably compact)?*

So we throw away a point and are hoping that what is left still has a decent C_p. Why do not we add one point? In general, adding a point can spoil C_p. For example, $C_p(\omega_1)$ is Lindelöf by Theorem 2.7, while $C_p(\omega_1 + 1)$ is not by Asanov's theorem [ASA]. Asanov's theorem implies that if $C_p(X)$ is Lindelöf then the tightness of X is countable (the *tightness* $t(X)$ of a space X is the smallest infinite cardinal number τ such that for any $A \subset X$ and any $x \in \bar{A}$ there exists $B \subset A$ of cardinality not exceeding τ such that $x \in \bar{B}$). That is, by adding one point $\{\omega_1\}$ we loose Lindelöfness of the function space. This observation motivates the following question.

Question 3.5 (Arhangelskii)**.** *Let $C_p(X \setminus \{x\})$ be Lindelöf and let x have countable tightness in X. Is $C_p(X)$ Lindelöf? What if X is first countable?*

Our next corollary is an answer to the Reznichenko's question whether Baturov's theorem [BAT] holds for countably compact spaces. Baturov's theorem states that $l(Y) = e(Y)$ for every $Y \subset C_p(X)$, where X is a Σ-Lindelöf space.

We answer Reznichenko's question by constructing a countably compact space X where the above equality fails to hold.

In the following example, by $[\alpha, \beta]_X$ we denote the set $[\alpha, \beta] \cap X$, where $\alpha, \beta \in \tau$ and $X \subset \tau$.

Example 3.6. Let $X = \{\alpha \leq \omega_2 : cf(\alpha) \neq \omega_1\}$. Then $l(C_p(X)) = \omega_2$ while $e(C_p(X)) = \omega$.

Proof of $e(C_p(X)) = \omega$:

It suffices to show that any $F \subset C_p(X)$ of cardinality ω_1 has a complete accumulation point in $C_p(X)$. Due to cofinality, there exists $\gamma < \omega_2$ such that f is constant on $[\gamma, \omega_2]_X$ for each $f \in F$. We can also choose γ with countable cofinality.

For each $f \in F$ let $f^* \in C_p(\gamma_\omega)$ be such that $f^* = f$ on $[0, \gamma]_{\gamma_\omega}$. Since $C_p(\gamma_\omega)$ is Lindelöf (Theorem 2.6), there exists $h^* \in C_p(\gamma_\omega)$ a complete accumulation point for $F^* = \{f^* : f \in F\}$. Define a function h as follows:

$$h(x) = \begin{cases} h^*(x) & \text{if } x \in [0, \gamma]_X, \\ h^*(\gamma) & \text{if } x \in [\gamma, \omega_2]_X. \end{cases}$$

No doubts, $h \in C_p(X)$. Let us show that h is a complete accumulation point for F. Let $h \in U = \{g \in C_p(X) : g(c_i) \in B_i\}$, where $c_1 < \cdots < c_n \in X$ and $B_1, \ldots, B_n$ are open in R. We need to show that $F \cap U$ is uncountable. It does not hurt if we make U smaller by assuming that $c_k = \gamma$ for some $k \leq n$. Since h is constant on $[\gamma, \omega_2]_X$ we may assume that $B_j = B_k$ for all $j \geq k$.

The set $U^* = \{g \in C_p(\gamma_\omega) : g(c_i) \in B_i, i \leq k\}$ is an open neighborhood of h^*. Since h^* is a complete accumulation point for F^*, $F^* \cap U^*$ is uncountable. If $f^* \in U^* \cap F^*$ then $f^*(c_k) \in B_k$. Therefore, for $j > k$, $f(c_j) = f(c_k) \in B_j$. And $f(c_j) \in B_j$ for $j \leq k$ because f coincides with f^* on $[0, \gamma]_X = [0, \gamma]_{\gamma_\omega}$. Therefore, $f \in F \cap U$ and $F \cap U$ is uncountable. □

Proof of $l(C_p(X)) = \omega_2$:

Asanov's theorem [ASA] implies that $t(X) \leq l(C_p(X))$. Since $t(X) = \omega_2$, $l(C_p(X)) \geq \omega_2$. And we actually have equality because the weight of X is ω_2.

□

In [BUZ], the author proves that $C_p(X)$ is hereditarily a D-space if X is compact. This result motivated the D-version of Reznichenko's question whether $C_p(X)$ is a hereditary D-space if X is countably compact. From the definition of a D-space it is easy to conclude that $l(X) = e(X)$ for every D-space X. Therefore, Example 3.6 serves as a counterexample to this question.

One of the central questions on D-spaces posed by van Douwen is *whether every Lindelöf space is a D-space*. In search for a counterexample (if there exists one) it might be worth to consider the following question.

Question 3.7. *Is $C_p(\tau_\omega)$ a D-space for $\tau \geq \omega_2$?*

Note that all theorems on D-spaces known so far do not cover the spaces in the above question.

After-Submission Remarks. After this paper was submitted, A. Dow and P. Simon answered Question 3.1 in negative. Therefore, it is reasonable to assume now that $C_p(X)$ in Question 3.2 and $C_p(X_n)$'s in Question 3.3 are Lindelöf.

Acknowledgment. The author would like to thank the referee for valuable remarks and suggestions.

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(*Received* May 29, 2003, *revised* October 10, 2003)