Differentiability of weak solutions of nonlinear second order parabolic systems with quadratic growth and nonlinearity $q \geq 2$

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Abstract. Let Ω be a bounded open subset of $\mathbb{R}^n$, let $X = (x, t)$ be a point of $\mathbb{R}^n \times \mathbb{R}^N$. In the cylinder $Q = \Omega \times (-T, 0)$, $T > 0$, we deduce the local differentiability result

$$u \in L^2(-a, 0, H^2(B(\sigma), \mathbb{R}^N)) \cap H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N))$$

for the solutions u of the class $L^q(-T, 0, H^{1,q}(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(\bar{Q}, \mathbb{R}^N)$ $(0 < \lambda < 1,\ N$ integer $\geq 1)$ of the nonlinear parabolic system

$$-\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du)$$

with quadratic growth and nonlinearity $q \geq 2$. This result had been obtained making use of the interpolation theory and an imbedding theorem of Gagliardo-Nirenberg type for functions u belonging to $W^{1,q} \cap C^{0,\lambda}$.

Keywords: differentiability of weak solution, parabolic systems, nonlinearity with $q > 2$

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1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^n$ $(n > 2)$ of generic point $x = (x_1, x_2, \dots, x_n)$, Q the cylinder $\Omega \times (-T, 0)$ $(0 < T < +\infty)$; here N is an integer > 1, $(\ |\ )_k$ and $\| \cdot \|_k$ are the scalar product and the norm in $\mathbb{R}^k$, respectively. We will drop the subscript k when there is no fear of confusion.

We define

$$B(x^0, \sigma) = \left\{ x \in \mathbb{R}^n : |x_i - x_i^0| < \sigma,\ i = 1, \dots, n \right\}.$$

If $u : Q \to \mathbb{R}^N$, we set $Du = (D_1 u, \dots, D_n u)$ where, as usual, $D_i = \frac{\partial}{\partial x_i}$. Clearly $Du \in \mathbb{R}^{nN}$ and we denote by $p = (p^1, \dots, p^n)$, $p^i \in \mathbb{R}^N$, a typical vector of $\mathbb{R}^{nN}$ and let $V(p) = \left(1 + \|p\|^2\right)^{\frac{1}{2}}$.

Let $u \in L^q(-T, 0, H^{1,q}(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(\bar{Q}, \mathbb{R}^N)$ $(0 < \lambda < 1)$ [(1)] be a solution in Q to the second order nonlinear parabolic system of variational type

$$-\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du) \tag{1.0}$$

in the sense that

$$\begin{aligned} &\int_Q \left\{ \sum_{i=1}^{n} \Big(a^i(X, u, Du) \,|\, D_i\varphi \Big) - \Big(u \,|\, \frac{\partial \varphi}{\partial t} \Big) \right\} dX \\ &= \int_Q \Big(B^0(X, u, Du) \,|\, \varphi \Big)\, dX, \quad \forall\, \varphi \in C_0^\infty(Q, \mathbb{R}^N), \end{aligned} \tag{1.1}$$

where $X = (x, t)$, $a^i(X, u, p)$, $i = 1, \dots, n$, and $B^0(X, u, p)$ are vectors of $\mathbb{R}^N$ defined on $\Lambda = Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$, satisfying the following conditions:

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(1) By $H^{m,p}(\Omega, \mathbb{R}^N)$, $m = 0, 1, 2, \dots$, $1 < p < \infty$, we will denote the usual Sobolev spaces

$$H^{0,p}(\Omega, \mathbb{R}^N) = L^p(\Omega, \mathbb{R}^N) \quad \text{and} \quad |u|_{0,p,\Omega} = \|u\|_{0,p,\Omega} = \left\{ \int_\Omega \|u\|^p \, dx \right\}^{1/p}, \quad 1 < p < \infty.$$

If $1 \le p < \infty$ and m, j are integers ≥ 0, we denote

$$|u|_{j,p,\Omega} = \left[\int_\Omega \left(\sum_{|\alpha| = j} \|D^\alpha u\|^2 \right)^{p/2} dx \right]^{1/p}, \quad \|u\|_{m,p,\Omega} = \left\{ \sum_{j=0}^{m} |u|_{j,p,\Omega}^p \right\}^{1/p}.$$

If $p = 2$, we shall use the notation $H^s, |\cdot|_{s,\Omega}, \|\cdot\|_{s,\Omega}$ simply.
By $H^{\theta,r}(\Omega, \mathbb{R}^N)$, $0 < \theta < 1$, $1 < r < \infty$, we will denote the Slobodecky space of those vectors $u \in L^r(\Omega, \mathbb{R}^N)$ such that

$$|u|_{\theta,r,\Omega}^r = \int_\Omega dx \int_\Omega \frac{\|u(x) - u(y)\|^r}{\|x - y\|^{n + \theta r}} dy < +\infty$$

By $H^{m+\theta,r}(\Omega, \mathbb{R}^N)$, $m = 1, 2, \dots$, $0 < \theta < 1$, $1 < r < \infty$ we will denote the space of those vectors $u \in H^{m,r}(\Omega, \mathbb{R}^N)$ such that $D^\alpha u \in H^{\theta,r}(\Omega, \mathbb{R}^N)$, $\forall\, |\alpha| = m$.
If $r = 2$ we shall use the notation $H^{m+\theta}$, $m = 0, 1, 2, \dots$, $0 \le \theta < 1$ instead of $H^{m+\theta,2}$.
By $C^{0,\lambda}(\Omega, \mathbb{R}^N)$, $0 < \lambda < 1$, we shall denote the space of those vectors $u \in C^0(\bar{\Omega}, \mathbb{R}^N)$ for which $[u]_{\lambda,\bar{\Omega}} = \sup\limits_{x,y \in \bar{\Omega}, x \ne y} \dfrac{\|u(x) - u(y)\|}{\|x - y\|^\lambda} < +\infty$.
In Q the Hölder continuity is considered with respect to the parabolic metric

$$d(X, Y) = \max \left\{ \|x - y\|, |t - \tau|^{1/2} \right\}, \quad X = (x, t), \quad Y = (y, \tau).$$

(1.2) the vector $B^0(X,u,p)$ is measurable in X, continuous in (u,p), and, for each $(X,u,p) \in \Lambda$, with $\|u\| \le k$, $p \in \mathbb{R}^{nN}$

$$\|B^0(X,u,p)\| + \sum_{s=1}^{n} \left\| \frac{\partial B^0}{\partial x_s} \right\| + \sum_{k=1}^{N} \left\| \frac{\partial B^0}{\partial u_k} \right\| \le M(k) V^q(p)$$

$$\sum_{k=1}^{N} \sum_{j=1}^{n} \left\| \frac{\partial B^0}{\partial p_k^j} \right\| \le c(k) V^{q-1}(p),$$

(1.3) the vectors $a^i(X,u,p)$, $i = 1,2,\dots,n$, are of class C^1 in $\bar{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ and, for each $(X,u,p) \in \Lambda$ with $\|u\| \le k$

$$\|a^i\| + \sum_{s=1}^{n} \left\| \frac{\partial a^i}{\partial x_s} \right\| + \sum_{k=1}^{N} \left\| \frac{\partial a^i}{\partial u_k} \right\| \le c(k) V^{q-1}(p), \quad i = 1,2,\dots,n$$

$$\sum_{k=1}^{N} \sum_{j=1}^{n} \left\| \frac{\partial a^i}{\partial p_k^j} \right\| \le M(k) V^{q-2}(p), \quad i = 1,2,\dots,n,$$

(1.4) there exists $\nu(k) > 0$ such that

$$\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} \frac{\partial a_k^i(X,u,p)}{\partial p_k^j} \xi_h^i \xi_k^j \ge \nu(k)\, V^{q-2}(p)\, \|\xi\|^2$$

for each $\xi = (\xi^1 \,|\, \xi^2 \,|\, \dots \,|\, \xi^n) \in \mathbb{R}^{nN}$ and for each $(X,u,p) \in \Lambda$ with $\|u\| \le k$.

In [4], the local differentiability was examined with respect to the spatial derivatives of the solutions

$$u \in L^q(-T,0,H^{1,q}(\Omega,\mathbb{R}^N)) \cap C^{0,\lambda}(Q,\mathbb{R}^N), \quad q \ge 2, \quad 0 < \lambda < 1 \tag{1.5}$$

to the system (1.1), proving that, under the assumptions of monotony and nonlinearity $q > 2$, for each cube $B(\sigma) = B(x^0,\sigma) \subset\subset \Omega$ and $\forall\, a \in (0,T)$ it results

$$u \in L^q(-a,0,H^{1+\theta,q}(B(\sigma),\mathbb{R}^N)), \quad \forall\, \theta \in \left(0, \frac{2}{q}\right)$$

and this result is analogous to that which we obtained in [3] under the assumptions of nonlinearity $q = 2$, under the boundedness conditions for the derivatives $\frac{\partial a^j}{\partial p_k^i}$ and of strong ellipticity.

In [5], we considered again the problem of differentiability, under assumptions of monotony and nonlinearity $1 < q < 2$, always achieving results of the same type.

The aim of this paper is to obtain for the solutions (1.5) of the system (1.1), under assumptions (1.2), (1.3), (1.4) and of nonlinearity $q > 2$, the result of

$$u \in L^2(-a, 0, H^2(B(\sigma), \mathbb{R}^N)) \cap H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N))$$

for each cube $B(\sigma) = B(x^0, \sigma) \subset\subset \Omega$ and $\forall\, a \in (0, T)$, making use of the interpolation theory and an imbedding theorem of Gagliardo-Nirenberg type for functions u belonging to $W^{1,q} \cap C^{0,\lambda}$.

This paper extends the result obtained by Marino-Maugeri [6] in the case of nonlinearity $q = 2$ and it is analogous to the regularity result which had been obtained by Campanato [2] for elliptic systems with nonlinearity $q > 2$.

2. Some notations and preliminary results

In this section we list a few lemmas that will be needed in the sequel and which are already well known in the mathematical literature.

Let $B(\sigma) = B(x^0, \sigma)$ $(x^0 \in \mathbb{R}^n,\ \sigma > 0)$ be a cube in $\mathbb{R}^n$ defined by

$$B(\sigma) = \{x \in \mathbb{R}^n : |x_i - x_i^0| < \sigma, \quad i = 1, \dots, n\}.$$

If $u : B(\sigma) \times (-T, 0) \to \mathbb{R}^N$ $(T > 0)$ and $X = (x, t) \in B(\tau\sigma) \times (-T, 0)$, $\tau \in (0, 1)$, $|h| < (1 - \tau)\sigma$, then we define

$$\tau_{i,h} u(X) = u(x + he^i, t) - u(X), \quad i = 1, 2, \dots, n,$$

where $\{e^s\}_{s=1,\dots,n}$ is the standard basis of $\mathbb{R}^n$.

Lemma 2.1. *If $u \in L^q(-b, -\rho, H^{1,q}(B(\sigma), \mathbb{R}^N))$, $q > 1$, $0 \le \rho < b$, then $\forall\, \tau \in (0, 1)$ and $\forall\, |h| < (1 - \tau)\sigma$*

$$\int_{-b}^{-\rho} dt \int_{B(\tau,\sigma)} \|\tau_{i,h} u\|^q \, dx \le |h|^q \int_{-b}^{-\rho} dt \int_{B(\sigma)} \|D_i u\|^q \, dx, \quad i = 1, 2, \dots, n.$$

See for instance [1, Cap. I, Lemma 3.VI].

Lemma 2.2. *If $v \in L^p(-a, 0, L^p(B(2\sigma), \mathbb{R}^N))$, $a, \sigma > 0$, $1 < p < +\infty$, and if there exists $M > 0$ such that*

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|\tau_{i,h} v\|^p \, dx \le |h|^p M, \quad \forall\, |h| < \sigma,\ i = 1, 2, \dots, n,$$

then $v \in L^p(-a, 0, H^{1,p}(B(\sigma), \mathbb{R}^N))$ and

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|D_i v\|^p \, dx \le M, \quad i = 1, 2, \dots, n.$$

The proof is the same as that of Theorem 3.X in [1].

Lemma 2.3. *Let N be a positive integer and Ω a cube of $\mathbb{R}^n$. If*

$$u \in H^{1+\theta,q}(\Omega, \mathbb{R}^N) \cap C^{0,\lambda}(\Omega, \mathbb{R}^N)$$

with $1 < r < \infty$, $0 < \theta < 1$ and $0 < \lambda < 1$, then $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and there exists a constant c (depending on $\Omega, \theta, \lambda, n, a, q$) such that:

$$\|u\|_{1,p,\Omega} \leq c\|u\|^{a}_{1+\theta,q,\Omega}\|u\|^{1-a}_{C^{0,\lambda}(\Omega,\mathbb{R}^N)},$$

where

$$\frac{1}{p} = \frac{1}{n} + a\left(\frac{1}{q} - \frac{1+\theta}{n}\right) - (1-a)\frac{\lambda}{n}, \quad \forall\, a \in \left]\frac{1-\lambda}{1+\theta-\lambda}, 1\right[.$$

In particular, if $1-\lambda < \theta < 1$, for $a = \frac{1}{2}$ we get

$$u \in W^{1,p}(\Omega, \mathbb{R}^N)$$

and there exists a constant c (depending on $\Omega, \theta, \lambda, n, a, q$) such that

$$\|u\|_{1,p,\Omega} \leq c\|u\|^{\frac{1}{2}}_{1+\theta,q,\Omega}\|u\|^{\frac{1}{2}}_{C^{0,\lambda}(\Omega,\mathbb{R}^N)}$$

where $p = 2q + \dfrac{2q^2(\theta+\lambda-1)}{n - q(\theta+\lambda-1)}$ $(> 2q)$.

The proof is the same as that of Theorem 2.2 in [6] for $m = 1$, $r = q$, $s = 0$, $j = 1$.

3. Differentiability of the solutions to the system (1.1)

Let $u \in L^q(-T, 0, H^{1,q}(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(\bar{Q}, \mathbb{R}^N)$, $0 < \lambda < 1$, $q \geq 2$, be a solution to the system (1.1) and suppose that assumptions (1.2), (1.3) and (1.4) are fulfilled; in what follows we shall set

$$k = \sup_{Q} \|u\|, \quad U = [u]_{\lambda,Q} = \sup_{X,Y\in Q, X\neq Y} \frac{\|u(X)-u(Y)\|}{d^{\lambda}(X,Y)},$$

where $d(X,Y)$ is the parabolic metric

$$d(X,Y) = \max\left\{\|x-y\|, |t-\tau|^{\frac{1}{2}}\right\}, \quad X = (x,t),\ Y = (y,\tau).$$

Now we show the following

Theorem 3.1. *If* $u \in L^q(-T,0,H^{1,q}(\Omega,\mathbb{R}^N)) \cap C^{0,\lambda}(\bar{Q},\mathbb{R}^N)$, $0<\lambda<1$, $q \geq 2$, *is a solution to the system* (1.1), *if assumptions* (1.2), (1.3) *and* (1.4) *hold, then,* $\forall\, B(3\sigma) = B(x^0,3\sigma) \subset\subset \Omega$, $\forall\, a,b \in (0,T)$, $a<b$, *we have*

$$u \in L^2(-a,0,H^2(B(\sigma),\mathbb{R}^N)) \cap H^1(-a,0,L^2(B(\sigma),\mathbb{R}^N)) \tag{3.1}$$

and the following estimate holds:

$$\begin{aligned}
&\int_{-a}^{0}\left(|u|^2_{2,B(\sigma)} + \left|\frac{\partial u}{\partial t}\right|^2\right) dt \\
&\leq c(\nu,k,U,\lambda,\sigma,q,a,b,n)\left\{1+\int_{-b}^{0}|u|^q_{1,q,B(3\sigma)}\,dt\right\}.
\end{aligned} \tag{3.2}$$

Proof: Given $B(3\sigma) = B(x^0,3\sigma) \subset\subset \Omega$, $a,b \in (0,T)$, with $a<b$, let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ be a real function which has the following properties:

$$0 \leq \psi \leq 1, \quad \psi = 1 \text{ in } B(\sigma), \quad \psi = 0 \text{ in } \mathbb{R}^n \setminus B(2\sigma), \quad \|D\psi\| \leq \frac{c}{\sigma}\,. \tag{3.3}$$

Let $\rho_m(t)$, with m integer $> 2/a$, be a function defined on $\mathbb{R}$ by

$$\rho_m(t) = \begin{cases} 1 & \text{if } -a \leq t \leq -\dfrac{2}{m}\,, \\ 0 & \text{if } t \geq -\dfrac{1}{m} \text{ or } t \leq b, \\ \dfrac{t+b}{b-a} & \text{if } -b < t < -a, \\ -(mt+1) & \text{if } -\dfrac{2}{m} < t < -\dfrac{1}{m}\,. \end{cases} \tag{3.4}$$

Finally let $\{g_s(t)\}$ be a sequence of symmetric mollifying functions

$$\begin{cases} g_s(t) \in C_o^\infty(\mathbb{R}), \quad g_s(t) \geq 0, \quad g_s(t) = g_s(-t) \\ \text{supp } g_s \subset \left[-\dfrac{1}{s}, \dfrac{1}{s}\right] \\ \int g_s(t)\,dt = 1. \end{cases} \tag{3.5}$$

Having fixed i integer, $1 \leq i \leq n$, and h such that $|h| < \min\left\{1, \dfrac{\sigma}{2}\right\}$, if we set $b^* = \dfrac{a+b}{2}$, let us assume in (1.1), for each $m > \dfrac{2}{a}$ and for each $s > \max\left\{m, \dfrac{1}{T-b}\right\}$,

$$\varphi = \tau_{i,-h}\{\psi^2 \rho_m[(\rho_m \tau_{i,h} u) * g_s]\}.$$

Then we get

$$\begin{aligned}(3.6)\qquad &\int_Q \sum_{j=1}^{n} \Big(\tau_{i,h} a^j(X,u,Du) \,|\, D_j\{\psi^2 \rho_m[(\rho_m \tau_{i,h} u) * g_s]\}\Big)\, dX \\ &= \int_Q \Big(\tau_{i,h} \,|\, \psi^2 \{\rho_m[(\rho_m \tau_{i,h} u) * g_s]\}'\Big)\, dX \\ &\quad + \int_Q \Big(B^0(X,u,Du) \,|\, \tau_{i,-h}\{\psi^2 \rho_m[(\rho_m \tau_{i,h} u) * g_s]\}\Big)\, dX.\end{aligned}$$

Furthermore

$$\begin{aligned}(3.7)\qquad &\tau_{i,h} a^j(X,u(X),Du(X)) \\ &= \int_0^1 \frac{\partial}{\partial \eta} a^j(x+\eta h e^i, t, u(X)+\eta \tau_{i,h} u(X), Du(X)+\eta \tau_{i,h} Du(X))\, d\eta \\ &= h \frac{\partial \tilde a^j}{\partial x_i} + \sum_{k=1}^{N} (\tau_{i,h} u_k(X)) \frac{\partial \tilde a^j}{\partial u_k} + \sum_{r=1}^{n} \sum_{k=1}^{N} (\tau_{i,h} D_r u_k(X)) \frac{\partial \tilde a^j}{\partial p_k^r}\end{aligned}$$

where, if $b = b(X,u,p)$ is a vector of $\mathbb{R}^N$, we set for the sake of simplicity

$$(3.8)\qquad \tilde b(x) = \int_0^1 b(x + h\eta e^i, t, u(X)+\eta \tau_{i,h} u(X), Du(X)+\eta \tau_{i,h} Du(X))\, d\eta.$$

Therefore, from (3.6) we obtain that

$$\begin{aligned}(3.9)\qquad &\int_Q \psi^2 \rho_m \sum_{j,r=1}^{n} \sum_{k=1}^{N} \Big((\tau_{i,h} D_r u_k(X)) \frac{\partial \tilde a^j}{\partial p_k^r} \,|\, (\rho_m \tau_{i,h} D_j u) * g_s\Big)\, dX \\ &= -2 \int_Q \psi \rho_m \sum_{j,r=1}^{n} \sum_{k=1}^{N} \Big((\tau_{i,h} D_r u_k(X)) \frac{\partial \tilde a^j}{\partial p_k^r} \,|\, D_j \psi[(\rho_m \tau_{i,h} u) * g_s]\Big)\, dX \\ &\quad - \int_Q \sum_{j,r=1}^{n} \sum_{k=1}^{N} \Big((\tau_{i,h} u_k(X)) \frac{\partial \tilde a^j}{\partial u_k} \,|\, D_j \Big\{\psi^2 \rho_m[(\rho_m \tau_{i,h} u) * g_s]\Big\}\Big)\, dX \\ &\quad - h \int_Q \sum_{j=1}^{n} \Big(\frac{\partial \tilde a^j}{\partial x_i} \,|\, D_j \Big\{\psi^2 \rho_m[(\rho_m \tau_{i,h} u) * g_s]\Big\}\Big)\, dX \\ &\quad + \int_Q \psi^2 \rho_m' (\tau_{i,h} u \,|\, (\rho_m \tau_{i,h} u) * g_s)\, dX \\ &\quad + \int_Q \Big(B^0(X,u,Du) \,|\, \tau_{i,-h}\{\psi^2 \rho_m[(\rho_m \tau_{i,h} u) * g_s]\}\Big)\, dX\end{aligned}$$

taking into account that

$$D_j\{\psi^2\rho_m[(\rho_m\tau_{i,h}u)*g_s]\} = \psi^2\rho_m[(\rho_m\tau_{i,h}D_ju)*g_s] + 2\psi\rho_m D_j\psi[(\rho_m\tau_{i,h}u)*g_s]$$

and that, by symmetry of the $g_s(t)$

$$\int_Q \Big(\tau_{i,h}u \,|\, \psi^2\rho_m[(\rho_m\tau_{i,h}u)*g_s]'\Big)\,dX = 0.$$

And so, from (3.9), taking the limit for $s\to+\infty$, we obtain that

$$\begin{aligned}
A &= \int_Q \psi^2\rho_m^2 \sum_{j,r=1}^{n}\sum_{k=1}^{N}\Big((\tau_{i,h}D_ru_k(X))\frac{\partial\tilde a^j}{\partial p_k^r} \,|\, \tau_{i,h}D_ju\Big)\,dX \\
&= -2\int_Q \psi\rho_m^2 \sum_{j,r=1}^{n}\sum_{k=1}^{N}\Big((\tau_{i,h}D_ru_k(X))\frac{\partial\tilde a^j}{\partial p_k^r} \,|\, D_j\psi\,\tau_{i,h}u\Big)\,dX \\
&\quad -\int_Q \sum_{j,r=1}^{n}\sum_{k=1}^{N}\Big((\tau_{i,h}u_k)\frac{\partial\tilde a^j}{\partial u_k} \,|\, D_j(\psi^2\rho_m^2\tau_{i,h}u)\Big)\,dX \\
&\quad - h\int_Q \sum_{j=1}^{n}\Big(\frac{\partial\tilde a^j}{\partial x_i} \,|\, D_j(\psi^2\rho_m^2\tau_{i,h}u)\Big)\,dX \qquad (3.10)\\
&\quad + \int_Q \psi^2\rho_m\rho_m'\|\tau_{i,h}u\|^2\,dX \\
&\quad + \int_Q \Big(B^0(X,u,Du) \,|\, \tau_{i,-h}(\psi^2\rho_m^2\tau_{i,h}u)\}\Big)\,dX \\
&= B+C+D+E+F.
\end{aligned}$$

By assumption (1.4) and from Lemma 2.VI of [2], the integral on the left-hand side can be estimated in the following way

$$\begin{aligned}
(3.11)\quad A &\ge \nu(k)\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2\rho_m^2\|\tau_{i,h}Du\|^2(1+\|Du\|^2)^{\frac{q-2}{2}}\,dx \\
&\ge \nu\, C(K,q)\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2\rho_m^2\|\tau_{i,h}Du\|^2(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q-2}\,dx.
\end{aligned}$$

On the other hand, from (3.8) and by assumption (1.3) it follows that

$$\sum_{k=1}^{N}\sum_{r=1}^{n}\left\|\frac{\partial\tilde a^j}{\partial p_k^r}\right\| \le M(k)(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q-2}, \qquad (3.12)$$

$$\left\| \frac{\partial \tilde{a}^j}{\partial x_i} \right\| + \sum_{k=1}^{N} \left\| \frac{\partial \tilde{a}^j}{\partial u_k} \right\| \le M(k)(1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-1}. \tag{3.13}$$

Then, we obtain that

$$\begin{aligned}
|B| &\le c(k,q,\sigma,n) \\
&\quad \cdot \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} u\|\, \|\tau_{i,h} Du\|\, dX \\
&\le c(k,q,\sigma,n) \\
&\quad \cdot \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} Du\|^2\, dx \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} u\|^2\, dx \right)^{\frac{1}{2}}
\end{aligned}$$

and from this inequality it follows, $\forall\, \epsilon > 0$, that

$$\begin{aligned}
|B| \le{}& \frac{\epsilon}{3} \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} Du\|^2\, dx \\
&+ c(k,q,\sigma,n,\epsilon) \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q} \|\tau_{i,h} u\|^2\, dx.
\end{aligned} \tag{3.14}$$

Analogously, we have

$$\begin{aligned}
|C| &\le c(k,q,\sigma,n) \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-1} \|\tau_{i,h} u\| \\
&\quad \cdot (\psi^2 \rho_m^2 \|\tau_{i,h} Du\| + c(\sigma) \psi \rho_m^2 \|\tau_{i,h} u\|)\, dx \\
&\le c(k,q,\sigma,n) \\
&\quad \cdot \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-1} \psi^2 \rho_m^2 \|\tau_{i,h} u\|\, \|\tau_{i,h} Du\|^2\, dx \\
&\quad + c(k,q,\sigma,n) \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-1} \psi \rho_m^2 \|\tau_{i,h} u\|^2\, dx \\
&\le c(k,q,\sigma,n) \\
&\quad \cdot \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} Du\|^2\, dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$\cdot \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)\, \|\tau_{i,h} u\|^2 \, dx \right)^{\frac{1}{2}}$$
$$+ c(k, q, \sigma, n) \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^q \|\tau_{i,h} u\|^2 \, dx.$$

Then, $\forall\, \epsilon > 0$ it follows

$$\begin{aligned} |C| \leq \frac{\epsilon}{3} \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} Du\|^2 \, dx \\ + c(k, q, \sigma, n, \epsilon) \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^q \|\tau_{i,h} u\|^2 \, dx. \end{aligned} \tag{3.15}$$

Moreover, by assumption (1.3) and from Lemma 2.1 we obtain that

$$\begin{aligned} |D| &\leq c(k, n, q)|h| \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-1} \\ &\quad \cdot (\psi^2 \rho_m^2 \|\tau_{i,h} Du\| + c(\sigma) \psi \rho_m^2 \|\tau_{i,h} u\|)\, dx \\ &\leq c(k, n, q)|h| \\ &\quad \cdot \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} Du\|^2 \, dx \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^q \, dx \right)^{\frac{1}{2}} \\ &\quad + c(k, \sigma, n, q)|h| \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi \rho_m^2 \|\tau_{i,h}\, u\|^q \, dx \right)^{\frac{1}{q}} \\ &\quad \cdot \left(\int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^q \, dx \right)^{\frac{q-1}{q}}. \end{aligned}$$

Then, $\forall\, \epsilon > 0$ it follows that

$$\begin{aligned} |D| \leq \epsilon \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2} \|\tau_{i,h} Du\|^2 \, dx \\ + c(k, q, n, \epsilon)|h|^2 \int_{-b^*}^{-1/m} dt \int_{B(3\sigma)} (1 + \|Du\|)^q \, dx \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
&+ c(k,q,n,\sigma)|h|^2 \left(\int_{-b^*}^{-1/m} dt \int_{B(3\sigma)} (1+\|Du\|)^q\,dx\right)^{\frac{1}{q}}\\
&\cdot\left(c(q)\int_{-b^*}^{-1/m} dt \int_{B(3\sigma)} (1+\|Du\|)^q\,dx\right)^{\frac{q-1}{q}}\\
&\le \epsilon \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2\rho_m^2 (1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q-2}\|\tau_{i,h}\,Du\|^2\,dx\\
&+ c(k,\sigma,n,q,\epsilon)|h|^2 \int_{-b^*}^{-1/m} dt \int_{B(3\sigma)} (1+\|Du\|)^q\,dx.
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
|E| &= \int_Q \psi^2\rho_m\rho_m'\|\tau_{i,h}u\|^2\,dx\\
&\le |h|^2 \int_{-b^*}^{-a} dt \int_{B(2\sigma)} \psi^2\rho_m\rho_m'\|Du\|^2\,dx \qquad (3.17)\\
&\le \frac{|h|^2}{b-a}\int_{-b^*}^{-a} dt \int_{B(2\sigma)} (1+\|Du\|)^q\,dx,
\end{aligned}
$$

taking into account that

$$
\rho_m\rho_m' \begin{cases} \le 0 & \text{if } \; -\dfrac{2}{m}\le t\le \dfrac{1}{m}\,,\\[2mm] =0 & \text{if }\; t\le -b \;\text{ or }\; t\ge -\dfrac{1}{m} \;\text{ or }\; -a\le t\le -\dfrac{2}{m}\,,\\[2mm] \le \dfrac{1}{b-a} & \text{if }\; -b\le t\le -a. \end{cases}
$$

By assumption (1.2), we have moreover

$$
\begin{aligned}
|F| &\le \int_Q \|B^0(X,u,Du)\|\,\|\tau_{i,-h}(\psi^2\rho_m^2\tau_{i,h}u)\|\,dX \qquad (3.18)\\
&\le c(k,q)\int_{-b^*}^{-1/m} dt \int_{B(\frac{5}{2}\sigma)} (1+\|Du\|^2)^{\frac{q}{2}}\,\|\tau_{i,-h}(\psi^2\tau_{i,h}\,u)\|\,dx.
\end{aligned}
$$

From (3.10)–(3.18), with $\epsilon = \dfrac{\nu}{6}$ in (3.14), (3.15), (3.16), it follows, for each integer

i, $1 \le i \le n$, and for each $|h| < \min\left\{1, \dfrac{\sigma}{2}\right\}$

$$
(3.19)\quad
\begin{aligned}
&\frac{\nu}{2} c(k,q) \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 \|\tau_{i,h}\, Du\|^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2}\, dx \\
&\le c(k,\sigma,q,a,b,n,\nu) |h|^2 \int_{-b^*}^{-1/m} dt \int_{B(3\sigma)} (1+\|Du\|)^q\, dx \\
&\quad + c(k,\sigma,q,n,\nu) \int_{-b^*}^{-1/m} dt \int_{B(2\sigma)} \|\tau_{i,h} u\|^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^q\, dx \\
&\quad + \int_{-b^*}^{-1/m} dt \int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2)^{\frac{q}{2}} \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\|\, dx.
\end{aligned}
$$

Let us consider now the last integral that appears at the right hand side of (3.19). From Theorem 3.III of [4] (with $\sigma_0 = 3\sigma, a = b^*$) we deduce that

$$
(3.20)\qquad u \in L^q(-b^*, 0, H^{1+\theta,q}(B(\frac{5}{2}\sigma), \mathbb{R}^N)), \quad \forall\, \theta \in \left(0, \frac{2}{q}\right)
$$

and

$$
(3.21)\quad
\begin{aligned}
&\int_{-b^*}^{0} |Du|^q_{\theta,q,B(\frac{5}{2}\sigma)}\, dt \\
&\le c(\nu,k,U,\theta,\lambda,\sigma,q,a,b,n) \int_{-b}^{0} dt \int_{B(3\sigma)} (1+\|Du\|)^q\, dx
\end{aligned}
$$

hence, thanks also to the assumption $u \in C^{0,\lambda}(Q, \mathbb{R}^N)$, it results for a.e. $t \in (-b^*, 0)$

$$
u(x,t) \in H^{1+\theta,q}(B(\frac{5}{2}\sigma), \mathbb{R}^N) \cap C^{0,\lambda}(B(\frac{5}{2}\sigma), \mathbb{R}^N), \quad \forall\, \theta \in \left(0, \frac{2}{q}\right).
$$

From Lemma 2.3 (with $\Omega = B(\frac{5}{2}\sigma)$ and $\theta = 1 - \frac{\lambda}{2}$) we get for a.e. $t \in (-b^*, 0)$

$$
(3.22)\qquad u(x,t) \in W^{1,p}(\Omega, \mathbb{R}^N) \ \text{ where } \ p = 2q + \frac{2q^2\lambda}{2n - \lambda q}
$$

and

$$
(3.23)\qquad \|u\|_{1,p,B(\frac{5}{2}\sigma)} \le c(\lambda,\sigma,n) \|u\|^{\frac{1}{2}}_{2-\frac{\lambda}{2},q,B(\frac{5}{2}\sigma)} \|u\|^{\frac{1}{2}}_{C^{0,\lambda}(B(\frac{5}{2}\sigma),\mathbb{R}^N)}.
$$

Now, since $p > 2q$, we obtain

$$W^{1,p}(B(\frac{5}{2}\sigma), \mathbb{R}^N) \subset W^{1,2q}(B(\frac{5}{2}\sigma), \mathbb{R}^N)$$

and this is an algebraic and topological inclusion; from which by (3.22) and (3.23), it follows for a.e. $t \in \left(-b^*, -\frac{1}{m}\right)$ that

$$u(x,t) \in W^{1,2q}(B(\frac{5}{2}\sigma), \mathbb{R}^N) \tag{3.24}$$

and

$$\begin{aligned} &\|u\|^{2q}_{1,2q,B(\frac{5}{2}\sigma)} \\ &\leq c(k,U,\lambda,\sigma,n)\|u\|^{q}_{2-\frac{\lambda}{2},q,B(\frac{5}{2}\sigma)} \\ &\leq c(k,U,\lambda,\sigma,n)\left\{1 + |u|^{q}_{1,q,B(\frac{5}{2}\sigma)} + |Du|^{q}_{1-\frac{\lambda}{2},q,B(\frac{5}{2}\sigma)}\right\}. \end{aligned} \tag{3.25}$$

This estimate holds in particular for a.e. $t \in \left(-b^*, -\frac{1}{m}\right)$; for such t therefore we obtain, $\forall\, \epsilon > 0$,

$$\begin{aligned} &c(k)\int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2)^{\frac{q}{2}} \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\|\, dx \\ &\leq \left(\int_{B(\frac{5}{2}\sigma)} |h|^{-2} \|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|^2\, dx\right)^{\frac{1}{2}} \left(c(k)\int_{B(\frac{5}{2}\sigma)} |h|^2 (1 + \|Du\|^2)^q\, dx\right)^{\frac{1}{2}} \\ &\leq \frac{\epsilon}{2}|h|^{-2} \int_{B(\frac{5}{2}\sigma)} \|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|^2\, dx + c(k,\epsilon)|h|^2 \int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2)^q\, dx \\ &\leq \frac{\epsilon}{2}\int_{B(2\sigma)} \|D(\psi^2\tau_{i,h}u)\|^2\, dx + c(k,\sigma,\epsilon)|h|^2 \left\{1 + \int_{B(\frac{5}{2}\sigma)} \|Du\|^{2q}\, dx\right\} \\ &\leq \epsilon \int_{B(2\sigma)} \psi^4 \|\tau_{i,h}Du\|^2\, dx + c(\sigma,\epsilon)\int_{B(2\sigma)} \psi^2\|\tau_{i,h}u\|^2\, dx \\ &\quad + c(k,\sigma,\epsilon)|h|^2 \left\{1 + \int_{B(\frac{5}{2}\sigma)} \|Du\|^{2q}\, dx\right\} \\ &\leq \epsilon \int_{B(2\sigma)} \psi^2 \|\tau_{i,h}Du\|^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^{q-2}\, dx \\ &\quad + c(\sigma,\epsilon)\int_{B(2\sigma)} \psi^2 \|\tau_{i,h}u\|^2 (1 + \|Du\| + \|\tau_{i,h}\, Du\|)^q\, dx \\ &\quad + c(k,\sigma,\epsilon)|h|^2 \left\{1 + \|u\|^{2q}_{1,2q,B(\frac{5}{2}\sigma)}\right\}. \end{aligned}$$

From this, for $\epsilon = \frac{\nu}{4}$, it follows that

$$\begin{aligned}
&c(k)\int_{B(\frac{5}{2}\sigma)}(1+\|Du\|^2)^{\frac{q}{2}}\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|\,dx\\
&\le \frac{\nu}{4}\int_{B(2\sigma)}\psi^2\|\tau_{i,h}Du\|^2(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q-2}\,dx\\
&\quad + c(\sigma,\nu)\int_{B(2\sigma)}\psi^2\|\tau_{i,h}u\|^2(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q}\,dx\\
&\quad + c(k,\sigma,\nu)|h|^2\left\{1+\|u\|^{2q}_{1,2q,B(\frac{5}{2}\sigma)}\right\}
\end{aligned}$$

and, from which, by multiplicating both members with ρ_m^2 and by integrating with respect to t over $\left(-b^*,-\frac{1}{m}\right)$ we deduce

$$\begin{aligned}
&c(k)\int_{-b^*}^{-\frac{1}{m}}\rho_m^2\,dt\int_{B(\frac{5}{2}\sigma)}(1+\|Du\|^2)^{\frac{q}{2}}\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|\,dx\\
&\le \frac{\nu}{4}\int_{-b^*}^{-\frac{1}{m}}dt\int_{B(2\sigma)}\psi^2\|\tau_{i,h}Du\|^2\rho_m^2(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q-2}\,dx\\
&\quad + c(\sigma,\nu)\int_{-b^*}^{-\frac{1}{m}}dt\int_{B(2\sigma)}\|\tau_{i,h}u\|^2(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q}\,dx\\
&\quad + c(\nu,k,U,\lambda,\sigma,n)|h|^2\int_{-b^*}^{-\frac{1}{m}}\left\{1+\|u\|^{2q}_{1,2q,B(\frac{5}{2}\sigma)}\,dx\right\}dt.
\end{aligned}\tag{3.26}$$

Let us consider the penultimate integral that appears at the right hand side of (3.26) and (3.19). Using the Hölder inequality and thanks to Lemma 2.1 and (3.25), we have, for a.e. $t\in(-b^*,0)$, that [2]

$$\begin{aligned}
&\int_{B(2\sigma)}(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q}\|\tau_{i,h}u\|^2\,dx\\
&\le\left(\int_{B(2\sigma)}(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{p}\,dx\right)^{\frac{q}{p}}\\
&\quad\cdot\left(\int_{B(2\sigma)}\|\tau_{i,h}u\|^{\frac{4n}{n+q(\theta+\lambda-1)}}\,dx\right)^{\frac{n+q(\theta+\lambda-1)}{2n}}\\
&\le c(q)\left\{1+\left(\int_{B(\frac{5}{2}\sigma)}\|Du\|^p\,dx\right)\right\}^{\frac{q}{p}}|h|^2\left(\int_{B(\frac{5}{2}\sigma)}|Du|^{\frac{4n}{n+q(\theta+\lambda-1)}}\,dx\right)^{\frac{n+q(\theta+\lambda-1)}{2n}}
\end{aligned}$$

(2) $\frac{4n}{n+q(\theta+\lambda-1)}=\frac{2p}{p-q}<p$ since $2<p-q$

$$\le c(n,q,\sigma,\theta,\lambda)|h|^2\left\{1+\left(\int_{B(\frac{5}{2}\sigma)}\|Du\|^p\,dx\right)\right\}^{\frac{q}{p}}\left\{1+\left(\int_{B(\frac{5}{2}\sigma)}\|Du\|^p\,dx\right)\right\}^{\frac{2}{p}}$$

$$\le c(q,\sigma,\theta,n,\lambda)|h|^2\left\{1+|Du|_{0,p,B(\frac{5}{2}\sigma)}\right\}^{q+2}$$

$$\le c(n,q,\sigma,\theta,\lambda)|h|^2\left\{1+\|u\|^{2q}_{1,p,B(\frac{5}{2}\sigma)}\right\}$$

$$\le c(n,q,\lambda,\theta,\sigma)|h|^2\left\{1+|u|^q_{1,q,B(\frac{5}{2}\sigma)}+|Du|^q_{1-\frac{\lambda}{2},q,B(\frac{5}{2}\sigma)}\right\}$$

from which, by integrating with respect to t over $(-b^*,0)$ we have

$$\int_{-b^*}^0 dt\int_{B(2\sigma)}(1+\|Du\|+\|\tau_{i,h}\,Du\|)^q\|\tau_{i,h}u\|^2\,dx \tag{3.27}$$
$$\le c(q,\sigma,n,\theta,\lambda,k,U,a,b)|h|^2\left\{1+\int_{-b^*}^0\left(|u|^q_{1,q,B(\frac{5}{2}\sigma)}+|Du|^q_{1-\frac{\lambda}{2},q,B(\frac{5}{2}\sigma)}\right)dt\right\}.$$

From (3.19), (3.25), (3.26), (3.27) and (3.21) (for $\theta=1-\frac{\lambda}{2}$) we deduce, for each integer i, $1\le i\le n$, and for each $|h|<\min\left\{1,\frac{\sigma}{2}\right\}$, taking the limit as $m\to\infty$, we get

$$\begin{aligned}\frac{\nu}{4}c(k,q)\int_{-a}^0 dt\int_{B(\sigma)}\|\tau_{i,h}Du\|^2(1+\|Du\|+\|\tau_{i,h}\,Du\|)^{q-2}\,dx\\ \le c(q,\sigma,n,\nu,\lambda,k,U,a,b)|h|^2\left\{1+\int_{-b}^0|u|^q_{1,q,B(3\sigma)}\,dt\right\}\end{aligned} \tag{3.28}$$

and then, for any h such that $|h|<\min\left\{1,\frac{\sigma}{2}\right\}$ we have

$$\begin{aligned}\int_{-a}^0 dt\int_{B(\sigma)}\|\tau_{i,h}Du\|^2\,dx\\ \le c(q,\sigma,n,\nu,\lambda,k,U,a,b)|h|^2\left\{1+\int_{-b}^0|u|^q_{1,q,B(3\sigma)}\,dt\right\}.\end{aligned} \tag{3.29}$$

The estimate (3.29) is trivial if $\min\left\{1,\frac{\sigma}{2}\right\}<h<\sigma$ and then (3.29) will be true for each integer i, $1\le i\le n$ and for each $|h|<\sigma$.

From (3.29) and by Lemma 2.2 we have that

$$Du\in L^2(-a,0,H^{1,2}(B(\sigma),\mathbb{R}^N)) \tag{3.30}$$

and then

$$u\in L^2(-a,0,H^2(B(\sigma),\mathbb{R}^N)) \tag{3.31}$$

and moreover

$$\int_{-a}^{0} |u|^2_{2,B(\sigma)}\,dt \le c(\nu,k,U,\sigma,a,b,q,n)\left\{1+\int_{-b}^{0}|u|^q_{1,q,B(3\sigma)}\,dt\right\}. \tag{3.32}$$

It remains to show that $u \in H^1(-a,0,L^2(B(\sigma),\mathbb{R}^N))$ and that the relative estimate holds. From (3.25) it follows, for a.e. $t \in (-a,0)$

$$\int_{B(\sigma)} \|D_i u\|^{2q}\,dx \le c(k,U,\lambda,\sigma,n)\left\{1+|u|^q_{1,q,B(\frac{5}{2}\sigma)}+|Du|^q_{1-\frac{\lambda}{2},q,B(\frac{5}{2}\sigma)}\right\},$$

$i = 1,2,\dots,n$, from which and from (3.21), by integrating with respect to t in $(-a,0)$ we deduce

$$D_i u \in L^{2q}(B(\sigma)\times(-a,0),\mathbb{R}^N), \quad i=1,2,\dots,n,$$

and

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|Du\|^{2q}\,dx \le c(\nu,k,U,\lambda,\sigma,n,a,b)\left\{1+\int_{-b}^{0}|u|^q_{1,q,B(3\sigma)}\,dt\right\}. \tag{3.33}$$

Now, by assumption (1.2)

$$\|B^0(X,u,Du)\| \le M(k,q)(1+\|Du\|^q)$$

and then, from (3.33) we deduce that

$$B^0(X,u,Du) \in L^2(B(\sigma)\times(-a,0),\mathbb{R}^N) \tag{3.34}$$

and

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|B^0(X,u,Du)\|^2\,dx \le c(k,q)\int_{-a}^{0} dt \int_{B(\sigma)} (1+\|Du\|^{2q})\,dx. \tag{3.35}$$

On the other hand, by assumption (1.3) we have that

$$D_i a^i(X,u,Du) \in L^2(B(\sigma)\times(-a,0),\mathbb{R}^N), \quad i=1,2,\dots,n, \tag{3.36}$$

and that

$$\begin{aligned}&\int_{-a}^{0} dt \int_{B(\sigma)} \sum_{i=1}^{n} \|D_i a^i(X,u,Du)\|^2\,dx\\ &\quad\le c(k,n,q)\int_{-a}^{0} dt \int_{B(\sigma)} \left(1+\|Du\|^{2q}+\sum_{i,j=1}^{n}\|D_{i,j}u\|^2\right)dx.\end{aligned} \tag{3.37}$$

Now, taking into account that u is a solution in Q (and then in $B(\sigma) \times (-a, 0)$) of the system (1.1), we deduce that, for each $\varphi \in C_0^\infty(B(\sigma) \times (-a, 0), \mathbb{R}^N)$

$$\int_{-a}^{0} dt \int_{B(\sigma)} \left(u \,|\, \frac{\partial \varphi}{\partial t}\right) dx$$
$$= -\int_{-a}^{0} dt \int_{B(\sigma)} \left(\left(\sum_{i=1}^{n} D_i a^i(X, u, Du) + B^0(X, u, Du)\right) |\, \varphi\right) dx$$

from which, by (3.34) and (3.36), it results that

$$\exists\, \frac{\partial u}{\partial t} \in L^2(B(\sigma) \times (-a, 0), \mathbb{R}^N) \tag{3.38}$$

and from (3.35), (3.37) it follows that

$$\int_{-a}^{0} dt \int_{B(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dx$$
$$\leq c(k, n) \int_{-a}^{0} dt \int_{B(\sigma)} \left(1 + \|Du\|^{2q} + \sum_{i,j=1}^{n} \|D_{i,j} u\|^2\right) dx$$

and then, by (3.32), (3.33) we deduce that

$$\begin{aligned} &\int_{-a}^{0} dt \int_{B(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dx \\ &\leq c(\nu, k, U, \lambda, \sigma, a, b, n) \left\{1 + \int_{-b}^{0} |u|^q_{1,q,B(3\sigma)}\, dt\right\}. \end{aligned} \tag{3.39}$$

Finally we deduce (3.1) and (3.2) from (3.31), (3.32), (3.38), (3.39) [3] . □

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[3] Theorem 3.1 can be proved by substituting (1.3) and (1.4) with the monotony assumptions (1.3), (1.4) of [4] and by following the technique in [4] instead of the one in [3].

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