A combinatorial property and power graphs of semigroups

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Abstract. Research on combinatorial properties of sequences in groups and semigroups originates from Bernhard Neumann's theorem answering a question of Paul Erdös. For results on related combinatorial properties of sequences in semigroups we refer to the book [3]. In 2000 the authors introduced a new combinatorial property and described all groups satisfying it. The present paper extends this result to all semigroups.

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1. Introduction

Combinatorial properties of groups and semigroups with all infinite sequences or subsets having certain unavoidable regularities have been actively investigated by many authors. This research originates from the well-known theorem due to Bernhard Neumann [15], who answered a question of Paul Erdös by proving that a group is centre-by-finite if and only if every infinite sequence contains a pair of elements that commute. For earlier results concerning combinatorial properties of this sort we refer to [3], [4], [6], [7], [9], and [14].

A new combinatorial property was introduced in [10] using power graphs. It is closely related to conditions considered earlier and leads to nontrivial interaction between graph, group and semigroup methods. All groups satisfying this property have been described in [10]. The present paper extends this result to all semigroups using known facts on the structure of epigroups, see [18].

Throughout, the word 'graph' means a directed graph without multiple edges and loops. The *power graph* $\mathrm{Pow}(S)$ of a semigroup S has all elements of S as vertices, and has edges (u, v) for all $u, v \in S$, $u \neq v$, such that v is a power of u. Let D be a finite graph. A semigroup S is said to be *power D-saturated* if and only if, for each infinite subset T of S, the power graph of S has a subgraph isomorphic to D with all vertices in T. Note that if a group is power D-saturated

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for a graph D with edges, then it satisfies the Bernhard Neumann's property too, because every element commutes with its powers.

In [10] the authors described all groups G and graphs D such that G is power D-saturated. A natural question that arises concerns the structure of semigroups that satisfy this property. This question was answered in [12] and [13] for the cases of commutative and linear semigroups, respectively. In the present paper we extend these results using a characterization of epigroups to obtain a complete description of all pairs (D, S), where D is a graph and S is a semigroup, such that S is power D-saturated. Semigroups have various valuable applications in combinatorics and computer science. To mention only one example, they occur as syntactic monoids of languages (see, for example, [4], [9] and [16]). Similar questions concerning divisibility graphs were considered in [12].

2. Main theorem

A graph is said to be *acyclic* if it has no directed cycles. It is *null* if it has no edges. Let S be a semigroup. The semigroup S with zero 0 adjoined is denoted by $S^0 = S \cup \{0\}$. If $I \supseteq J \neq \emptyset$ are ideals of S, then the *factor* I/J of S is the semigroup on the set $(I \setminus J) \cup 0$ with multiplication defined by

$$xy = \begin{cases} xy & \text{if } \; xy \notin J \\ 0 & \text{otherwise.} \end{cases}$$

Let H be a group, Λ, I two nonempty sets, and let $Q = [q_{\lambda i}]$ be a $(\Lambda \times I)$-matrix with entries in H^0 such that no row or column consists entirely of zeros. Then the *Rees matrix semigroup* $M^0(H; I, \Lambda; Q)$ over H^0 with *sandwich-matrix* Q consists of zero 0 and all triples $(h; i, \lambda)$, for $i \in I$, $\lambda \in \Lambda$, and $h \in H^0$, where all triples $(0, i, \lambda)$ are identified with 0, and multiplication is defined by the rule

$$(h_1; i_1, \lambda_1)(h_2; i_2, \lambda_2) = (h_1 q_{\lambda_1 i_2} h_2; i_1, \lambda_2).$$

Let G be a group and p a prime such that p divides $|G|$. Then the elements with order of some power of p form a subgroup of G called a *primary component* of G. The *quasicyclic p-group* is the infinite group with generators $g_1, g_2, \ldots$ such that $g_1^p = e$ and $g_i^p = g_{i-1}$, for all $i > 1$, where e is the identity of the group. A group is *torsion* is all elements have finite order. A group G is *centre-by-finite* if the quotient group $|G/C(G)|$ is finite.

Obviously, finite semigroups are power D-saturated for all graphs D, and all semigroups are power D-saturated for null graphs. Therefore the following theorem describes all nontrivial pairs (D, S) such that S is power D-saturated.

Theorem 1. *Let D be a finite directed graph that is not null, and let S be an infinite semigroup. Then the following conditions are equivalent:*

(i) *the power graph of S is D-saturated;*

(ii) *D is acyclic and S^0 has a finite ideal series*

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0 \tag{1}$$

where every infinite factor S_j/S_{j-1} is a Rees matrix semigroup that has a finite sandwich matrix with entries in a centre-by-finite torsion group H_j such that each primary component of the center of H_j is finite or quasicyclic, the center of H_j has only a finite number of primary components, and the index of the center is not divisible by p for each quasicyclic p-subgroup of H_j.

3. Preliminaries

For notation and terminology of graph, group and semigroup theories not mentioned in this paper the reader is referred to [1], [5], [8], [9] and [17].

Let S be a semigroup. An element s of S is said to be *periodic* if there exist positive integers m, n such that $s^{m+n} = s^m$. The semigroup S is *periodic* if all elements of S are periodic. If $a, b \in S$, then we write $a\mathcal{H}b$ if $S^1a = S^1b$ and $aS^1 = bS^1$, where $S^1 = S \cup \{1\}$ stands for the semigroup S with identity 1 adjoined. The relation $\mathcal{H}$ is an equivalence relation (see [5, §2.1]).

A semigroup is said to be *completely simple* if it has no proper ideals and has a minimal idempotent. Similarly, a semigroup with zero is *completely* 0*-simple* if it has no proper nonzero ideals and has a minimal nonzero idempotent. It is well known that every completely simple semigroup is isomorphic to a Rees matrix semigroup $M(H; I, \Lambda; P)$ over a group H (see [5, Theorem 3.3.1]), and every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup $M^0(H; I, \Lambda; P)$ over a group H with zero adjoined. Conversely, every semigroup $M(H; I, \Lambda; P)$ is completely simple, and a semigroup $M^0(H; I, \Lambda; P)$ is completely 0-simple if and only if each row and column of P contains at least one nonzero entry (see [5, Theorem 3.2.3]).

Let H be a group, $G = M(H; I, \Lambda; P)$, and let $i \in I$, $\lambda \in \Lambda$. Then we put $G_{i\lambda} = \{(h; i, \lambda) \mid h \in H\}$. The set $G_{i\lambda}$ is an $\mathcal{H}$-class of G. When $G = M^0(H; I, \Lambda; P)$ we include zero in all of these sets, that is, we put $G_{i\lambda} = \{0\} \cup \{(h; i, \lambda) \mid h \in H\}$. In this case the set $G_{i\lambda} \setminus \{0\}$ is an $\mathcal{H}$-class of G. If an entry $p_{\lambda i}$ in the sandwich matrix P is zero, then the set $G_{i\lambda}^2 = 0$ by [5, Lemma 2.2.5]. If $p_{\lambda i} \neq 0$, then $G_{i\lambda}$ is a maximal subgroup of G isomorphic to H, by [5, Lemma 2.2.5].

An *epigroup* is a semigroup such that a power of each element belongs to a subgroup (see [18]). We use the following technical lemma.

Lemma 2 ([18, Proposition 11.1]).

An epigroup with finitely many idempotents has a finite ideal series in which each factor is either completely simple, completely 0*-simple or a nilsemigroup and,* *in the first two of these cases, the Rees matrix semigroups have finite sandwich matrices.*

A *complete symmetric graph* K_∞ is a graph with edges (u, v) and (v, u) for all distinct $u, v \in K_\infty$. An *infinite ascending* (resp., *descending*) *chain* A_∞ (resp., D_∞) is the graph with the set $\mathbb{Z}^+$ of all positive integers as vertices and with edges (i, j) such that $i < j$ (resp., $i > j$).

The following lemma found in [11] is needed for the proof of the main theorem.

Lemma 3 ([11, Lemma 4.3])**.** *Every infinite graph contains an infinite set of vertices which induces a null subgraph, an infinite ascending chain, an infinite descending chain or an infinite complete symmetric graph.*

We also need the following result obtained in [10].

Proposition 4 ([10])**.** *Let $D = (V, E)$ be a graph that is not null and let H be an infinite group. Then H is power D-saturated if and only if H is a centre-by-finite torsion group, the center $C(H)$ has a finite number of primary components, each primary component of $C(H)$ is finite or quasicyclic, the index of the center $C(H)$ is not divisible by p for each quasicyclic p-subgroup of H, and D is acyclic.*

4. Proofs

Lemma 5. *If D is a graph and S is a power D-saturated semigroup, then all subsemigroups and all quotient semigroups of S are power D-saturated, too.*

Proof: is straightforward and we omit it. □

Lemma 6. *If S is an infinite nil semigroup, then S has an infinite subset that induces a null subgraph in the power graph of S.*

Proof: Let 0 be the zero of S. Suppose to the contrary that $\mathrm{Pow}(S)$ has no infinite subsets which induce null subgraphs. Evidently, all elements of every infinite ascending chain or complete symmetric graph of a power graph of a semigroup are not nil. Therefore $\mathrm{Pow}(S)$ does not contain infinite ascending chains or complete symmetric graphs. It follows from Lemma 3 that $\mathrm{Pow}(S)$ has an infinite descending chain, say, $x_1, x_2, \ldots$ with edges (x_j, x_i) for all $1 \le i < j$. We may assume that $x_1 \neq 0$, since otherwise we can start the path with x_2.

Consider two elements x_2x_i and x_2x_j in S, where $2 < i < j$. Since x_1, x_2, x_i and x_j are vertices of our chain, we know that there exist positive integers $k_1, k_2, k_3, k_4 > 1$ such that

$$x_2^{k_1} = x_1 \tag{2}$$

$$x_i^{k_2} = x_j^{k_3} = x_2 \tag{3}$$

$$x_j^{k_4} = x_i. \tag{4}$$

Then $x_1 = x_2x_i(x_i^{k_2-1}x_2^{k_1-2})$ and $x_2x_j(x_j^{k_4-1}) = x_2x_i$ and so x_2x_i and x_2x_j are non-zero elements in S.

Suppose that x_2x_i and x_2x_j are adjacent in the power graph $\mathrm{Pow}(S)$. If

$$(x_2x_j)^k = x_2x_i,$$

for some positive integer k, then by combining equations (3) and (4) and equating indices we get

$$k(k_3 + 1) = k_3 + k_4.$$

If $k = 1$ then $k_4 = 1$, which is a contradiction. If $k > 1$, then

$$k(k_3 + 1) > 2(k_3) > k_3 + k_4 = k(k_3 + 1)$$

and again we have a contradiction.

On the other hand, if

$$(x_2x_i)^k = x_2x_j,$$

for some $k > 1$ we get

$$k(k_3 + k_4) = k_3 + 1.$$

Then

$$k(k_3 + k_4) > k_3 + k_4 > k_3 + 1 = k(k_3 + k_4),$$

and another contradiction arises.

These contradictions show that all elements $x_2x_3, x_2x_4, x_2x_5, \ldots$ are not adjacent in $\mathrm{Pow}(S)$, which completes the proof.

□

Proof of Theorem 1: (i)$\Rightarrow$(ii): If S has an element s that is not periodic, then the vertices $s^2, s^3, s^5, \ldots$ are not adjacent in $\mathrm{Pow}(S)$. Obviously, if an infinite sequence of vertices induces a null subgraph of $\mathrm{Pow}(S)$, then S is not power D-saturated. This contradiction shows that S is periodic, and so it is an epigroup.

If S contains infinitely many idempotents, then the idempotents are not adjacent in $\mathrm{Pow}(S)$, a contradiction. Therefore S contains a finite number of idempotents.

By Lemma 2, S has a finite ideal series

$$\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S,$$

where each factor S_j/S_{j-1} is nil, completely simple or completely 0-simple with finite sandwich matrix. Hence S^0 has a finite ideal series (1) where each factor S_j/S_{j-1} is nil or completely 0-simple with finite sandwich matrix.

Consider an infinite factor S_j/S_{j-1}. If S_j/S_{j-1} is nil, then Lemma 6 tells us that S_j/S_{j-1} contains an infinite subset that induces a null subgraph in

$\mathrm{Pow}(S_j/S_{j-1})$. Hence S has a quotient semigroup which is not D-saturated, and it follows by Lemma 5 that S is not D-saturated. Thus we see that every infinite factor of the ideal series is completely 0-simple.

Suppose that the sandwich matrix of an infinite $S_j/S_{j-1} = M^0(H; I, \Lambda; P)$ has a zero entry $p_{\lambda i}$. By Lemma 2, P is finite, and so H is infinite. Then $G_{i\lambda}^2 = 0$. Hence $s^\ell \in S_{j-1}$ for all $s \in G_{i\lambda}$ and $\ell > 1$. This means that the elements of $G_{i\lambda}$ induce an infinite null subgraph in $\mathrm{Pow}(S)$ and S is not D-saturated. This contradiction shows that all entries of P are nonzero. Therefore all subsets $G_{i\lambda} \in S_j/S_{j-1}$ are maximal subgroups of S_j/S_{j-1} isomorphic to H.

All subsemigroups of S are power D-saturated by Lemma 5. Thus we know by Proposition 4 that, if S_j/S_{j-1} is infinite, then every $\mathcal{H}$-class of S_j/S_{j-1} is an isomorphic copy of a centre-by-finite torsion group H, such that the center $C(H)$ has a finite number of primary components, each primary component of $C(H)$ is finite or quasicyclic and the order of $H/C(H)$ is not divisible by p for each quasicyclic p-subgroup of H.

Take any infinite $\mathcal{H}$-class. It contains a quasicyclic subgroup C_∞. Let $p_1, p_2, \dots$ be generators of C_∞, such that $p_1^p = e$ and $p_i^p = p_{i-1}$. Then it is routine to verify that the set $\{p_1, p_2, \dots\}$ induces a subgraph that is isomorphic to D_∞, which is of course acyclic. By the power D-saturation of S, we see that D embeds in this subgraph, and so D is acyclic too.

(ii) $\Rightarrow$(i): Pick any infinite subset T of S. Clearly, $T = \bigcup\{T \cap (S_j \setminus S_{j-1}) \mid 1 \le j \le n\}$, and so at least one of $T \cap (S_j \setminus S_{j-1})$ is infinite. Since all entries of the sandwich matrix of the Rees matrix semigroup $S_j \setminus S_{j-1} = M^0(H; I, \Lambda; P)$ are nonzero, we see that $S_j \setminus S_{j-1}$ is the union of 0 and a finite number of subgroups isomorphic to H. All of these subgroups are power D-saturated by Proposition 4. Hence at least one of these subgroups has an infinite intersection X with T. It follows that D embeds in the subgraph induced by the vertices of X. Therefore S is power D-saturated. $\square$

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