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Estimation variances for parameterized marked Poisson processes and for parameterized Poisson segment processes

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Abstract. A complete and sufficient statistic is found for stationary marked Poisson processes with a parametric distribution of marks. Then this statistic is used to derive the uniformly best unbiased estimator for the length density of a Poisson or Cox segment process with a parametric primary grain distribution. It is the number of segments with reference point within the sampling window divided by the window volume and multiplied by the uniformly best unbiased estimator of the mean segment length.

Keywords: complete statistic, compact sets process, intensity estimation, marked point process, Poisson process, random closed sets, Rao-Blackwell Theorem, segment process, spatial statistic, stochastic geometry, sufficient statistic

Classification: Primary 60D05; Secondary 62B05

1. Introduction

In [5], we found the minimum variance unbiased estimator for the function of the process intensity (for example: length density) in the case of a Poisson or Cox process of compact sets in $\mathbb{R}^d$ with known distribution of primary grain. This estimator was based on the number of compact sets in the window. In this note we extend the result for the case of parametric primary grain distribution.

The complete and sufficient statistic for the stationary marked Poisson process with a parametric distribution of mark with unknown parameter is found in Section 2.

The main aim of this note is to find a minimum variance unbiased estimator of the length density in the case of a stationary Poisson process of compact sets with a parametric primary grain distribution. If we consider only grains with reference point within the observation window and if we assume that we know these compact sets exactly then the problem should be reduced to the stationary marked point process with the distribution of marks equal to the distribution of the primary grain. There are cases when we know these compact sets exactly. For example in the process of discs or in the Stochastic Restoration Estimation [1] or in minus-sampling [6].

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In Section 3, we consider a stationary segment process with a parametric length distribution with unknown parameters and unknown arbitrary direction distribution. We assume again that we know exact length of each segment with reference point within the observation window. We propose to estimate the length density by the means of the number of segments with reference point within the sampling window multiplied by the uniformly best unbiased estimator of mean segment length and divided by the window volume. This estimator, however, does not use all information available in the window.

In Section 4 we compare the variance of the proposed estimator with the variance of the natural estimator (i.e., the estimator which is based on the total length of the segments in the observation window) in certain special cases. The edge effect has nearly no power in a sufficiently large window therefore our proposed estimator, based on the complete and sufficient statistic, has smaller variance in such a window.

2. Complete, sufficient statistic for a stationary marked Poisson process with a parametric distribution of marks

Consider a stationary marked Poisson process Φ on $\mathbb{R}^d$ with marks from a Polish space M (for details see for example [2] or [6]). Let α denote the intensity of Φ and $\Lambda_0(\theta)$ the parametric distribution of marks with an unknown parameter $\theta \in \Theta$. Suppose that every $\Lambda_0(\theta)$ is absolutely continuous with respect to some σ-finite measure ξ. Let $f_\theta(m), m \in M$, denote the density of the mark distribution and let $f_\theta(m_1, \dots, m_j) = f_\theta(m_1) \cdot \ldots \cdot f_\theta(m_j)$ denote the density of the joint distribution of j independent marks. Denote the distribution of Φ by $P_{\alpha,\theta}$ and the expectation with respect to $P_{\alpha,\theta}$ by $\mathbb{E}_{P_{\alpha,\theta}}$. Furthermore, let ϕ denote a realization of Φ and let $\mathcal{N}$ denote the set of all possible realizations.

Let $T_j : \mathcal{X} \subseteq M^j \to \mathbb{R}^l$ be a statistic for an independent j-tuple of marks which is complete and sufficient for θ. Assume, further, that the distribution of T_j is absolutely continuous with respect to some σ-finite measure ζ with density $f_{T_j}(\theta, t)$, $\theta \in \Theta$, $t \in T_j(\mathcal{X})$.

The model characteristic to be estimated is a function $\tau(\alpha, \theta) : \mathbb{R}^+ \times \Theta \to \mathbb{R}$. Suppose that the process is observed through a bounded measurable observation window $W \subseteq \mathbb{R}^d$ of positive Lebesgue measure. Let $\mathcal{E}_W$ be the set of all estimators which depend only on points from W. We are looking for an unbiased estimator $e \in \mathcal{E}_W$ of $\tau(\alpha, \theta)$ which is the minimum variance unbiased estimator from $\mathcal{E}_W$.

Let $S_T : \mathcal{N} \to \binom{\mathbb{N}}{\mathbb{R}^l}$; $\phi \mapsto \binom{\phi(W)}{T_{\phi(W)}}$ denote the statistic of Φ. S_T has the joint density

$$f(\Phi(W) = j, T_{\Phi(W)} = t) = \exp(-\alpha\mu^d(W)) \frac{(\alpha\mu^d(W))^j}{j!} \cdot f_{T_j}(\theta, t),$$

where μ^d is the d-dimensional Lebesgue measure.

Theorem 1. *The statistic S_T is a complete and sufficient for (α, θ).*

Proof: 1) The proof of sufficiency will be based on Factorization criterion [3, Theorem 1.5.2].

The intensity measure Λ is absolutely continuous with respect to $\mu^d \times \xi$ and $\widetilde{\Phi} = \Phi|(W \times M)$ is a finite marked point process in a bounded region, hence we can write its density on $\bigcup_{j=0}^{\infty}(W \times M)^j$ with respect to $\nu = \sum_{j=0}^{\infty}(\mu^d \times \xi)^j$ as

$$f(\{z_1, m_1\}, \dots, \{z_j, m_j\}) = \exp(-\alpha\mu^d(W))\frac{(\alpha\mu^d(W))^j}{j!} \cdot g(z_1, m_1) \cdot \ldots \cdot g(z_j, m_j),$$

where $g = \dfrac{\widetilde{\Lambda}(d(z,m))}{\mu^d(dz) \times \xi(dm)}$. We have $g(z,m) = I_W(z) \cdot f_\theta(m)$ since Φ is a stationary marked Poisson process. Factorization criterion for the sufficient statistic T_j says that there exist non-negative measurable functions such that

$$f_\theta(m_1) \cdot \ldots \cdot f_\theta(m_j) = g_j(\theta, T_j(m_1, \dots, m_j)) \cdot h_j(m_1, \dots, m_j).$$

Together we can write the density as

$$\begin{aligned} f(\{z_1, m_1\}, \dots, \{z_j, m_j\}) = \exp(-\alpha\mu^d(W))\ \frac{(\alpha\mu^d(W))^j}{j!}\ g_j(\theta, T_j(m_1, \dots, m_j)) \times \\ \times I_W(z_1) \dots I_W(z_j)\ h_j(m_1, \dots, m_j) \end{aligned}$$

and by Factorization Criterion the statistic S_T is sufficient for α, θ.

2) Let $h(j,t) : (\mathbb{N} \times T_j(\mathcal{X})) \to \mathbb{R}$ be a real integrable function such that $E_{P_{\alpha,\theta}} h(\Phi(W), T_{\Phi(W)}) = 0$ for all α, θ. We have

$$\sum_{j=0}^{\infty} \int \frac{(\alpha\mu^d(W))^j}{j!} e^{-\alpha\mu^d(W)} \cdot f_{T_j}(\theta, t) \cdot h(j,t) \zeta(dt) = 0 \quad \forall \alpha > 0, \theta \in \Theta,$$

$$e^{-\alpha\mu^d(W)} \cdot \sum_{j=0}^{\infty} \frac{(\alpha\mu^d(W))^j}{j!} \int f_{T_j}(\theta, t) \cdot h(j,t) \zeta(dt) = 0 \quad \forall \alpha > 0, \theta \in \Theta.$$

Since $e^{-\alpha\mu^d(W)} > 0$ for any $\alpha > 0$ all coefficients of the power series in α must vanish. Thus

$$\int f_{T_j}(\theta, t) \cdot h(j,t) \zeta(dt) = 0 \quad \forall \theta \in \Theta,\ \forall j \in \mathbb{N}.$$

T_j is a complete statistic, hence $h(j,t) = 0$ $\forall j \in \mathbb{N}$, $\forall t \in \mathbb{R}^+$, *a.s.* and S_T is complete for (α, θ). □

The Rao-Blackwell theorem [3] together with the previous theorem yields the next result.

Corollary 1. *Let $\tau(\alpha, \theta) : \mathbb{R}^+ \times \Theta \to \mathbb{R}$ be a function of the parameters α, θ which we want to estimate. If e is an unbiased estimator of $\tau(\alpha, \theta)$ then*

$$\mathbb{E}\left[e(\Phi) \mid S_T\right]$$

is the minimum variance unbiased estimator among all unbiased estimators from $\mathcal{E}_W$.

Remark 1. It is easy to show that Corollary 1 holds for a stationary marked mixed Poisson process [6, Chapter 5.2] as well.

3. Poisson segment processes

The method introduced above will be demonstrated on a stationary Poisson segment process now. For detailed introduction of the segment process we refer to [5].

Suppose now that we know exact length and direction of every segment which has the reference point inside of the observation window. Then we consider this process as a marked Poisson point process with the distribution of marks $\Lambda_0(\theta)$ which lives on the space $M = \mathbb{R}^+ \times \mathcal{U}_d$, where $\mathcal{U}_d$ is the space of all 1-dimensional subspaces in $\mathbb{R}^d$ and it is isomorphic to the unit semisphere in $\mathbb{R}^d$. Denote a mark by a pair (r, β), where $r \in \mathbb{R}^+$ is the segment length and $\beta \in \mathcal{U}_d$ is the segment direction. Suppose again that each $\Lambda_0(\theta)$ is absolutely continuous with respect to some σ-finite measure ξ on $\mathbb{R}^+ \times \mathcal{U}_d$. Thus there exists a density of marks which we denote again $f_\theta(r, \beta)$, $r \in \mathbb{R}^+$, $\beta \in \mathcal{U}_d$.

Let $T_j : \mathbb{R}^{+j} \to \mathbb{R}^l$ be a statistics of the independent j-tuple of the marks and assume that T_j is complete and sufficient for θ and that its distribution is absolutely continuous with respect to some σ-finite measure η. Hence, for example, we can determine the minimum variance unbiased estimator e_j of $\mathbb{E}r$, where $\mathbb{E}r$ denotes the expectation of the segment length.

Let $\lambda = \mathbb{E}\sum_{S\in\Phi} H^1(S \cap [0,1]^d)$ denote the length density of a stationary segment process where H^1 is the 1-dimensional Hausdorff measure.

We want to find a minimum variance unbiased estimator for λ of a stationary Poisson segment process with a parametric distribution of primary grain with unknown parameters from the set of estimators $\mathcal{E}_W$. It is easy to see that $\lambda = \tau(\alpha, \theta) = \alpha\mathbb{E}r$.

Let e_λ denote the estimator of the length density $e_\lambda : \mathcal{N} \to \mathbb{R}^+; \quad \phi \mapsto \frac{\phi(W)}{\mu^d(W)} \cdot e_{\phi(W)}$, where $\phi(W)$ denotes the number of segment reference points in the observation window W.

Theorem 2. *The estimator e_λ of the length density λ of a stationary Poisson segment process with a parametric distribution of primary grain with unknown parameters has minimum variance among all unbiased estimators from $\mathcal{E}_W$.*

PROOF: We can express the expectation of e_λ because we know the density of the process.

$$\mathbb{E}_{P_{\alpha,\theta}} e_\lambda(\Phi) = \sum_{j=0}^{\infty} \exp(-\alpha\mu^d(W)) \frac{\alpha^j}{j!} \times \Big[\overbrace{\int_M \cdots \int_M}^{j} \overbrace{\int_W \cdots \int_W}^{j} \frac{j}{\mu^d(W)} \times$$
$$\times\, e_j((r_1,\beta_1),\dots,(r_j,\beta_j))$$
$$f_\theta(r_1,\beta_1)\dots f_\theta(r_j,\beta_j)(dz_1)\dots(dz_j)\ \xi(d(r_1,\beta_1))\dots\xi(d(r_j,\beta_j))\Big]$$
$$= \frac{1}{\mu^d(W)} \sum_{j=0}^{\infty} j \cdot \exp(-\alpha\mu^d(W)) \frac{\alpha^j \mu^d(W)^j}{j!} \times \overbrace{\int_M \cdots \int_M}^{j} \times$$
$$\times\, e_j((r_1,\beta_1),\dots,(r_j,\beta_j))$$
$$f_\theta(r_1,\beta_1)\dots f_\theta(r_j,\beta_j)\xi(d(r_1,\beta_1))\dots\xi(d(r_j,\beta_j)) = \alpha \cdot \mathbb{E}r.$$

Thus e_λ is an unbiased estimator of λ. We finish the proof by applying Corollary 1. □

Assume additionally that the typical segment length and direction are independent random variables. It means that Λ_0 corresponds to the distribution product $\mathcal{D} \times \rho$. If $\mathcal{D}$ is the exponential distribution with parameter μ (ρ is arbitrary) then $\sum_{i=1}^{j} r_i$ is the complete and sufficient statistic for μ. Hence the minimum variance unbiased estimator of the length density λ is

$$e_{\lambda_1} = \frac{\phi(W)}{\mu^d(W)} \frac{\sum_{i=1}^{\phi(W)} r_i}{\phi(W)} = \frac{\sum_{i=1}^{\phi(W)} r_i}{\mu^d(W)}.$$

If $\mathcal{D}$ is the uniform distribution $\mathrm{U}(0,A)$ (ρ is arbitrary) then $\max_{i=1,\dots,j} r_i$ is the complete and sufficient statistic for A. Hence the minimum variance unbiased estimator of the length density λ is

$$e_{\lambda_2} = \frac{\phi(W)}{\mu^d(W)} \frac{\phi(W)+1}{2\cdot\phi(W)} \max_{i=1,\dots,\phi(W)} r_i = \frac{\phi(W)+1}{2\cdot\mu^d(W)} \max_{i=1,\dots,\phi(W)} r_i.$$

Theorem 3. *The variances of the estimators defined above are*

$$\mathrm{var}(e_{\lambda_1}) = \frac{\alpha\ \mathbb{E}r^2}{\mu^d(W)},$$
$$\mathrm{var}(e_{\lambda_2}) = \frac{A^2}{4}\left[\frac{\alpha}{\mu^d(W)} + \frac{1}{\mu^d(W)^2} - \frac{2}{\alpha\mu^d(W)^3} + \frac{2}{\alpha^2\mu^d(W)^4} - \frac{2}{\alpha^2\mu^d(W)^4} e^{-\alpha\mu^d(W)}\right].$$

Proof: Let $r(S)$ denote the length of the segment S. Using [5, Lemma 2] we get

$$\begin{aligned}
\operatorname{var}[e_{\lambda_1}] &= \int \left(\frac{r(S)}{\mu^d(W)} I_{\mathcal{W}_1} S\right)^2 \Lambda_P(dS) = \frac{1}{(\mu^d(W))^2} \int_{\mathcal{W}_1} [r(S)]^2 \, \Lambda_P(dS) \\
&= \frac{\alpha}{(\mu^d(W))^2} \int_{\mathcal{U}_d} \int_{\mathbb{R}+} \int_W r^2 \, dz \, \Lambda_0(d(r,\beta)) \\
&= \frac{\alpha \mu^d(W)}{(\mu^d(W))^2} \int_{\mathbb{R}+} \int_{\mathcal{U}_d} r^2 f_\theta(r,\beta) \, \xi(d(r,\beta)) \\
&= \frac{\alpha}{\mu^d(W)} \int_{\mathbb{R}+} r^2 f_\theta(r) \int_{\mathcal{U}_d} f_\theta(\beta) \, \zeta(d\beta) \, \vartheta(dr) \\
&= \frac{\alpha}{\mu^d(W)} \int_{\mathbb{R}+} r^2 f_\theta(r) \, \vartheta(dr) = \frac{\alpha \, \mathbb{E} r^2}{(\mu^d(W))}.
\end{aligned}$$

We cannot use [5, Lemma 2] in the second part of the proof therefore we have to compute $\mathbb{E}(e_{\lambda_2})^2$ directly. After longer but straightforward computation we received the result [4]. □

Remark 2. We can improve the estimator e_λ if we select other reference points, e.g. the lexicographic maximum. Then we get another estimator. We have more estimators thus we can average them and get even lower variance. This variance may be computed using [5, Lemma 2] in some special cases. This will be illustrated in the next section.

Remark 3. It is easy to show that Theorems 2, 3 and Remark 2 can be extended to Mixed Poisson processes.

4. Comparison of some estimators

So far we have supposed that we know the lengths of all segments. It means that we have used some information outside the window W, otherwise the estimator e_λ does not use all segments hitting W. Furthermore, e_λ is easily implementable and does not depend on the directional distribution. If we consider all segments hitting W in the estimator, then we use all information which we know, and the minimum variance unbiased estimator on this set of estimators will be the best among all unbiased estimators. But this estimator is too complicate and it is hardly applicable in practice. Now a question arises whether the commonly used estimator

$$\tilde{e}(\Phi) = \frac{1}{\mu^d(W)} \sum_{S \in \Phi : S \cap W \neq \emptyset} H^1(S \cap W)$$

which uses all segments hitting W can have lower variance then the minimum variance unbiased estimator e_λ from $\mathcal{E}_W$. Here $H^1(S \cap W)$ is the length of the visible part of the segment S.

We consider a stationary Poisson segment process on $\mathbb{R}^2$. Assume for simplicity that the observation window W is a square of side length a. Under this assumptions the variance of the common unbiased estimator $\tilde{e}$ is [5]

$$\text{(3)} \quad \mathrm{var}[\tilde{e}] = \frac{\alpha}{a^4}\left[a^2\mathbb{E}r^2 - \frac{1}{3}a\mathbb{E}r^3\mathbb{E}(\sin|\beta| + \cos|\beta|) + \frac{1}{3}\mathbb{E}r^4\mathbb{E}(\sin|\beta|\cos|\beta|)\right].$$

Example 1. Comparison of the variances of the estimators e_{λ_1} and $\tilde{e}$.

Let Φ be a Poisson segment process in $\mathbb{R}^2$ with the exponential length distribution with parameter μ and arbitrary direction distribution. Let the typical segment length and the direction be independent random variables. If the observation window W is a square of side length a then Theorem 3 and formula (3) give the variances of e_{λ_1} and $\tilde{e}$. After comparison of the variances of these two estimators we receive that there exists an $a_0 \geq 0$ such that $\mathrm{var}[e_{\lambda_1}] > \mathrm{var}[\tilde{e}]$ whenever $a > a_0$ and there exists an $a'_0 \geq 0$ such that $\mathrm{var}[\frac{e_{\lambda_1}+e'_{\lambda_1}}{2}] < \mathrm{var}[\tilde{e}]$ whenever $a > a'_0$, where e'_{λ_1} is the same estimator as e_{λ_1} but based on the lexicographic maximum reference points. The variance of $\frac{e_{\lambda_1}+e'_{\lambda_1}}{2}$ was computed via [5, Lemma 2]. Moreover all these three estimators are asymptotically equivalent as $a \to \infty$.

Table 1. Comparison of the variances of the estimators $\tilde{e}$ and e_{λ_1} in some special cases. Deterministic direction is chosen in such a way that the direction is parallel to one side of the square.

ρ	$\mathrm{var}(\tilde{e})$	$\mathrm{var}(e_{\lambda_1})$	$\mathrm{var}(\frac{e_{\lambda_1}+e'_{\lambda_1}}{2})$	a_0	a'_0
Deterministic	$\frac{2\alpha}{\mu^2a^2} - \frac{2\alpha}{\mu^3a^3}$	$\frac{2\alpha}{\mu^2a^2}$	$\frac{2\alpha}{\mu^2a^2} - \frac{3\alpha}{\mu^3a^3} + O(\frac{\alpha e^{-\mu a}}{2a\mu})$	0	$1.15\mathbb{E}r$
Unif.$(-\frac{\pi}{2}, \frac{\pi}{2}]$	$\frac{2\alpha}{\mu^2a^2} - \frac{8\alpha}{\pi\mu^3a^3} + \frac{8\alpha}{\pi\mu^4a^4}$	$\frac{2\alpha}{\mu^2a^2}$	$\frac{2\alpha}{\mu^2a^2} - \frac{12\alpha}{\pi\mu^3a^3} + \frac{12\alpha}{\pi\mu^4a^4} + o(\alpha e^{-\mu a})$	$\mathbb{E}r$	$\approx \mathbb{E}r$

Example 2. Comparison of the variances of the estimators e_{λ_2} and $\tilde{e}$.

Let Φ be a Poisson segment process in $\mathbb{R}^2$ with a parametric length distribution which is the uniform distribution $U(0, A)$ and an arbitrary direction distribution. Let the typical segment length and the direction be independent random variables. If the observation window W is a square of side length a then Theorem 3 and formula (3) give the variances of e_{λ_2} and $\tilde{e}$. After comparison of the variances of these two estimators we receive that there exists an $a_0 \geq 0$ such that $\mathrm{var}[e_{\lambda_2}] < \mathrm{var}[\tilde{e}]$ whenever $a > a_0$. Moreover, $\frac{e_{\lambda_2}+e'_{\lambda_2}}{2}$ has even lower variance then e_{λ_2} where e'_{λ_2} is the same estimator as e_{λ_2} but based on the lexicographic maximum reference points. The variance of $\frac{e_{\lambda_2}+e'_{\lambda_2}}{2}$ was computed via [5, Lemma 2]. The estimators e_{λ_2} and $\frac{e_{\lambda_2}+e'_{\lambda_2}}{2}$ are asymptotically equivalent as $a \to \infty$, but

$$\lim_{a\to\infty} \frac{\mathrm{var}(e_{\lambda_2})}{\mathrm{var}(\tilde{e})} = \frac{3}{4}\,.$$

Table 2. Comparison of the variances of the estimators $\tilde{e}$ and e_{λ_2} in some special cases. Deterministic direction is chosen in such a way that the direction is parallel to one side of the square.

ρ	$\mathrm{var}(\tilde{e})$	$\mathrm{var}(e_{\lambda_2})$	a_0
Deterministic	$\frac{A^2\alpha}{3a^2} - \frac{A^3\alpha}{12a^3}$	$\frac{A^2\alpha}{4a^2} + \frac{A^2}{4a^4} - \frac{A^2}{2\alpha a^6} + \frac{A^2}{2\alpha a^8} - \frac{A^2 e^{-\alpha a^2}}{2\alpha a^8}$	$a_0 \leq A + \sqrt{\frac{3}{\alpha}}$
Unif. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\frac{A^2\alpha}{3a^2} - \frac{A^3\alpha}{3\pi a^3} + \frac{A^4\alpha}{15\pi a^4}$	$\frac{A^2\alpha}{4a^2} + \frac{A^2}{4a^4} - \frac{A^2}{2\alpha a^6} + \frac{A^2}{2\alpha a^8} - \frac{A^2 e^{-\alpha a^2}}{2\alpha a^8}$	$a_0 \leq \frac{4}{\pi}A + \sqrt{\frac{3}{\alpha}}$

Remark 4. The estimator e_{λ_2} is a function of the complete, sufficient statistic while the estimator $\tilde{e}$ is not. Hence the estimator e_{λ_2} has asymptotically lower variance then $\tilde{e}$. All the estimators e_{λ_1}, $\frac{e_{\lambda_1}+e'_{\lambda_1}}{2}$ and $\tilde{e}$ in Example 1 are functions of the complete, sufficient statistic therefore all the estimators are asymptotically equivalent.

We can see from these comparisons that using the estimator e_{λ_1} for the exponential length distribution does not bring nearly any improvement for the practice. Except the case where it is difficult to measure the length of the segments. On the other hand, the use of the estimator e_{λ_2} for the uniform length distribution brings a big improvement of the variance of the estimator even in the minus-sampling case as we can see in the following simulations.

We used a Poisson segment process with intensity λ, the uniform length distribution on $[0, H]$ and the uniform distribution of direction independent of the length distribution. We simulated this process in a square window with side length A. We estimated λ by the common estimator $\tilde{e}$ and by our estimator $\frac{e_{\lambda_2}+e'_{\lambda_2}}{2}$.

Because we do not know exact length of all segments from the realization we used a smaller window for estimating by e_{λ_2}. The window is decreased by H from 3 sides (upper, bottom, right) so that every segment which has reference point inside of this smaller window must have both endpoints inside W. Similarly we estimated e'_{λ_2}.

Table 3. Comparison of the variances of the estimators $\tilde{e}$ and $\frac{e_{\lambda_2}+e'_{\lambda_2}}{2}$ done by 1000 simulations for various parameters. Because there is 1 parameter more than it is necessary in the model, we will vary only λ and A. We set $H = 0.1$ throughout all simulations. The column N shows average number of segments which has reference point inside W and the column *ratio* shows $\frac{\mathrm{var}(\frac{e_{\lambda_2}+e'_{\lambda_2}}{2})}{\mathrm{var}(\tilde{e})}$.

λ	A	N	$\mathbb{E}(\tilde{e})$	$\mathbb{E}(\frac{e_{\lambda_2}+e'_{\lambda_2}}{2})$	$\mathrm{var}(\tilde{e})$	$\mathrm{var}(\frac{e_{\lambda_2}+e'_{\lambda_2}}{2})$	*ratio*
0.5	1	10	0.506365	0.505619	0.0324177	0.0379858	1.17
0.5	2	40	0.501595	0.502476	0.0082196	0.00754585	0.92
0.5	3	90	0.499526	0.498896	0.00379548	0.00319849	0.84
0.5	5	250	0.498365	0.498943	0.00141376	0.00114489	0.81
0.5	10	1000	0.500593	0.500492	0.000328948	0.000253791	0.77
1	1	20	1.0031	1.01682	0.0672616	0.0749364	1.11
1	2	80	0.996187	0.998461	0.0168793	0.0150844	0.89
1	3	180	1.00021	1.00096	0.00809763	0.00645902	0.80
1	5	500	0.995941	0.996918	0.0027469	0.00217183	0.79
1	10	2000	1.00001	1.00029	0.000657966	0.000508336	0.77
2.5	1	50	2.49811	2.50566	0.157888	0.173362	1.10
2.5	2	200	2.50665	2.5043	0.0411783	0.0358533	0.87
2.5	3	450	2.4985	2.49637	0.0192654	0.0150338	0.78
2.5	5	1250	2.50226	2.50029	0.00643597	0.00484536	0.75
5	2	400	4.99813	5.00004	0.078849	0.0689221	0.87
10	2	800	9.99727	10.0032	0.169924	0.145386	0.855
15	2	1200	14.9922	14.9813	0.250196	0.215067	0.86

We can see from the table that *ratio* converges to $3/4$ as the window increases for any intensity even for minus-sampling case. The table shows that *ratio* depends mainly on the ratio of H and A. Furthermore the table shows that the border effects have less influence on the estimator than the use of a sufficient statistic in sufficiently large observation windows.

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