# On conditions for the boundedness of the Weyl fractional integral on weighted $L^p$ spaces

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*Abstract.* In this paper we give a sufficient condition on the pair of weights $(w, v)$ for the boundedness of the Weyl fractional integral $I_\alpha^+$ from $L^p(v)$ into $L^p(w)$. Under some restrictions on $w$ and $v$, this condition is also necessary. Besides, it allows us to show that for any $p : 1 \leq p < \infty$ there exist non-trivial weights $w$ such that $I_\alpha^+$ is bounded from $L^p(w)$ into itself, even in the case $\alpha > 1$.

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## 1. Introduction and main results

Let $0 < \alpha < 1$. Given a locally integrable function $f$ on $\mathbb{R}$, its Weyl fractional integral is defined by

$$I_\alpha^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\, dy. \tag{1.1}$$

Similarly, the Riesz fractional integral is given by

$$I_\alpha^- f(x) = \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}}\, dy. \tag{1.2}$$

By a weight $w$ we mean a locally integrable, non-negative function defined on $\mathbb{R}$. For any Lebesgue measurable set $E \subseteq \mathbb{R}$ we denote the $w$-measure of $E$ by $w(E) = \int_E w(x)\, dx$, and the characteristic function of $E$ by $\chi_E$.

Throughout the paper, $C$ shall be a positive constant not necessarily the same at each occurrence.

Let $w$ and $v$ be two weights on $\mathbb{R}$ and $1 < p < \infty$. We consider the weighted norm inequality,

$$\left[\int_{-\infty}^{+\infty} |I_\alpha^+ f(x)|^p w(x)\, dx\right]^{1/p} \leq C \left[\int_{-\infty}^{+\infty} |f(x)|^p v(x)\, dx\right]^{1/p}, \tag{1.3}$$

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for every $f$ in $L^p(v)$. If we denote $\sigma(x) = v(x)^{1-p'}$, where $1/p + 1/p' = 1$, then (1.3) is equivalent to

$$\left[\int_{-\infty}^{+\infty} |I_\alpha^+(f\sigma)(x)|^p w(x)\,dx\right]^{1/p} \le C\left[\int_{-\infty}^{+\infty} |f(x)|^p \sigma(x)\,dx\right]^{1/p}, \tag{1.4}$$

for every $f$ in $L^p(\sigma)$.

The fractional maximal operator,

$$M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(t)|\,dt$$

satisfies the inequality $M_\alpha^+ f(x) \le I_\alpha^+(|f|)(x)$. The boundedness of $M_\alpha^+$ from $L^p(v)$ into $L^p(w)$ implies that there exists a constant $C > 0$ such that for every $a < b$,

$$\left(\int_{-\infty}^{a} \frac{w(y)}{(b-y)^{(1-\alpha)p}}\,dy\right)^{1/p} \left(\int_a^b \sigma(y)\,dy\right)^{1/p'} \le C, \tag{1.5}$$

see proof of Theorem 3 in [4]. Then, this condition (1.5) is necessary for the inequality (1.4) to hold. The following theorem gives a sufficient condition for (1.4), which is also necessary in some cases.

**Theorem 1.1.** *Let $w$ and $\sigma$ be two weights on $\mathbb{R}$. Let $1 < p < \infty$ and $0 < \alpha < 1$. Then* (1.4) *holds if $I_\alpha^- w$ belongs to $L^{p'}_{\mathrm{loc}}(\sigma)$ and*

$$I_\alpha^-[(I_\alpha^- w)^{p'}\sigma](x) \le C I_\alpha^- w(x) \qquad \sigma\text{-a.e.} \tag{1.6}$$

**Theorem 1.2.** *Let $1 < p < \infty$ and $0 < \alpha < 1$. If $w$ and $\sigma$ satisfy*

$$\sup_{r>0} \left[\int_{2r}^{\infty} \frac{w([x-\rho, x-2r])}{\rho^{1-\alpha}} \frac{d\rho}{\rho}\right]^{1/p} \left[\int_0^r \frac{\sigma([x-\rho, x])}{\rho^{1-\alpha}} \frac{d\rho}{\rho}\right]^{1/p'} \le C, \tag{1.7}$$

*for all $x \in \mathbb{R}$, then condition* (1.6) *is necessary for the inequality* (1.4) *to hold.*

Let $w$ be any weight and $\sigma = v^{1-p'} = (I_\alpha^- w)^{-p'} w$. Clearly, the pair $(w, \sigma)$ satisfies condition (1.6). Therefore, the inequality

$$\int_{-\infty}^{+\infty} |I_\alpha^+(f)(x)|^p w(x)\,dx \le C \int_{-\infty}^{+\infty} |f(x)|^p w(x)^{1-p} I_\alpha^-(w)(x)^p\,dx$$

holds. If $w$ is a power weight, for instance $w(x) = x^\gamma \chi_{(0,\infty)}(x)$, $\gamma > -1$, it is easy to see that $w(x)^{1-p} I_\alpha^-(w)(x)^p \approx C x^{\gamma+\alpha p}\chi_{(0,\infty)}(x)$ and therefore

$$\int_0^\infty |I_\alpha^+(f)(x)|^p x^\gamma \, dx \le C \int_0^\infty |f(x)|^p x^{\gamma+\alpha p}\, dx.$$

A similar result for $I_\alpha^-$ was obtained by E. Hernández in [3]. Furthermore, if the weight $w$ satisfies $I_\alpha^- w(x) \le C w(x)$ almost everywhere, then $I_\alpha^+$ maps $L^p(w)$ boundedly into itself. It is easy to check that $w(x) = e^x$ satisfies this condition. Therefore, the class of weights $w$ such that $I_\alpha^+$ maps $L^p(w)$ boundedly into itself, is not empty. This is in sharp contrast with the case of the two-sided operator $I_\alpha f(x) = \int_{-\infty}^{+\infty} \frac{f(y)}{|y-x|^{1-\alpha}}\, dy$, for which this class is trivial. Indeed, there does not exist a non-zero weight $w$ satisfying the condition

$$(A_{p,\alpha}) \qquad w(I)^{1/p} w^{1-p'}(I)^{1/p'} \le C|I|^{1-\alpha}$$

for all intervals $I$, which is necessary for the boundedness of $I_\alpha$.

**Remark 1.3.** We can consider the operators $I_\alpha^+$ and $I_\alpha^-$ defined as in (1.1) and (1.2) for every $\alpha > 0$. In the case $\alpha \ge 1$ the weights for these operators were studied by F.J. Martín Reyes and E. Sawyer in [5].

**Definition 1.4.** For fixed $1 \le p < \infty$ and $0 < \alpha$, we say that the weight $w$ belongs to the class $F_{p,\alpha}^+$, respectively $F_{p,\alpha}^-$, if the operator $I_\alpha^+$, respectively $I_\alpha^-$, maps $L^p(w)$ boundedly into itself.

We have seen above that these classes are non-trivial, at least in the case $1 < p < \infty$, $0 < \alpha < 1$. The following theorems give us a characterization of them.

**Theorem 1.5.** *Let $0 < \alpha < 1$. The following are equivalent:*

1. $w \in F_{1,\alpha}^+$.
2. *There exists a constant $C$ such that for any $f$*

$$\int_{-\infty}^{+\infty} M_\alpha^+(f)(x) w(x)\, dx \le C \int_{-\infty}^{+\infty} |f(x)| w(x)\, dx.$$

3. *There exists a constant $C$ such that $I_\alpha^- w(x) \le C w(x)$ a.e.*

Actually the result is true for pairs of weights.

**Theorem 1.6.** *Let $v$ and $w$ be two weights and $0 < \alpha < 1$. The following are equivalent:*

1. *There exists a constant $C$ such that for any $f$*

$$\int_{-\infty}^{+\infty} |I_\alpha^+(f)(x)| w(x)\, dx \le C \int_{-\infty}^{+\infty} |f(x)| v(x)\, dx.$$

2. *There exists a constant $C$ such that for any $f$*

$$\int_{-\infty}^{+\infty} M_\alpha^+(f)(x)w(x)\,dx \le C\int_{-\infty}^{+\infty} |f(x)|v(x)\,dx.$$

3. *There exists a constant $C$ such that $I_\alpha^- w(x) \le Cv(x)$ a.e.*

**Remark 1.7.** By a duality argument, parts (1) and (3) of the previous theorem are equivalent even in the case $\alpha \ge 1$.

**Theorem 1.8.** *Let $1 < p < \infty$ and $\alpha > 0$. The following are equivalent:*

1. $w \in F_{p,\alpha}^+$.
2. *There exist two weights $w_0 \in F_{1,\alpha}^+$ and $w_1 \in F_{1,\alpha}^-$ such that $w = w_0 w_1^{1-p}$.*

Clearly we obtain similar theorems for $I_\alpha^-$ reversing the orientation of the real line.

## 2. Proof of Theorems 1.1 and 1.2

Let $w$ and $\sigma$ be two weights on $\mathbb{R}$. If $I_\alpha^- w$ belongs to $L_{\text{loc}}^{p'}(\sigma)$, we denote

$$\nu = (I_\alpha^- w)^{p'}\sigma. \tag{2.1}$$

Then, we can write condition (1.6) in the form

$$I_\alpha^- \nu \le C I_\alpha^- w \qquad \sigma\text{-}a.e. \tag{2.2}$$

The following three lemmas shall be needed in the proof of Theorem 1.1.

**Lemma 2.1.** *Let $1 < p < \infty$ and $\nu$ be defined by* (2.1).

(i) *Suppose that*

$$\left[\int_{-\infty}^{+\infty} |I_\alpha^-(g\nu)|^{p'}\sigma\right]^{1/p'} \le C\left[\int_{-\infty}^{+\infty} |g|^{p'}\nu\right]^{1/p'}, \tag{2.3}$$

*for all $g \in L^{p'}(\nu)$. Then, for any $r : 1 < r \le p'$ the inequality*

$$\left[\int_{-\infty}^{+\infty} \left|\frac{I_\alpha^-(g\nu)}{I_\alpha^- w}\right|^r \nu\right]^{1/r} \le C\left[\int_{-\infty}^{+\infty} |g|^r\nu\right]^{1/r}, \tag{2.4}$$

*holds for all $g \in L^r(\nu)$.*

(ii) *If* (2.2) *holds, then* (2.4) *holds for all* $r : 1 < r \leq \infty$. (*In the case* $r = \infty$, *inequality* (2.4) *is to be interpreted in the* $L^\infty(d\nu)$ *norm.*)

Proof: In order to prove (i) we will make use of the theory of interpolation in the setting of Lorentz spaces. We recall that for $0 < p < \infty$, $0 < q \leq \infty$, the space $L^{p,q}(\nu)$ is defined as the set of all measurable functions $f$ for which

$$\|f\|_{p,q} = \|t^{\frac{1}{p}} f^*(t)\|_{L^q(dt/t)} < \infty$$

where $f^*$ is the decreasing rearrangement of $f$ with respect to the measure $\nu$. It is known that if $1 < p < \infty$ then the associate space of $L^{p,1}(\nu)$ is $L^{p',\infty}(\nu)$ and that if a quasilinear operator $T$ maps $L^{p,1}(\nu)$ boundedly into $L^p(\nu)$ and $L^q(\nu)$ into $L^q(\nu)$, where $1 < p < q \leq \infty$ then $T$ is a bounded operator on $L^s(\nu)$ for any $p < s < q$ (see [1]).

We define the operator $A$ by

$$Ag = \frac{I_\alpha^-(g\nu)}{I_\alpha^- w}\,. \tag{2.5}$$

Taking into account (2.3) we have that

$$\|Ag\|_{L^{p'}(\nu)} \leq C\|g\|_{L^{p'}(\nu)}. \tag{2.6}$$

That is, the operator $A$ is bounded from $L^{p'}(\nu)$ to $L^{p'}(\nu)$. We shall show that for all $1 < r < p'$,

$$\|Ag\|_{L^r(\nu)} \leq C\|g\|_{L^{r,1}(\nu)}. \tag{2.7}$$

The adjoint operator of $A$ is defined by

$$A^* f = I_\alpha^+[f(I_\alpha^- w)^{-1}]\nu,$$

and (2.7) can be rewritten as

$$\|I_\alpha^+[f(I_\alpha^- w)^{-r}\nu]\|_{L^{r',\infty}(\nu)} \leq C\|f\|_{L^{r'}((I_\alpha^- w)^{-r}\nu)}. \tag{2.8}$$

This inequality is equivalent to

$$\begin{aligned}\|I_\alpha^+ g\|_{L^{r',\infty}(\nu)} &\leq C\|(I_\alpha^- w)^r \nu^{-1} g\|_{L^{r'}((I_\alpha^- w)^{-r}\nu)} \\ &= C\|g\|_{L^{r'}((I_\alpha^- w)^{r'}\nu^{1-r'})}.\end{aligned}$$

This is the same as asserting that $I_\alpha^+$ is bounded from $L^{r'}((I_\alpha^- w)^{r'}\nu^{1-r'})$ to $L^{r',\infty}(\nu)$. By Theorem 2 in [4] this is equivalent to the existence of a constant $C>0$ such that for any interval $I$,

$$\int_I \left|\frac{I_\alpha^-(\chi_I \nu)}{I_\alpha^- w}\right|^r \nu \le C\nu(I). \tag{2.9}$$

Using (2.3) with $g=\chi_I$, we get

$$\int_I |I_\alpha^-(\chi_I\nu)|^{p'}\sigma \le C\nu(I). \tag{2.10}$$

Applying Hölder's inequality with exponents $p'/r$ and its conjugate, by (2.10) we have that

$$\begin{aligned}\int_I \left[\frac{I_\alpha^-(\chi_I \nu)}{I_\alpha^- w}\right]^r \nu &\le \left[\int_I \left[\frac{I_\alpha^-(\chi_I \nu)}{I_\alpha^- w}\right]^{p'} \nu\right]^{r/p'} \nu(I)^{1-r/p'} \\ &= \left[\int_I \left[I_\alpha^-(\chi_I\nu)\right]^{p'}\sigma\right]^{r/p'} \nu(I)^{1-r/p'} \\ &\le C\nu(I).\end{aligned}$$

Then (2.9) holds, and it implies (2.8). Therefore, by duality we have (2.7). Now, by (2.6) and an interpolation theorem for $L^{r,1}(\nu)$, we obtain (2.4) for all $1<r<p'$.

(ii) By inequality (2.2), the operator $A$ defined in (2.5) is bounded on $L^\infty(\nu)$ that is,

$$\|Ag\|_{L^\infty(\nu)} \le C\|g\|_{L^\infty(\nu)}. \tag{2.11}$$

On the other hand, (2.2) implies that

$$\int_I \left[I_\alpha^-(\chi_I\nu)\right]^{p'}\sigma \le C\int_I (I_\alpha^- w)^{p'}\sigma = C\nu(I),$$

for any interval $I$. Then (2.10) holds and, as in part (i), (2.7) holds for all $r\le p'$. Now, interpolating (2.11) and (2.7) we have that (2.4) holds for all $1<r<\infty$. The case $r=\infty$ is straightforward and left to the reader. □

**Lemma 2.2.** *Let $w$ and $\sigma$ be two weights defined on $\mathbb{R}$. Let $0<\alpha<1$. Then, for every positive integer $m$, the inequality*

$$I_\alpha^-[(I_\alpha^+\sigma)^m w] \le C\left\{(I_\alpha^- w)(I_\alpha^+\sigma)^m + I_\alpha^-[(I_\alpha^- w)(I_\alpha^+\sigma)^{m-1}\sigma]\right\} \tag{2.12}$$

*holds with a constant $C$ depending on $\alpha$ and $m$.*

Proof: Taking into account that $m > 0$ we get,

$$\begin{aligned} I_\alpha^-[(I_\alpha^+\sigma)^m w](x) &= \int_{-\infty}^x \frac{I_\alpha^+\sigma(y)^m}{(x-y)^{1-\alpha}} w(y)\,dy \\ &= \int_{-\infty}^x \frac{1}{(x-y)^{1-\alpha}} \left(\int_y^\infty \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m w(y)\,dy \\ &\le 2^m \left[\int_{-\infty}^x \frac{1}{(x-y)^{1-\alpha}} \left(\int_x^\infty \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m w(y)\,dy\right] \\ &\quad + 2^m \left[\int_{-\infty}^x \frac{1}{(x-y)^{1-\alpha}} \left(\int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m w(y)\,dy\right] \\ &= A_m + B_m. \end{aligned} \tag{2.13}$$

We have the estimate

$$\begin{aligned} A_m &\le C \int_{-\infty}^x \frac{1}{(x-y)^{1-\alpha}} \left(\int_x^\infty \frac{\sigma(z)dz}{(z-x)^{1-\alpha}}\right)^m w(y)\,dy \\ &= C I_\alpha^+\sigma(x)^m I_\alpha^- w(x). \end{aligned}$$

Then, in order to prove (2.12), by (2.13), it will be enough to show that

$$B_m \le C I_\alpha^-[(I_\alpha^- w)(I_\alpha^+\sigma)^{m-1}\sigma](x). \tag{2.14}$$

We can write $B_m$ in the form

$$B_m = C\int_{-\infty}^x \left(\int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m \int_{x-y}^\infty \frac{dt}{t^{2-\alpha}} w(y)\,dy.$$

Applying Fubini's Theorem we have that

$$B_m = C\int_0^\infty \frac{1}{t^{2-\alpha}} \int_{x-t}^x \left(\int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m w(y)\,dy\,dt. \tag{2.15}$$

If we prove that for every positive integer $m$, the inequality

$$\int_{x-t}^x \left(\int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m w(y)\,dy \le C\int_{x-2^m t}^x I_\alpha^- w(y) I_\alpha^+\sigma(y)^{m-1}\sigma(y)\,dy, \tag{2.16}$$

holds with a constant $C$ depending on $m$ and $\alpha$ only, then by (2.15) and Fubini's Theorem, we obtain (2.14). We shall show (2.16) by induction. If $m = 1$, changing the order of integration,

$$\begin{aligned} \int_{x-t}^x \left(\int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right) w(y)\,dy\,dt &= \int_{x-t}^x \int_{x-t}^z \frac{w(y)dy}{(z-y)^{1-\alpha}} \sigma(z)\,dz \\ &\le \int_{x-2t}^x I_\alpha^- w(z)\sigma(z)\,dz. \end{aligned}$$

That is, (2.16) holds in the case $m = 1$.

Let $m > 1$ and assume that (2.16) holds for $m-1$. Integrating by parts, we observe that

$$\left(\int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m = m\int_y^x \frac{1}{(u-y)^{1-\alpha}}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1}\sigma(u)\,du.$$

Then applying Fubini's Theorem,

$$\begin{aligned} I_m &= \int_{x-t}^x \left(\int_y^x \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^m w(y)\,dy \\ &= m\int_{x-t}^x\int_y^x \frac{1}{(u-y)^{1-\alpha}}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1}\sigma(u)\,du\,w(y)\,dy \qquad (2.17)\\ &= m\int_{x-t}^x\int_{x-t}^u \frac{1}{(u-y)^{1-\alpha}}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1} w(y)\,dy\,\sigma(u)\,du. \end{aligned}$$

By (2.17), we can write $I_m$ in the form

$$I_m = C\int_{x-t}^x\int_{x-t}^u\int_{u-y}^{2(u-y)} \frac{ds}{s^{2-\alpha}}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1} w(y)\,dy\,\sigma(u)\,du.$$

Changing the order of integration and enlarging the domain we have that

$$\begin{aligned} I_m &= C\int_{x-t}^x\int_0^{u-x+t}\frac{1}{s^{2-\alpha}}\int_{u-s}^{u-s/2}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1} w(y)\,dy\,ds\,\sigma(u)\,du \\ &\quad + C\int_{x-t}^x\int_{u-x+t}^{2(u-x+t)}\frac{1}{s^{2-\alpha}}\int_{x-t}^{u-s/2}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1} w(y)\,dy\,ds\,\sigma(u)\,du \\ &\le C\int_{x-t}^x\int_0^{u-x+t}\frac{1}{s^{2-\alpha}}\int_{u-s}^{u}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1} w(y)\,dy\,ds\,\sigma(u)\,du \\ &\quad + C\int_{x-t}^x\int_{u-x+t}^{2(u-x+t)}\frac{1}{s^{2-\alpha}}\int_{u-s}^{u}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1} w(y)\,dy\,ds\,\sigma(u)\,du \\ &= C\int_{x-t}^x\int_0^{2(u-x+t)}\frac{1}{s^{2-\alpha}}\int_{u-s}^{u}\left(\int_y^u \frac{\sigma(z)dz}{(z-y)^{1-\alpha}}\right)^{m-1} w(y)\,dy\,ds\,\sigma(u)\,du. \end{aligned}$$

Using (2.16) in the case $m-1$, we get

$$I_m \le C\int_{x-t}^x\int_0^{2(u-x+t)}\frac{1}{s^{2-\alpha}}\int_{u-2^{m-1}s}^{u} I_\alpha^- w(y) I_\alpha^+\sigma(y)^{m-2}\sigma(y)\,dy\,ds\,\sigma(u)\,du.$$

Applying Fubini's Theorem, we obtain the estimate

$$\begin{aligned} I_m &\le C\int_{x-t}^{x}\int_{u-2^m(u-x+t)}^{u} I_\alpha^- w(y) I_\alpha^+\sigma(y)^{m-2}\sigma(y)\int_{\frac{u-y}{2^{m-1}}}^{2(u-x+t)}\frac{ds}{s^{2-\alpha}}\sigma(u)\,du\,dy \\ &\le C\int_{x-t}^{x}\int_{u-2^m(u-x+t)}^{u}\frac{I_\alpha^- w(y) I_\alpha^+\sigma(y)^{m-2}}{(u-y)^{1-\alpha}}\sigma(y)\,dy\,\sigma(u)\,du. \end{aligned}$$

Changing the order of integration again, we have that

$$\begin{aligned} I_m \le C&\int_{x-2^m t}^{x-t} I_\alpha^- w(y) I_\alpha^+\sigma(y)^{m-2}\int_{\frac{y-2^m(x-t)}{1-2^m}}^{x}\frac{\sigma(u)}{(u-y)^{1-\alpha}}\sigma(y)\,dy\,du \\ &+ C\int_{x-t}^{x} I_\alpha^- w(y) I_\alpha^+\sigma(y)^{m-2}\int_{y}^{x}\frac{\sigma(u)}{(u-y)^{1-\alpha}}\sigma(y)\,dy\,du. \end{aligned}$$

Enlarging the domain of integration in the first term on the right hand,

$$\begin{aligned} I_m &\le C\int_{x-2^m t}^{x} I_\alpha^- w(y)\ I_\alpha^+\sigma(y)^{m-2}\ I_\alpha^+\sigma(y)\ \sigma(y)\,dy \\ &= C\int_{x-2^m t}^{x} I_\alpha^- w(y)\ I_\alpha^+\sigma(y)^{m-1}\ \sigma(y)\,dy. \end{aligned}$$

This shows that (2.16) holds for every positive integer $m$, and finishes the proof of this lemma. □

The following two lemmas are simple variants of Lemma 4 and Lemma 5 in [6], therefore we omit their proofs.

**Lemma 2.3.** *Let $w$ and $\sigma$ be two weights, $0<\alpha<1$ and $1<p<\infty$. We assume that $m<p\le m+1$, where $m$ is a positive integer. Let $\delta=(p-1)/m$. Then, the inequality*

$$\begin{aligned} (2.18)\quad & I_\alpha^-[(I_\alpha^+\sigma)^{p-1}w] \\ &\le C\left\{(I_\alpha^- w)(I_\alpha^+\sigma)^{p-1}+(I_\alpha^- w)^{1-\delta}[I_\alpha^-[(I_\alpha^- w)(I_\alpha^+\sigma)^{m-1}\sigma]]^\delta\right\} \end{aligned}$$

*holds, with a constant $C$ depending on $\alpha, p$ and $m$.*

Let $w$ and $\sigma$ be two weights on $\mathbb{R}$ and $1<p<\infty$. We define the operator $B_p$ in the form

$$B_p(f)=I_\alpha^-[|I_\alpha^+(f\sigma)|^{p-1}w], \tag{2.19}$$

for each $f\in L^p(\sigma)$.

**Lemma 2.4.** *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Suppose that for every $f \in L^p(\sigma)$, we have the inequality*

$$\int_{-\infty}^{+\infty} [B_p(f)]^{p'} \sigma \leq C\|f\|^p_{L^p(\sigma)}. \tag{2.20}$$

*Then,* (1.4) *holds.*

Proof of Theorem 1.1: Let $\nu$ be defined as in (2.1), that is $\nu = (I^-_\alpha w)^{p'}\sigma$. Condition (1.5) is $I^-_\alpha \nu \leq C I^-_\alpha w$, $\sigma - a.e.$ Then, by Lemma 2.1(ii), we get (2.4) for every $r : 1 < r \leq \infty$. In the case $r = p'$ we have that the inequality

$$\|I^-_\alpha(g\nu)\|_{L^{p'}(\sigma)} \leq C\|g\|_{L^{p'}(\nu)} \tag{2.21}$$

holds for every $g \in L^{p'}(\nu)$. By duality (2.21) is equivalent to

$$\|I^+_\alpha(f\sigma)\|_{L^p(\nu)} \leq C\|f\|_{L^p(\sigma)}, \tag{2.22}$$

for every $f \in L^p(\sigma)$. We shall show that (2.22) implies (2.21). Thus, applying Lemma 2.4 we obtain (1.4).

Let $f \in L^p(\sigma)$, $f \geq 0$. We consider the operator $B_p$ defined in (2.19). First of all, we prove that (2.20) holds for all positive integers $p \geq 2$. By Lemma 2.2 with $m = p - 1$,

$$B_p(f) \leq C\left\{(I^-_\alpha w)(I^+_\alpha(f\sigma))^{p-1} + I^-_\alpha[(I^-_\alpha w)(I^+_\alpha(f\sigma))^{p-2} f\sigma]\right\}.$$

Raising both sides of this inequality to the power $p'$ and integrating with respect to the weight $\sigma$,

$$\int_{-\infty}^{+\infty} B_p(f)^{p'}\sigma \leq C\int_{-\infty}^{+\infty} I^-_\alpha(w)^{p'}(I^+_\alpha(f\sigma))^p\sigma$$
$$+C\int_{-\infty}^{+\infty} I^-_\alpha[(I^-_\alpha w)(I^+_\alpha(f\sigma))^{p-2}f\sigma]^{p'}\sigma.$$

By (2.22), the first term on the right hand is bounded by $C\|f\|^p_{L^p(\sigma)}$. To estimate the second term we consider the function

$$g = (I^-_\alpha w)^{1-p'}[I^+_\alpha(f\sigma)]^{p-2}f.$$

Using (2.21), we have that

$$\int_{-\infty}^{+\infty} I^-_\alpha[(I^-_\alpha w)(I^+_\alpha(f\sigma))^{p-2}f\sigma]^{p'}\sigma = \|I^-_\alpha(g\nu)\|^{p'}_{L^{p'}(\sigma)}$$
$$\leq C\|g\|^{p'}_{L^{p'}(\nu)} = C\int_{-\infty}^{+\infty}(I^-_\alpha w)^{(2-p')p'}(I^+_\alpha(f\sigma))^{(p-2)p'}f^{p'}\sigma.$$

This inequality gives (2.20) for $p=2$. From now on, assume that $p>2$. By Hölder's inequality with exponents $\frac{p-1}{p-2}$ and $\frac{p}{p'}$ and using the identity $(2-p')p'\frac{p-1}{p-2} = p'$, we obtain that the last expression is bounded by

$$C\left[\int_{-\infty}^{+\infty} I_\alpha^+(f\sigma)^p (I_\alpha^- w)^{p'}\sigma\right]^{\frac{p-2}{p-1}} \|f\|_{L^p(\sigma)}^{p'} \le C\|f\|_{L^p(\sigma)}^p.$$

In consequence, (2.20) holds for every positive integer $p$.

Now, we suppose that $p$ is not an integer and $m<p<m+1$ with $m$ a positive integer. By Lemma 2.3 we get

$$B_p(f) \le C\left\{(I_\alpha^- w)(I_\alpha^+ f\sigma)^{p-1} + (I_\alpha^- w)^{1-\delta}[I_\alpha^-[(I_\alpha^- w)(I_\alpha^+ f\sigma)^{m-1} f\sigma]]^\delta\right\},$$

where $\delta = \frac{p-1}{m}$. Raising both sides of this inequality to the power $p'$ and integrating against $\sigma(x)dx$, we have

$$\int_{-\infty}^{+\infty} B_p(f)^{p'}\sigma \le C\int_{-\infty}^{+\infty} (I_\alpha^+(f\sigma))^p \nu + C\int_{-\infty}^{+\infty} (I_\alpha^- w)^{p'(1-\delta)}\left\{I_\alpha^-[(I_\alpha^- w)(I_\alpha^+ f\sigma)^{m-1} f\sigma]\right\}^{p'\delta}\sigma.$$

Using (2.22), the first term on the right hand is bounded by $C\|f\|_{L^p(\sigma)}^p$. Now, let $r = p'\delta < p'$ and

$$g = (I_\alpha^- w)^{1-p'}[I_\alpha^+(f\sigma)]^{m-1} f.$$

Applying Lemma 2.1(i), more precisely using (2.4) we have that

$$\begin{aligned}\int_{-\infty}^{+\infty} &(I_\alpha^- w)^{p'(1-\delta)}\left\{I_\alpha^-[(I_\alpha^- w)(I_\alpha^+ f\sigma)^{m-1} f\sigma]\right\}^{p'\delta}\sigma \\ &= \int_{-\infty}^{+\infty}\left|\frac{I_\alpha^-(g\nu)}{I_\alpha^- w}\right|^r \nu \le C\int_{-\infty}^{+\infty} g^r \nu \\ &= C\int_{-\infty}^{+\infty}(I_\alpha^- w)^{p'(1-\frac{1}{m})}(I_\alpha^+ f\sigma)^{(1-\frac{1}{m})p} f^{\frac{p}{m}}\sigma.\end{aligned}$$

If $1<p<2$ then $m=1$ and the proof is complete in this case. Suppose $p>2$. Applying Hölder's inequality with exponent $m$ and its conjugate, and taking into account (2.22) we obtain that

$$\int_{-\infty}^{+\infty}(I_\alpha^- w)^{p'(1-\frac{1}{m})}(I_\alpha^+ f\sigma)^{(1-\frac{1}{m})p} f^{\frac{p}{m}}\sigma \le \left[\int_{-\infty}^{+\infty}(I_\alpha^- w)^{p'} I_\alpha^+(f\sigma)^p\sigma\right]^{1-\frac{1}{m}} \|f\|_{L^p(\sigma)}^{\frac{p}{m}} \le C\|f\|_{L^p(\sigma)}^p.$$

Thus, (2.20) is proved for every $1<p<\infty$. This completes the proof of Theorem 1.1. $\square$

**Remark 2.5.** We observe that applying Lemma 2.4 we have proved that (2.22) implies (1.4).

We observe that by duality (1.4) is equivalent to

$$\|I_\alpha^-(fw)\|_{L^{p'}(\sigma)} \leq C\|f\|_{L^{p'}(w)}. \tag{2.23}$$

Proof of Theorem 1.2: Let us assume that (1.4) and (1.7) hold. We can write

$$I_\alpha^-[(I_\alpha^- w)^{p'}\sigma](x) = C\int_0^\infty \frac{[(I_\alpha^- w)^{p'}\sigma]([x-r,x])}{r^{1-\alpha}}\frac{dr}{r}\,.$$

For each $r>0$, let $w = w_{1,r} + w_{2,r}$ where,

$$w_{1,r} = w\chi_{[x-2r,x]} \quad \text{and} \quad w_{2,r} = w - w_{1,r}.$$

Then,

$$I_\alpha^- w = I_\alpha^-(w_{1,r}) + I_\alpha^-(w_{2,r}),$$

and it follows that

$$\begin{aligned}
&I_\alpha^-[(I_\alpha^- w)^{p'}\sigma](x)\\
&\quad\leq C\int_0^\infty \frac{[(I_\alpha^- w_{1,r})^{p'}\sigma]([x-r,x])}{r^{1-\alpha}}\frac{dr}{r} + C\int_0^\infty \frac{[(I_\alpha^- w_{2,r})^{p'}\sigma]([x-r,x])}{r^{1-\alpha}}\frac{dr}{r}\\
&\quad= A+B.
\end{aligned}$$

By (2.23), we have the estimate

$$\begin{aligned}
[(I_\alpha^- w_{1,r})^{p'}\sigma]([x-r,x]) &= \int_{x-r}^x I_\alpha^-(\chi_{[x-2r,x]}w)(y)^{p'}\sigma(y)\,dy\\
&\leq C\int_{-\infty}^{+\infty} |\chi_{[x-2r,x]}(y)|^{p'} w(y)\,dy = Cw([x-2r,x]).
\end{aligned}$$

Therefore,

$$A \leq C\int_0^\infty \frac{w([x-2r,x])}{r^{1-\alpha}}\frac{dr}{r} = CI_\alpha^- w(x).$$

On the other hand, taking into account the definition of $w_{2,r}$, for each $z\in[x-r,x]$ we have that

$$\begin{aligned}
I_\alpha^-(w_{2,r})(z) &= C\int_r^\infty \frac{w_{2,r}([z-\rho,z])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\\
&\leq C\int_r^\infty \frac{w([x-2\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\\
&= C\int_{2r}^\infty \frac{w([x-\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\,.
\end{aligned}$$

Then,

$$\int_{x-r}^{x} I_\alpha^-(w_{2,r})(z)^{p'}\sigma(z)\,dz \le C\bigg(\int_{2r}^{\infty}\frac{w([x-\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\bigg)^{p'}\int_{x-r}^{x}\sigma(z)\,dz.$$

In consequence,

$$B \le C\int_0^\infty \frac{\sigma([x-r,x])}{r^{1-\alpha}}\left(\int_{2r}^{\infty}\frac{w([x-\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\right)^{p'}\frac{dr}{r}. \tag{2.24}$$

Applying Fubini's Theorem, we observe that

$$\begin{aligned} g(r) &= \int_{2r}^{\infty}\frac{w([x-\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho} = \int_{2r}^{\infty}\frac{1}{\rho^{2-\alpha}}\int_{x-\rho}^{x-2r} w(z)\,dz\,d\rho \\ &= \int_{-\infty}^{x-2r} w(z)\int_{x-z}^{\infty}\frac{d\rho}{\rho^{2-\alpha}}\,dz = C\int_{-\infty}^{x-2r}\frac{w(z)}{(x-z)^{1-\alpha}}\,dz. \end{aligned}$$

Thus, the derivative $g'(r)$ is equal to $-C\frac{w(x-2r)}{r^{1-\alpha}}$. Integrating by parts from (2.24) it follows that we can dominate $B$ by

$$\begin{aligned} C\int_0^\infty\int_0^r \frac{\sigma([x-\rho,x])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\left(\int_{2r}^{\infty}\frac{w([x-\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\right)^{p'/p}\frac{w(x-2r)}{r^{1-\alpha}}\,dr& \\ \le C\int_0^\infty \frac{w(x-2r)}{r^{1-\alpha}}\,dr& \end{aligned}$$

since (1.7) implies that

$$\sup_{r>0}\int_0^r \frac{\sigma([x-\rho,x])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\bigg(\int_{2r}^{\infty}\frac{w([x-\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\bigg)^{p'/p} \le C.$$

Then,

$$B \le C\int_{-\infty}^{x}\frac{w(y)}{(x-y)^{1-\alpha}}\,dy = CI_\alpha^- w(x),$$

and the proof of this theorem is complete. □

In order to state the next proposition, we need to introduce the following definition.

**Definition 2.6.** Let $\beta > 0$. We say that a weight $w$ belongs to $RD^-(\beta)$ if there exists a constant $C > 0$, such that

$$w([x-\rho,x]) \le C\left(\frac{\rho}{r}\right)^{\beta} w([x-r,x]),$$

for all $x \in \mathbb{R}$, $r > 0$ and $0 < \rho < r$.

**Proposition 2.7.** *Let $1 < p < \infty$. Let $w$ and $\sigma$ be two weights on $\mathbb{R}$. If $\sigma \in RD^-(\beta)$ for some $\beta > 1-\alpha$, then* (1.5) *implies condition* (1.7)*.*

Proof: We suppose that $w$ and $\sigma$ satisfy condition (1.5) and $\sigma \in RD^-(\beta)$ with $\beta > 1-\alpha$. For each $r > 0$ we have that

$$\begin{aligned}\int_0^r \frac{\sigma([x-\rho,x])}{\rho^{1-\alpha}}\frac{d\rho}{\rho} &\leq C\frac{\sigma([x-r,x])}{r^\beta}\int_0^r \rho^{\beta+\alpha-2}\,d\rho\\ &= C\frac{\sigma([x-r,x])}{r^{1-\alpha}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^r \frac{\sigma([x-\rho,x])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\left(\int_{2r}^\infty \frac{w([x-\rho,x-2r])}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\right)^{p'-1} \qquad (2.25)\\ &\leq C\frac{1}{r^{1-\alpha}}\left(\int_{2r}^\infty \frac{w([x-\rho,x-2r])\ \sigma([x-r,x])^{p/p'}}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\right)^{p'-1}\\ &\leq C\frac{1}{r^{1-\alpha}}\left(\int_{2r}^\infty \frac{w([x-\rho,x-r])\ \sigma([x-r,x+\rho-2r])^{p/p'}}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\right)^{p'-1}.\end{aligned}$$

On the other hand (1.5) implies condition $A^+_{p,\alpha}$, that is, there exists a constant $C$ such that for every $a \in \mathbb{R}$ and $h > 0$

$$(w([a-h,a]))^{1/p}(\sigma([a,a+h]))^{1/p'} \leq Ch^{1-\alpha}.$$

Applying condition $A^+_{p,\alpha}$, it follows that (2.25) is bounded by

$$C\frac{1}{r^{1-\alpha}}\left(\int_{2r}^\infty \frac{(\rho-r)^{(1-\alpha)p}}{\rho^{1-\alpha}}\frac{d\rho}{\rho}\right)^{p'-1} \leq C\frac{1}{r^{1-\alpha}}r^{(1-\alpha)(p-1)(p'-1)} = C.$$

Then, $w$ and $\sigma$ satisfy (1.7). □

**Corollary 2.8.** *Let $\sigma$ belong to $RD^-(\beta)$ for some $\beta > 1-\alpha$. Then* (1.6) *is a necessary and sufficient condition for the inequality* (1.4) *to hold.*

Proof: It is an immediate consequence of Proposition 2.7 part (ii) and Theorem 1.2.

□

**Remark 2.9.** As an application of these results we consider the existence of non-negative solution of the non-linear integral equation

$$u = I_\alpha^-(u^q\sigma) + I_\alpha^- w \quad \sigma\text{ -}a.e., \tag{2.26}$$

where we suppose that $I_\alpha^- w < \infty$ $\sigma$ -*a.e.* and we have the following result:

Let $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $A(p) = (p-1)p^{-q}$ and $0 < \alpha < 1$. Let $w$ and $\sigma$ be two locally integrable weights.

(i) If $I_\alpha^- w$ belongs to $L^q_{\mathrm{loc}}(\sigma)$ and the inequality

$$I_\alpha^-[(I_\alpha^- w)^q\sigma](x) \le A(p) I_\alpha^- w(x) \quad \sigma\text{ -}a.e. \tag{2.27}$$

holds, then equation (2.26) has a non-negative solution in $L^q_{\mathrm{loc}}(\sigma)$.

(ii) Assume that there exists a constant $C$ such that

$$\int_0^r \frac{\sigma([x-\rho,x])}{\rho^{1-\alpha}}\,d\rho \le C\frac{\sigma([x-r,x])^{1/q}\;\sigma([x-2r,x-r])^{1/p}}{r^{1-\alpha}}, \tag{2.28}$$

for all $x \in \mathbb{R}$ and $r > 0$. If (2.26) has a non-negative solution in $L^q_{\mathrm{loc}}(\sigma)$, then $I_\alpha^- w$ belongs to $L^q_{\mathrm{loc}}(\sigma)$ and there exists a constant $A > 0$ such that

$$I_\alpha^-[(I_\alpha^- w)^q\sigma](x) \le A I_\alpha^- w(x) \quad \sigma\text{ -}a.e. \tag{2.29}$$

The proof is similar to the one in [6].

**Definition 2.10.** We say that a weight $w$ belongs to $D^-$ if there exists a constant $C > 0$, such that for all $x$ belonging to $\mathbb{R}$ and $r > 0$,

$$w([x,x+r]) \le Cw([x-r,x]).$$

Taking into account Definition 2.10 we state the next proposition.

**Proposition 2.11.** *Let* $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $0 < \alpha < 1$. *If* $\sigma$ *belongs to* $D^-$ *with a constant* $C : 0 < C < (2^{1-\alpha} - 1)^{-1}$ *then condition* (2.28) *holds.*

Proof: Since $\sigma \in D^-$ with constant $C$ we have that

$$(1+C)\sigma([x,x+r]) \le C\sigma([x-r,x+r]).$$

Then,

$$\sigma([x,x+r]) \le \frac{C}{1+C}\sigma([x-r,x+r]), \tag{2.30}$$

for every $x \in \mathbb{R}$ and $r > 0$. Let $\beta > 1 - \alpha$ such that

$$0 < C \leq \frac{1}{2^{\beta} - 1}\,. \tag{2.31}$$

We shall show that $\sigma \in RD^{-}(\beta)$ with constant $\frac{1+C}{C} = A^{-1}$. Let $x \in \mathbb{R}$ and $r > 0$. Fixing $\rho : 0 < \rho < r$, there exists a positive integer $i$ such that, $2^{-i}r \leq \rho < 2^{-i+1}r$. Then, using (2.30) we have that

$$\begin{aligned} \sigma([x-\rho, x]) &\leq \sigma([x - 2^{-i+1}r, x]) \leq A^{i-1}\sigma([x-r, x]) \\ &\leq A^{-1}(A2^{\beta})^{i}\left(\frac{\rho}{r}\right)^{\beta}\sigma([x-r,x]). \end{aligned} \tag{2.32}$$

Taking into account (2.31) we have that $A = \frac{C}{1+C} \leq \frac{1}{2^{\beta}}$. Then, by (2.32) we obtain that

$$\sigma([x-\rho, x]) \leq A^{-1}\left(\frac{\rho}{r}\right)^{\beta}\sigma([x-r,x]).$$

Since $\beta + \alpha > 1$ we have the estimate

$$\begin{aligned} \int_0^r \frac{\sigma([x-\rho,x])}{\rho^{1-\alpha}}\frac{d\rho}{\rho} &\leq \frac{A^{-1}}{r^{\beta}}\sigma([x-r,x])\int_0^r \rho^{\beta+\alpha-2}\,d\rho \\ &\leq \frac{A^{-1}}{\beta+\alpha-1}\,\frac{\sigma([x-r,x])}{r^{1-\alpha}}\,. \end{aligned} \tag{2.33}$$

From the hypothesis $\sigma \in D^{-}$ it follows that $\sigma([x-r,x]) \leq C\sigma([x-2r, x-r])$. Then, applying (2.33), we have that

$$\int_0^r \frac{\sigma([x-\rho,x])}{\rho^{1-\alpha}}\frac{d\rho}{\rho} \leq \frac{A^{-1}}{\beta+\alpha-1}C^{1/p}\frac{\sigma([x-r,x])^{1/q}\ \sigma([x-2r,x-r])^{1/p}}{r^{1-\alpha}}\,.$$

This shows that (2.28) holds and completes the proof of the proposition. □

## 3. The case of equal weights

As we have observed in Section 1, the class of weights $w$ such that $I_{\alpha}^{+}$ maps $L^{p}(w)$, $1 < p < \infty$, boundedly into itself, is non-empty. In fact, it is non-empty even in the case $p = 1$.

Proof of Theorem 1.5: (1) ⇒(2): It follows from the inequality $M_{\alpha}^{+}(f)(x) \leq I_{\alpha}^{+}(|f|)(x)$.

(2) ⇒(3): We assume that

$$\int M_{\alpha}^{+}f(x)w(x)\,dx \leq C\int |f(x)|w(x)\,dx.$$

Let $x$ be a Lebesgue point for $w$, $h>0$ and $I=(x,x+h)$. If $a=\operatorname{ess\,inf}_{y\in I} w(y)$ and $\varepsilon>0$ we consider the set $E=\{x\in I : w(y)\le a+\varepsilon\}$ and the function $f=|E|^{-1}\chi_E$. It is clear that for any $y<x$

$$M_\alpha^+ f(y)\ge \frac{1}{(x+h-y)^{1-\alpha}}\int_y^{x+h} f(y)\,dy=\frac{1}{(x+h-y)^{1-\alpha}}\,.$$

Therefore,

$$\int_{-\infty}^{x}\frac{w(y)}{(x+h-y)^{1-\alpha}}\,dy\le\frac{C}{|E|}\int\chi_E w\le C(a+\varepsilon).$$

Thus,

$$\int_{-\infty}^{x}\frac{w(y)}{(x+h-y)^{1-\alpha}}\,dy\le Ca\le\frac{1}{h}\int_x^{x+h} w(y)\,dy.$$

When $h$ goes to zero the left hand side converges to $I_\alpha^-(f)(x)$ while the right hand side converges to $w(x)$.

(3) $\Rightarrow$(1): Indeed,

$$\begin{aligned}\int_{-\infty}^{+\infty}|I_\alpha^+(f)|w\,dx&\le\int_{-\infty}^{+\infty}I_\alpha^+(|f|)w\,dx\\&=\int_{-\infty}^{+\infty}|f|I_\alpha^-(w)\,dx\le C\int_{-\infty}^{+\infty}|f|w\,dx.\end{aligned}$$

$\square$

Proof of Theorem 1.6: The proof is similar to the previous one and we omit it. $\square$

Proof of Theorem 1.8: (1) $\Rightarrow$(2): By duality $w\in F_{p,\alpha}^+$ is equivalent to $w^{1-p'}\in F_{p',\alpha}^-$. It follows easily that the operators $M_1(g)=[w^{1/p}I_\alpha^+(w^{-1/p}|g|^{p'})]^{1/p'}$ and $M_2(g)=[w^{-1/p}I_\alpha^-(w^{1/p}|g|^p)]^{1/p}$ are bounded from $L^{pp'}(\mathbb{R})$ into itself. Applying the Rubio de Francia algorithm, see [2, Lemma 5.1, p. 434], we can obtain a weight $v$ such that

$$M_1(v)\le Cv\quad\text{ and }\quad M_2(v)\le Cv.$$

Then, $w_0=w^{1/p}v^p$ belongs to $F_{1,\alpha}^+$ and $w_1=w^{-1/p}v^{p'}$ belongs to $F_{1,\alpha}^-$. Clearly $w=w_0w_1^{1-p}$.

(2) $\Rightarrow$(1): We suppose that $w=w_0w_1^{1-p}$, with $w_0\in F_{1,\alpha}^+$ and $w_1\in F_{1,\alpha}^-$. It follows easily from Hölder's inequality that

$$|I_\alpha^+(f)(x)|^p\le I_\alpha^+(|f|^pw_1^{1-p})(x)I_\alpha^+(w_1)(x)^{p-1}.$$

Therefore, by duality

$$\begin{aligned}\int_{-\infty}^{+\infty} |I_\alpha^+(f)(x)|^p w(x)\,dx &\le \int_{-\infty}^{+\infty} I_\alpha^+(|f|^p w_1^{1-p})(x) I_\alpha^+(w_1)(x)^{p-1} w(x)\,dx \\ &= \int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} I_\alpha^-[I_\alpha^+(w_1)^{p-1}w](x)\,dx \\ &\le C\int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} I_\alpha^-[w_1^{p-1}w](x)\,dx \\ &= C\int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} I_\alpha^-(w_0)(x)\,dx \\ &\le C\int_{-\infty}^{+\infty} |f(x)|^p w_1(x)^{1-p} w_0(x)\,dx \\ &= C\int_{-\infty}^{+\infty} |f(x)|^p w(x)\,dx.\end{aligned}$$

□

In the rest of the paper we will make some remarks about the classes $F^+_{p,\alpha}$.

**Proposition 3.1.** *Let $w$ be a weight and $0<\alpha<1$. Then*

(a) $F^+_{1,\alpha} \subset F^+_{p,\alpha}$ *for $1<p<\infty$;*
(b) *if $w \in F^+_{1,\alpha}$ and $f$ is a non-negative increasing function then $fw \in F^+_{1,\alpha}$;*
(c) *there exists a weight $u_0 \in F^+_{1,\alpha}$ for all $0<\alpha<1$, that is not essentially increasing;*
(d) *for any $1<p<\infty$ there exists a weight $u \in F^+_{p,\alpha} \setminus F^+_{1,\alpha}$;*
(e) *there exists an increasing weight $w$ such that $w \notin F^+_{1,\alpha}$.*

Proof: (a) Theorem 1.5 states that $w \in F^+_{1,\alpha}$ is equivalent to $I_\alpha^- w \le Cw$. Therefore $(I_\alpha^- w)^{p'} w^{1-p'} \le Cw$ and the result follows from Theorem 1.1.

In order to prove (b) we observe that it is easy to check that if $w$ satisfies part (3) of Theorem 1.5 then so does $fw$ for any non-negative increasing $f$.

(c) Simple computations show that the function $u$ defined by

$$u(x) = \sum_{n=1}^{\infty} 2^n \chi_{(2^{-2n},2^{-2n+2}]}(x) + e^x \chi_{(1,\infty)}(x),$$

satisfies $I_\alpha^-(u)(x) \le Cu(x)$ almost everywhere and it is clearly not increasing.

(d) Let $u_0$ be the function defined in part (c) and $u_1(x)=u_0(1-x)$. From the equality $I_\alpha^+(u_1)(x) = I_\alpha^-(u_0)(1-x)$ it follows that $u_1$ belongs to $F^-_{1,\alpha}$. By Theorem 1.8 we have that $w = u_0 u_1^{1-p}$ belongs to the class $F^+_{p,\alpha}$.

We shall show that there does not exist a constant $C$ such that $I_\alpha^-(w)(x) \leq Cw(x)$ *a.e.* Let $x$ be such that $2^{-2n_0} < 1-x \leq 2^{-2n_0+2}$ for some $n_0 > 1$. Then, $u(x) = 2^{n_0(1-p)+1}$, while for any $x \in [3/4, 1)$

$$\begin{aligned} I_\alpha^-(w)(x) &\geq \int_0^{1/4} \frac{w(y)}{(x-y)^{1-\alpha}}\,dy = \sum_{n=2}^{\infty} 2^{n+1-np} \int_{2^{-2n}}^{2^{-2n+2}} \frac{1}{(x-y)^{1-\alpha}}\,dy \\ &\geq 3\sum_{n=2}^{\infty} 2^{n+1-np}\, 2^{-2n} = A > 0. \end{aligned}$$

In consequence the inequality $I_\alpha^-(w)(x) \leq Cw(x)$ *a.e.* would imply $0 < A < 2^{n_0(1-p)+1}$ for every $n_0 > 1$.

The function $w(x) = \chi_{[0,\infty)}(x)$ satisfies that $I_\alpha^-(w)(x) = \frac{x^\alpha}{\alpha}\chi_{[0,\infty)}(x)$ and (e) follows. □

**Proposition 3.2.** *Let $w$ be a weight. Then,*

(a) *for any $0 < \gamma < 1$, there exists $u$ satisfying:*
  (i) *$u \in F_{1,\alpha}^+$ for every $\alpha : \gamma < \alpha < 1$,*
  (ii) *$u \notin F_{1,\alpha}^+$ for every $0 < \alpha \leq \gamma$;*

(b) *let $\alpha,\ \beta > 0$. If $w \in F_{1,\alpha}^+$ then $I_\beta^-(w) \in F_{1,\alpha}^+$;*

(c) *for every $1 \leq p < \infty$, if $0 < \alpha < \beta$ then $F_{p,\alpha}^+ \subset F_{p,\beta}^+$.*

**Remark 3.3.** It follows from (a) of Proposition 3.1 and (c) of Proposition 3.2 that for any $0 < \alpha < 1 < \beta$ and $1 < p < \infty$ we have $F_{1,\alpha}^+ \subset F_{p,\beta}^+$. This inclusion provides easy examples of equal weights satisfying conditions (1.4) and (1.5) in [5, p. 728].

Proof: In order to prove (a) we consider the sequence $a_n = 1 - \frac{1}{2^n}$, $n \geq 0$ and we define the function

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{(x-a_{n-1})^\gamma}\,\chi_{(a_{n-1},a_n]}(x) + e^x\chi_{[1,\infty)}(x).$$

It is an easy but tedious computation to check that $I_\alpha^-(u)(x) \leq Cu(x)$ for any $\gamma < \alpha < 1$. On the other hand, for $0 < \alpha \leq \gamma$ and any positive integer $n_0$, if $1 < x < 1 + 2^{-n_0}$ we have

$$\begin{aligned} I_\alpha^-(u)(x) &\geq \int_0^1 \frac{u(y)}{(x-y)^{1-\alpha}}\,dy \\ &\geq \sum_{n=1}^{n_0} \int_{a_{n-1}}^{a_n} \frac{dy}{(y-a_{n-1})^\gamma(1+2^{-n_0}-y)^{1-\alpha}}\,. \end{aligned}$$

A change of variables gives

$$I_\alpha^-(u)(x) \geq C \sum_{n=1}^{n_0} 2^{n(\gamma-\alpha)}.$$

Therefore, the inequality $I_\alpha^-(u)(x) \leq Cu(x)$ almost everywhere for $1 < x < 1 + 2^{-n_0}$ would imply $\sum_{n=1}^{n_0} 2^{n(\gamma-\alpha)} \leq Ce^2$ for every $n_0 > 1$.

Part (b) is a consequence of the equality $I_\alpha^- \circ I_\beta^-(w) = I_\beta^- \circ I_\alpha^-(w)$.

We shall prove part (c). Let us assume that $w \in F_{p,\alpha}^+$. There exists a positive integer $n > 1$ such that $\alpha < \beta < n\alpha$. Then, for any positive $f$ we have

$$\begin{aligned} I_\beta^+(f)(x) &= \int_x^\infty \frac{f(y)}{(y-x)^{1-\beta}}\,dy \\ &\leq \int_{x+1}^\infty \frac{f(y)}{(y-x)^{1-n\alpha}}\,dy + \int_x^{x+1} \frac{f(y)}{(y-x)^{1-\alpha}}\,dy \\ &\leq I_\alpha^+ \circ I_\alpha^+ \circ \cdots \circ I_\alpha^+(f)(x) + I_\alpha^+(f)(x), \end{aligned}$$

which implies that $I_\beta^+$ is bounded from $L^p(w)$ into itself. □

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