Perimeter preservers of nonnegative integer matrices

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Abstract. We investigate the perimeter of nonnegative integer matrices. We also characterize the linear operators which preserve the rank and perimeter of nonnegative integer matrices. That is, a linear operator T preserves the rank and perimeter of rank-1 matrices if and only if it has the form $T(A) = P(A \circ B)Q$, or $T(A) = P(A^t \circ B)Q$ with appropriate permutation matrices P and Q and positive integer matrix B, where $\circ$ denotes Hadamard product.

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1. Introduction and preliminaries

Nonnegative integer matrices are combinatorially interesting matrices. So it has been a subject of many research works (see [5]). In [1], Beasley and Pullman defined the perimeter of a Boolean rank-1 matrix in order to characterize the linear operators that preserve Boolean rank. In this paper, we consider the nonnegative integer matrices of rank-1 and their perimeters. We also characterize the linear operators that preserve the rank and perimeter of the rank-1 matrices over nonnegative integers.

Let $\mathbb{Z}_+$ be a semiring of nonnegative integers and let $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{Z}_+$. The *rank* or *factor rank* [2], $r(A)$, of a nonzero matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = BC$. The rank of a zero matrix is zero. If $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ has rank 1, there exist nonzero vectors $\mathbf{u} \in \mathcal{M}_{m,1}(\mathbb{Z}_+)$ and $\mathbf{v} \in \mathcal{M}_{n,1}(\mathbb{Z}_+)$ such that $A = \mathbf{u}\mathbf{v}^t$. The *perimeter* [1] of this rank 1 matrix A, $p(A)$ is defined as $|\mathbf{u}| + |\mathbf{v}|$ for arbitrary factorization $A = \mathbf{u}\mathbf{v}^t$, where $|\mathbf{u}|$ denotes the number of nonzero entries in $\mathbf{u}$. It is clear that the perimeter of a rank 1 matrix is uniquely determined by the given matrix. Let $A \circ B$ denote the Hadamard (or Schur) product, the (i, j) entry of $A \circ B$ is $a_{ij}b_{ij}$.

A matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ is called a *cell* [3] if it has exactly one nonzero entry, that being a 1. We denote the cell whose nonzero entry is in the (i, j)th position by E_{ij}. Let $\mathbb{E}_{m,n} = \{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$. For $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we define $A^* = [{a_{ij}}^*]$ to be the $m \times n$ $(0, 1)$-matrix whose (i, j)th entry is 1 if and only if $a_{ij} > 0$.

It follows from the definition that $p(A) = p(A^*)$ and $(AB)^* = A^*B^*$, $(B + C)^* = B^* +_B C^*$, where $1 +_B 1 = 1$ is Boolean arithmetic, for all $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ and all $B, C \in \mathcal{M}_{n,r}(\mathbb{Z}_+)$.

If A and B are in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we say that A *dominates* B (written $B \leq A$ or $A \geq B$) if $a_{ij} = 0$ implies $b_{ij} = 0$ for all i, j ([4]). Then we can obtain the fact that $A \geq B$ if and only if $(A + B)^* = A^*$ for any $m \times n$ matrices A and B.

2. Perimeter preservers

A mapping $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is called a *linear operator* if T satisfies

$$T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$$

for all $A, B \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ and for all $\alpha, \beta \in \mathbb{Z}_+$.

In this section, we will characterize the linear operators that preserve both the rank and the perimeter of every rank-1 matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$.

Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then

(1) T is a (P, Q, B)*-operator* if there exist permutation matrices $P \in \mathcal{M}_{m,m}(\mathbb{Z}_+)$, $Q \in \mathcal{M}_{n,n}(\mathbb{Z}_+)$ and a positive matrix $B \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ with $r(B) = 1$ such that $T(A) = P(A \circ B)Q$ for all A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, or $m = n$ and $T(A) = P(A^t \circ B)Q$ for all A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$;
(2) T *preserves rank* 1 if $r(T(A)) = 1$ whenever $r(A) = 1$ for all $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$;
(3) T *preserves perimeter* k of rank-1 matrices if $p(T(A)) = k$ whenever $p(A) = k$ for all $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ with $r(A) = 1$.

Theorem 2.1. *If T is a (P, Q, B)-operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, then T preserves both rank and perimeter of every rank-1 matrix.*

PROOF: Since the operators Hadamard product, transpose and permutational equivalence preserve the rank and perimeter of every rank-1 matrix, the theorem follows. □

We note that an $m \times n$ matrix has perimeter 2 if and only if it is a positive integer multiple of a cell. We say that A is *a row* (*column*) *matrix* if A has nonzero entries only in one row (column, respectively). Thus we have the following lemma:

Lemma 2.2. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. If T preserves rank 1 and perimeter 2 of every rank-1 matrix, then the following statements hold:*

(1) *there exist positive integers u_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and a mapping $f : \mathbb{E}_{m,n} \to \mathbb{E}_{m,n}$ such that for $A = [a_{ij}]$, $T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} u_{ij} f(E_{ij})$;*
(2) *T maps a row (column) matrix to a row (column) matrix or if $m = n$, a row (column) matrix to a column (row) matrix.*

Proof: (1) Since T preserves perimeter 2, T maps a cell into a positive integer multiple of a cell.

(2) If not, then there exist two distinct cells E_{ij}, E_{ih} in some ith row such that $T(E_{ij})$ and $T(E_{ih})$ lie in two different rows and different columns. Then the rank of $E_{ij}+E_{ih}$ is 1 but that of $T\left(E_{ij}+E_{ih}\right)=T(E_{ij})+T(E_{ih})$ is 2. Therefore T does not preserve rank 1, a contradiction. □

An example follows of a linear operator that preserves rank 1 and perimeter 2 of a rank-1 matrix, but the operator does not preserve perimeter 3 and is not a (P,Q,B)-operator.

Example 2.3. Let $T:\mathcal{M}_{2,2}(\mathbb{Z}_+)\to\mathcal{M}_{2,2}(\mathbb{Z}_+)$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=(a+b+c+d)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It is easy to verify that T is a linear operator which preserves rank 1 and perimeter 2. But T does not preserve perimeter 3 and hence it is not a (P,Q,B)-operator. □

Let $R_i=\{E_{ij}\,|\,1\le j\le n\}$, $C_j=\{E_{ij}\,|\,1\le i\le m\}$, $\mathcal{R}=\{R_i\,|\,1\le i\le m\}$ and $\mathcal{C}=\{C_j\,|\,1\le j\le n\}$. For a linear operator T on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, define $T^*(A)=[T(A)]^*$ for all A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Let $T^*(R_i)=\{T^*(E_{ij})\,|\,1\le j\le n\}$ for all $i=1,\dots,m$ and $T^*(C_j)=\{T^*(E_{ij})\,|\,1\le i\le m\}$ for all $j=1,\dots,n$.

Lemma 2.4. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Suppose that T preserves rank 1 and perimeters 2 and p (≥ 3) of every rank-1 matrix. Then*

(1) *T maps two distinct cells in a row (or column) into positive multiples of two distinct cells in a row or in a column;*
(2) *for the case $m=n$, if T maps some R_i into a row (column) matrix then T maps every row matrix into a row (column) matrix, and if T maps some C_j into a row (column) matrix then T maps every column matrix into a row (column) matrix.*

Proof: (1) Suppose $T(E_{ij})=\alpha E_{rl}$ and $T(E_{ih})=\beta E_{rl}$ for some cells $E_{ij}\ne E_{ih}$ and some positive integers $\alpha,\beta\in\mathbb{Z}_+$. Then T maps the ith row of a matrix A into rth row or lth column by Lemma 2.2. Without loss of generality, we assume the former. Thus for any rank-1 matrix A with perimeter p (≥ 3) which dominates $E_{ij}+E_{ih}$, we can show that $T(A)$ has perimeter at most $p-1$, a contradiction. Thus T maps two distinct cells in a row into two distinct cells in a row or in a column.

(2) If not, then there exist rows R_i and R_j such that $T^*(R_i) \subseteq R_r$ and $T^*(R_j) \subseteq C_s$ for some r, s. Consider a rank-1 matrix $D = E_{ip} + E_{iq} + E_{jp} + E_{jq}$ with $p \neq q$. Then we have

$$T(D) = T(E_{ip} + E_{iq}) + T(E_{jp} + E_{jq}) = (\alpha_1 E_{rp'} + \alpha_2 E_{rq'}) + (\beta_1 E_{p''s} + \beta_2 E_{q''s})$$

for some $p' \neq q'$ and $p'' \neq q''$ and some positive integers $\alpha_i, \beta_i \in \mathbb{Z}_+$ by (1). Therefore $r\,(T(D)) \neq 1$ and T does not preserve rank 1, a contradiction. Hence T maps each row of A into a row (or a column) of $T(A)$. Similarly, T maps each column of A into a column (or a row) of $T(A)$. □

Now we have an interesting example:

Example 2.5. Consider a linear operator T on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ with $m \geq 3$ and $n \geq 4$ such that

$$T(A) = B = [b_{ij}]$$

where $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, $b_{ij} = 0$ if $i \geq 2$ and $b_{1j} = \sum_{i=1}^{m} a_{ir}$ with $r \equiv i + (j-1) \pmod{n}$ and $1 \leq r \leq n$. Then T maps each row and each column into the first row with some positive integer multiplication. And T preserves both rank and perimeters $2, 3$ and $n+1$ of rank-1 matrices. But T does not preserve perimeters k $(k \geq 4$ and $k \neq n+1)$ of rank-1 matrices: For if $4 \leq k \leq n$, then we can choose a $2 \times (k-2)$ submatrix with perimeter k which is mapped to distinct k positions in the first row of B under T. Then this $1 \times k$ submatrix has perimeter $k+1$. Therefore T does not preserve perimeter k of rank-1 matrices. □

Lemma 2.6. *Let T be a linear operator defined by*

$$T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} u_{ij} f(E_{ij})$$

for some function $f : \mathbb{E}_{m,n} \to \mathbb{E}_{m,n}$ and for some positive integers u_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$. If T preserves both rank and perimeters 2 and k $(k \geq 4, k \neq n+1)$ of rank-1 matrices, then the corresponding map f is a bijection on $\mathbb{E}_{m,n}$.

Proof: By Lemma 2.2, $T(E_{ij}) = b_{ij} E_{rl}$ for some $E_{rl} \in \mathbb{E}_{m,n}$ and some positive integer $b_{ij} \in \mathbb{Z}_+$. Without loss of generality, we may assume that T maps the ith row of a matrix into the rth row with positive integer multiplication. Suppose $f(E_{ij}) = f(E_{pq})$ for some distinct pairs $E_{ij}, E_{pq} \in \mathbb{E}_{m,n}$. Then we have $T(E_{ij}) = b_{ij} E_{rl}$ and $T(E_{pq}) = c_{pq} E_{rl}$ for some positive integers $b_{ij}, c_{pq} \in \mathbb{Z}_+$. If $i = p$ or $j = q$, then we have contradictions by Lemma 2.4. So let $i \neq p$ and $j \neq q$.

If $4 \leq k \leq n$, we will show that we can choose a $2 \times (k-2)$ submatrix from the ith and pth row whose image under T has a $1 \times k$ submatrix in the rth row as follows: Since $T(E_{ij}) = b_{ij} E_{rl}$ and $T(E_{pq}) = c_{pq} E_{rl}$, T maps the ith row and the pth row into the rth row. But T maps distinct cells in each row (or column) to distinct cells by Lemma 2.4. Now, choose E_{ij}, E_{pj} but do not choose E_{iq}, E_{pq}. Since there is a cell E_{ph} $(h \neq j, q)$ in the pth row such that $f(E_{ph}) = f(E_{iq})$ but $f(E_{ih}) \neq f(E_{pj})$, we choose the 2×2 submatrix $E_{ij} + E_{ih} + E_{pj} + E_{ph}$ whose image under T is a 1×4 submatrix in the rth row. And we can choose a cell E_{ps} $(s \neq q, j, h)$ such that $f(E_{is}) \neq f(E_{pj}), f(E_{pq}), f(E_{ph})$. Then we have a 2×3 submatrix $E_{ij} + E_{ih} + E_{is} + E_{pj} + E_{ph} + E_{ps}$ whose image under T is a 1×5 submatrix in the rth row. Similarly, we can choose a $2 \times (k-2)$ submatrix whose image under T is a $1 \times k$ submatrix in the rth row. This shows that T does not preserve the perimeter k of a rank-1 matrix, a contradiction.

If $k = n + k' \geq n + 2$, consider the matrix

$$D = \sum_{s=1}^{n} E_{is} + \sum_{t=1}^{n} E_{p\,t} + \sum_{h=1}^{k'-2} \sum_{g=1}^{n} E_{hg}$$

with rank 1 and perimeter $n + k' = k$. Then T maps the ith and pth row of D into the rth row with positive integer multiplication by Lemma 2.4. Thus the perimeter of $T(D)$ is less than $n + k' = k$, a contradiction.

Hence $f(E_{ij}) \neq f(E_{pq})$ for any two distinct cells $E_{ij}, E_{pq} \in \mathbb{E}_{m,n}$. Therefore f is a bijection. □

We obtain the following characterization theorem for linear operators preserving the rank and the perimeter of rank-1 matrices over nonnegative integers.

Theorem 2.7. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then the following are equivalent:*

(1) *T is a (P, Q, B)-operator;*
(2) *T preserves both rank and perimeter of rank-1 matrices;*
(3) *T preserves both rank and perimeters 2 and k $(k \geq 4, k \neq n+1)$ of rank-1 matrices.*

Proof: (1) implies (2) by Theorem 2.1. It is obvious that (2) implies (3). We now show that (3) implies (1). Assume (3). Then T induces a bijection $f : \mathbb{E}_{m,n} \to \mathbb{E}_{m,n}$ by Lemma 2.6. By Lemma 2.4, there are two cases; (a) T^* maps $\mathcal{R}$ onto $\mathcal{R}$ and maps $\mathcal{C}$ onto $\mathcal{C}$ or (b) T^* maps $\mathcal{R}$ onto $\mathcal{C}$ and $\mathcal{C}$ onto $\mathcal{R}$.

Case (a). We note that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all i, j, where σ and τ are permutations of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. Let P and Q be the permutation matrices corresponding to σ and τ, respectively. Then for any $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = b_{ij} E_{\sigma(i)\tau(j)}$ for some positive integer $b_{ij} \in \mathbb{Z}_+$. Now we claim that $B = (b_{ij})$ has rank 1. For, consider an $m \times n$ matrix J, all of whose entries are 1's. Then we have

$$T(J) = T\left(\sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} T(E_{ij}) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{\sigma(i)\tau(j)} = PBQ.$$

Since J has rank 1, it follows that $r(T(J)) = 1$ and hence $r(B) = 1$ since permutational equivalences preserve rank. Therefore for any $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we have

$$\begin{aligned} T(A) &= T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}T(E_{ij}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}E_{\sigma(i)\tau(j)} = P(A \circ B)Q. \end{aligned}$$

Thus T is a (P, Q, B)-operator.

Case (b). We note that $m = n$, $T^*(R_i) = C_{\sigma(i)}$ and $T^*(C_j) = R_{\tau(j)}$ for all i, j, where σ and τ are permutations of $\{1, \ldots, m\}$. By an argument similar to case (a), we obtain that $T(A)$ is of the form $T(A) = P(A^t \circ B)Q$. Thus T is a (P, Q, B)-operator. □

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ *strongly preserves* perimeter k of rank-1 matrices if $p\,(T(A)) = k$ if and only if $p(A) = k$.

Consider a linear operator T on $\mathcal{M}_{2,2}(\mathbb{Z}_+)$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b+c+d)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then T preserves both rank and perimeter 2 of rank-1 matrices but does not strongly preserve perimeter 2.

Theorem 2.8. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then T preserves both rank and perimeter of rank-1 matrices if and only if it preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices.*

PROOF: Suppose T preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices. Then T maps each row of a matrix into a row or a column (if $m = n$) with positive integer multiplication. Since T strongly preserves perimeter 2, T maps each cell onto a positive integer multiple of a cell. This means that T induces a bijection f on $\mathbb{E}_{m,n}$. Thus T preserves both rank and perimeter of rank-1 matrices by a method similar to that in the proof of Theorem 2.7.

The converse is immediate. □

Thus we have characterizations of the linear operators that preserve both rank and perimeter of rank-1 matrices over nonnegative integers.

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