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On loops whose inner permutations commute

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Abstract. Multiplication groups of (finite) loops with commuting inner permutations are investigated. Special attention is paid to the normal closure of the abelian permutation group.

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1. Introduction

A loop Q is a quasigroup with a neutral element. The left and right translations L_a and R_a defined by $L_a(x) = ax$ and $R_a(x) = xa$, respectively, are permutations of Q and the permutation group $M(Q) = \langle A, B\rangle$, where $A = \{L_a/a \in Q\}$, $B = \{R_a/a \in Q\}$ is called the multiplication group of Q. The stabiliser $I(Q)$ of the neutral element is the inner permutation group of the loop Q. R.H. Bruck showed in [1] that $I(Q)$ is abelian, provided that Q is nilpotent of class 2. The converse assertion does not seem to be true even for finite loops. In order to solve this question it may help to get some information on the structure of $M(Q)$ for hypothetical counterexample Q. This is just the purpose of the present short note. Here we pay particular attention to the subgroup $M(Q)' \cdot I(Q)$ which is the normal closure of $I(Q)$ in $M(Q)$.

We start with the following well-known folklore result:

Theorem 1.1. *A group G is isomorphic to the multiplication group $M(Q)$ of a loop Q if and only if there exist a corefree subgroup H of G (then $H \cong I(Q)$) and transversals A, B to H in G such that $G = \langle A, B\rangle$ and $[A, B] \subseteq H$.*

Additional information on the multiplication groups can be found in [2]–[15].

2. Preliminary results

In this section, let H be a subgroup of a group G such that there exist H-connected transversals A, B to H in G (it means that $AH = G = BH$, $A^{-1}A \cap H = 1 = B^{-1}B \cap H$ and $[A, B] \subseteq H$). Further let L and N denote the core $\mathrm{core}_G(H)$ and the normalizer $N_G(H)$ of H in G, respectively.

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Proposition 2.1. (i) $(A \cup B) \cap H \subseteq L$.

(ii) $Z(G) \subseteq AL \cap BL$.

(iii) *If* $C \subseteq A \cup B$, *then* $C \subseteq \mathrm{Core}_G(\langle H, C\rangle)$.

(iv) $N = HV$, *where* $V/L = Z(G/L)$.

(v) *If* $H \subseteq E \trianglelefteq F \subseteq G$, *then* $F' \subseteq E$.

(vi) *If* $M \trianglelefteq G$ *and* $M \subseteq A \cap B$, *then* $[A \cup B, M] \subseteq L$.

Proof: See [2, 3.11, 3.12, 3.18, 3.19]. □

Corollary 2.2. *Assume that* $L = 1$. *Then*

(i) $A \cap H = 1 = B \cap H$;

(ii) $Z(G) \subseteq A \cap B$;

(iii) $N = H \times Z(G)$;

(iv) *if* $M \trianglelefteq G$ *and* $M \subseteq A \cap B$, *then* $M \subseteq Z(\langle A, B\rangle)$ (*and hence* $M \subseteq Z(G)$, *provided that* $G = \langle A, B\rangle$).

Lemma 2.3. (i) $C_G(B)A \subseteq AL$ *and* $C_G(A)B \subseteq BL$.

(ii) $C_G(B)$ *is a subgroup of* AL.

(iii) $C_G(A)$ *is a subgroup of* BL.

Proof: (i) If $c \in C_G(B)$ and $a \in A$, then $ca = a_1u$ for some $a_1 \in A$ and $u \in H$. Now, $u = a_1^{-1}ca$ and $u^b = (a_1^{-1})^b \cdot c^b \cdot a^b = [b, a_1] \cdot a_1^{-1} \cdot c^b \cdot a \cdot [a, b] = [b, a_1]u[a, b] \in H$ for every $b \in B$. Since $G = BH$, it follows that $u \in L$.

(ii) and (iii). Use (i). □

Lemma 2.4. *Let* $a \in A$ *be such that* $aA \subseteq A$. *Then* $[a, B] \subseteq L$.

Proof: We have $[a, b]^{a_1} \cdot [a_1, b] = [aa_1, b] \in H$ for all $a_1 \in A$ and $b \in B$. Since $G = AH$, it follows that $[a, b] \in L$. □

Corollary 2.5. *Assume that* $L = 1$. *Then:*

(i) $C_G(B) = \{a \in A \mid aA = A\}$ *is a subgroup of* A;

(ii) $C_G(A) = \{b \in B \mid bB = B\}$ *is a subgroup of* B;

(iii) $zA = A$ *and* $zB = B$ *for every* $z \in Z(G)$.

Lemma 2.6. *Assume that* $L = 1$. *If* K *is a subgroup of* G *such that* $H \subseteq K$, *then* $Z(K) = (Z(G) \cap K) \times (Z(K) \cap H)$.

Proof: Clearly, $Z(K) \subseteq N = H \times Z(G)$ (2.2(iii)) and the rest is clear. □

Corollary 2.7. *Assume that* $L = 1$. *If* K *is a subgroup of* G *such that* $H \subseteq K$, *then* $Z(K) \subseteq H$ *if and only if* $Z(G) \cap K = 1$.

Proposition 2.8. *Assume that* H *is a finite abelian group. Then:*

(i) G *is soluble;*

(ii) *if* $G = \langle A, B\rangle$, *then* H *is subnormal in* G;

(iii) *if* $L = 1 \neq G = \langle A, B\rangle$, *then* $Z(G) \neq 1$.

Proof: (i) See [10, 4.1].

(ii) See [5, 3.1].

(iii) Since $G \neq 1 = L$ and H is subnormal in G, we have $H \neq N$. Now, $Z(G) \neq 1$ follows from 2.2(iii). □

Lemma 2.9. *Assume that $L = 1$ and H is subnormal in G. If $1 \neq H \leq K \trianglelefteq G$, then $K \cap Z(G) \neq 1$.*

Proof: Put $N_0 = H$ and $N_{i+1} = N_G(N_i)$ for every $i \geq 0$.
Then $H = N_0 \trianglelefteq N_1 = N \trianglelefteq N_2 \trianglelefteq \dots$ and, by [7, 3.3], we have $N_n = G$ for some $n \geq 1$. Now, assume on the contrary that $K \cap Z(G) = 1$. Since $N_1 = N = H \times Z(G)$, we get $K \cap N_1 = H$. On the other hand, $K \neq H$, and so $K \cap N_n \neq H$. Let m be the greatest number with $1 \leq m < n$ and $K \cap N_m = H$. Then $K \trianglelefteq G$ and $N_m \trianglelefteq N_{m+1}$ implies $H = K \cap N_m \trianglelefteq N_{m+1}$ and $N_{m+1} \subseteq N_1$. Consequently, $N_1 = G$ and $H \trianglelefteq G$, a contradiction with $L = 1 \neq H$. □

Lemma 2.10. *Assume that $L = 1$, $G = \langle A, B\rangle$ and $[A, B] = 1$. Then both A and B are normal subgroups of G.*

Proof: Using 2.5 and the equality $[A, B] = 1$, we conclude that $A = C_G(B)$ and $B = C_G(A)$. Then A is a subgroup of G, $A \cup B \subseteq N_G(A)$, and hence $N_G(A) = G$, quite similarly $N_G(B) = G$. □

3. The normal closure

This section is an immediate continuation of the preceding one. We keep the notation and we denote by K the normal closure of H in G.

Proposition 3.1. *$K = G'H$.*

Proof: We have $H \subseteq K \trianglelefteq G$, and so $G' \subseteq K$ by 2.1(v). Thus $G'H \subseteq K$. On the other hand, $H \subseteq G'H \trianglelefteq G$ implies $G'H = K$. □

In the remaining part of this section, we assume that $L = 1$.

Lemma 3.2. *$Z(K) = (Z(G) \cap K) \times (Z(K) \cap H)$ and $HZ(K) = H(Z(G) \cap K) = N_K(H)$.*

Proof: See 2.6 and 2.2(iii). □

Lemma 3.3. *$Z(K) = 1$ if and only if $Z(G) \cap K = 1$ and if and only if $N_K(H) = H$.*

Proof: The direct implication follows from 3.2. Conversely, if $Z(G) \cap K = 1$, then $Z(K) \subseteq H$ again by 3.2. But $K \trianglelefteq G$ implies $Z(K) \trianglelefteq G$ and $Z(K) \subseteq L = 1$. Thus $Z(K) = 1$. □

Lemma 3.4. *Let F be a subgroup of G such that $H \subseteq C_G(F)$. Then $K \subseteq C_G(V)$, $V \subseteq C_G(K)$ and $V \cap K \subseteq Z(K)$, where $V = \mathrm{Core}_G(F)$.*

Proof: We have $H \subseteq C_G(V) \trianglelefteq N_G(V) = G$, and so $K \subseteq C_G(V)$. The rest is clear. □

Lemma 3.5. *If H is abelian, then*

$$Z(K) = \mathrm{Core}_G((Z(G) \cap K)H) = \mathrm{Core}_G(Z(K)H).$$

Proof: Put $F = (Z(G) \cap K)H$ and $V = \mathrm{Core}_G(F)$. By 3.2, $Z(K) \subseteq F$ and, since $Z(K) \trianglelefteq G$, we get $Z(K) \subseteq V$. Further, since H is abelian, we have $H \subseteq C_G(F)$, and therefore $V \cap K \subseteq Z(K)$ by 3.4. Finally, $V \subseteq F \subseteq K$ implies $V = Z(K)$. □

Proposition 3.6. *Assume that H is a non-trivial finite abelian group and $G = \langle A, B \rangle$. Then:*

(i) *$Z(G) \cap K \neq 1 \neq Z(K)$ and $Z(K)H \neq H$;*
(ii) *if $K \not\subseteq N$, then $K \neq G$.*

Proof: (i) By 2.8(ii), H is subnormal in G. By Lemma 2.9 $K \cap Z(G) \neq 1$.

(ii) If $G \not\subseteq N$, then $N \neq G$ and, since H is subnormal in G, $N \subseteq W$ for a proper normal subgroup W of G. Then $G' \subseteq W$, and $K \subseteq W$. □

Proposition 3.7. *Assume that H is a finite abelian group and $G = \langle A, B \rangle$. Then K is finite.*

Proof: The index $[G : Z(G)]$ is finite ([5, 3.5]). □

Lemma 3.8. *Assume that H is abelian, $G = \langle A, B \rangle$ and $[A, B] \subseteq Z(K)$. Then both $AZ(K)$ and $BZ(K)$ are normal subgroups of G and $G' \subseteq AZ(K) \cap BZ(K) \cap K$.*

Proof: By 3.5, $Z(K) = \mathrm{Core}_G(HZ(K))$. Now, both $AZ(K)$ and $BZ(K)$ are normal subgroups of G by 2.10. Further, since $G = AH = AZ(K)H$ and H is abelian, $G/AZ(K)$ is so and we get $G' \subseteq AZ(K) \cap K = (A \cap K)Z(K)$. □

Lemma 3.9. *If $G' \subseteq Z(K)$, then $K \subseteq N$.*

Proof: $G' \subseteq Z(K)$ implies $G' \subseteq N_G(H)$. □

Lemma 3.10. *If H is abelian, $G = \langle A, B \rangle$, $[A, B] \subseteq Z(K)$ and $G' \cap AZ(K) \cap BZ(K) \subseteq Z(K)$, then $K \subseteq N$.*

Proof: Combine 3.8 and 3.9. □

4. Further results (a)

Again, we keep the notation introduced in the aforegoing two sections. Moreover, we will assume that H is abelian, $L = 1$ and that $PH \trianglelefteq K$ for a subgroup P of $Z(G) \cap K$ (notice that $PH = P \times H$). We put $U = \mathrm{Core}_G(PH)$ and $H_0 = U \cap H$.

Lemma 4.1. $P \times H_0 = U \subseteq Z(K)$.

PROOF: We have $U = \mathrm{Core}_G(PH) \subseteq \mathrm{Core}_G((Z(G) \cap K)H) = Z(K)$ by 3.5. $\square$

Lemma 4.2. *Let $a \in A \cap K$ and $b \in B$ be such that $a^{-1}b \in H$. Then $a^{-1}b \in H_0$.*

PROOF: We have $a = bu$, $u \in H$. Now, let $a_1 \in A$, $a_1 = b_1u_1$, $b_1 \in B$, $u_1 \in H$. Then $a^{a_1} = a^{b_1u_1} = (av)^{u_1}$, $v = [a, b_1] \in H$, and $(av)^{u_1} = a \cdot (u_1^{-1})^a \cdot vu_1 \in aPH$, since $PH \trianglelefteq K$. Thus $a^{a_1} \in aPH$. On the other hand, $a^{a_1} = (bu)^{a_1} = b \cdot [b, a_1] \cdot u^{a_1} = au^{-1}wu^{a_1}$, $w = [b, a_1] \in H$. It follows $u^{a_1} \in PH$ for every $a_1 \in A$. Consequently, $u \in \mathrm{Core}_G(PH) \cap H = H_0$. $\square$

Lemma 4.3. $[A \cap K, B] \cup [B \cap K, A] \subseteq H_0$.

PROOF: Let $a \in A \cap K$ and $b \in B$. We have $a^b = au$, $u = [a, b] \in H$, and, if $b_1 \in B$, then $bb_1 = b_2v$ for some $b_2 \in B$ and $v \in H$. Moreover, $a^{bb_1} = (au)^{b_1} = awu^{b_1}$, $w = [a, b_1] \in H$, and so $a^{b_2v} = awu^{b_1}$. Consequently, $a^{b_2} = a^zw^zu^{b_1z}$, $z = v^{-1}$. Since $PH \trianglelefteq K$, we have $[a, z] \in PH$ and $a^z \in aPH$. Of course, $a^{b_2} \in aH$, and hence $u^{b_1z} = (w^z)^{-1}(a^z)^{-1}a^{b_2} \in PH$, $u^{b_1} \in PH$. Thus $u \in \mathrm{Core}_G(PH) \cap H = H_0$. $\square$

Corollary 4.4. $[A \cap K, B] \cup [B \cup K, A] \subseteq Z(K)$.

Lemma 4.5. *Both $A \cap K$ and $B \cap K$ are abelian subgroups of G.*

PROOF: Firstly, if $a_1, a_2 \in A \cap K$, $a_3 \in A$, $u \in H$ are such that $a_1a_2 = a_3u$, then $a_3 \in A \cap K$ and $(a_1a_2)^b = a_1a_2v$, $v = [a_1, b] \cdot [a_2, b] \in H_0$, for every $b \in B$ (use 4.1, 4.3 and 4.4). Further, $a_3^b = a_3w$, $w = [a_3, b] \in H_0$, and $u^b = (a_3^{-1}a_1a_2)^b = w^{-1}va_3^{-1}a_1a_2 = w^{-1}vu \in H$. Consequently, $u \in L = 1$ and $a_1a_2 = a_3 \in A \cap K$. Proceeding similarly, we get $a_1^{-1} = a_4z_1$, $a_4 \in A \cap K$, $z_1 \in H$, $(a_1^{-1})^b = z_2a_1^{-1}$, $z_2 \in H_0$, $(a_1^{-1})^b = a_4z_3z_1^b$, $z_3 = [a_4, b] \in H_0$, and, finally, $z_1^b = z_3^{-1}z_2a_4^{-1}a_1^{-1} = z_3^{-1}z_2z_1 \in H$. Thus $z_1 = 1$ and $a_1^{-1} = a_4 \in A \cap K$. We have shown that $A \cap K$ is a subgroup of G. It remains to show that it is abelian.

We have $a_2 = b_2u_2$, $b_2 \in B$, $u_2 \in H_0$ (4.2) and $a_1^{a_2} = a_1^{b_2u_2} = (a_1v_1)^{u_2} = a_1v_1$, $v_1 = [a_1, b_2] \in H_0$ (since $H_0 \subseteq Z(K)$). Thus $v_1 \in A \cap H = 1$, $a_1^{a_2} = a_1$ and $a_1a_2 = a_2a_1$. $\square$

Lemma 4.6. *$V = (A \cap K) \times H_0 = (B \cap K) \times H_0$ is a normal abelian subgroup of G.*

Proof: The equality $(A \cap K) \times H_0 = (B \cap K) \times H_0$ follows from 4.1 and 4.2. Further, $Z(G) \cap K \subseteq A \cap K$, and hence $Z(G) \cap K \subseteq \mathrm{Core}_G(V)$, $PH \subseteq V$, $P \times H_0 = U \subseteq \mathrm{Core}_G(V)$ and, finally, $V \subseteq \mathrm{Core}_G(V)$ by 4.4. The subgroup V is abelian by 4.5. □

Lemma 4.7. $K' \subseteq U \subseteq V$.

Proof: Clearly, $U \subseteq V$ and we have to show that $K' \subseteq U$. But $H \subseteq PH \trianglelefteq K$ implies $K' \subseteq PH$, and, since $K' \trianglelefteq G$, we get $K' \subseteq \mathrm{Core}_G(PH) = U$. □

Corollary 4.8. $G''' = 1$.

Lemma 4.9. *For all $u \in H$ and $x \in G$ there exists $a_u \in A \cap K$ such that $a_u u \in H^x$.*

Proof: A is a two-sided transversal to every subgroup conjugate to H. Consequently, there exists $a_u \in A$ such that $u^{-1} \in H^x a_u$, i.e., $a_u u \in H^x$. Since $H \cup H^x \subseteq K$, we have $a_u \in K$. □

Lemma 4.10. $\mathrm{Core}_K(H) = Z(K) \cap H$.

Proof: Clearly, $Z(K) \cap H \subseteq \mathrm{Core}_K(H)$. Now, let $w \in \mathrm{Core}_K(H)$. Firstly, we show that $w^b \in N_K(H)$ for every $b \in B$.

By 4.9, $a_w w \in H^{b^{-1}}$ for some $a_w \in A \cap K$. Similarly, if $u \in H$, then $a_u u \in H^{b^{-1}}$, $a_u \in A \cap K$, and we have $a_w w \cdot a_u u = a_u u \cdot a_w w$, since $H^{b^{-1}}$ is abelian. Consequently, $(a_w w)^{a_u} = (a_w w)^{u^{-1}}$. Now, $a_w w^{a_u} w^{-1} = (a_w w)^{a_u} w^{-1} = (a_w w)^{u^{-1}} w^{-1} = u a_w w u^{-1} w^{-1} = a_w^{u^{-1}}$, since both $A \cap K$ and H are abelian. We have shown that $w^{a_u} = zw$, $z = [a_w, u^{-1}]$. By 4.6, $z = a_1 z_1$, where $a_1 \in A \cap K$ and $z_1 \in H_0$. On the other hand, $w \in \mathrm{Core}_K(H)$ implies $w^{a_u} \in H$. It follows $a_1 \in A \cap H = 1$, $a_1 = 1$, and hence $w^{a_u} \in wH_0$. Now we have $a_w^{u^{-1}} = (a_w w)^{a_u} w^{-1} = a_w w^{a_u} w^{-1} \in a_w H$, $[a_w, u^{-1}] \in H$ and $u^{a_w} \in uH = H$ for every $u \in H$. This means that $a_w \in N_G(H) = N$. But then $a_w \in K \cap N = N_K(H) = HZ(K)$ (3.2). We have $a_w w \in H^{b^{-1}}$, $a_w^b w^b \in H \subseteq N_K(H)$, $a_w^b \in a_w H_0$ by 4.3, $a_w^b \in N_K(H)$ (since $a_w \in N_K(H)$), and therefore $w^b \in N_K(H)$ for every $b \in B$.

We have shown our claim. Now, it follows that $w \in \mathrm{Core}_G(N_K(H)) = \mathrm{Core}_G(Z(K)H) = Z(K)$ by 3.5. Thus $w \in Z(K) \cap H$ and the proof is finished.

□

Proposition 4.11. *Assume that H is abelian, $L = 1$ and that $PH \trianglelefteq K$ for a subgroup P of $Z(G) \cap K$. Then:*

(i) $\mathrm{Core}_G(PH) = P \times H_0 \subseteq Z(K)$, *where* $H_0 = H \cap \mathrm{Core}_G(PH) \subseteq Z(K) \cap H$;

(ii) $a^{-1}b \in H_0$ *for all* $a \in A \cap K$ *and* $b \in B$ *such that* $a^{-1}b \in H$;
(iii) $b^{-1}a \in H_0$ *for all* $b \in B \cap K$ *and* $a \in A$ *such that* $b^{-1}a \in H$;
(iv) $[A \cap K, B] \cup [B \cap K, A] \subseteq H_0 \subseteq Z(K)$;
(v) *both* $A \cap K$ *and* $B \cap K$ *are abelian subgroups of* G *and* $K = (A \cap K)H = (B \cap K)H$;
(vi) $(A \cap K) \times H_0 = (B \cap K) \times H_0$ *is a normal abelian subgroup of* C;
(vii) $(A \cap K)Z(K) = (B \cap K)Z(K)$ *is a normal abelian subgroup of* G;
(viii) $K' \subseteq P \times H_0 = \mathrm{Core}_G(PH) \subseteq (A \cap K) \times H_0$;
(ix) $G''' = 1$ *and* K *is nilpotent of class at most* 2;
(x) $\mathrm{Core}_K(H) = Z(K) \cap H$.

Proof: See the preceding lemmas. □

Corollary 4.12. *Let* $H_0 = 1$. *Then:*

(i) $\mathrm{Core}_G(PH) = P$;
(ii) $A \cap K = B \cap K = Z(G) \cap K$;
(iii) $K = (A \cap K) \times H = (B \cap K) \times H$ *is abelian;*
(iv) $H \trianglelefteq K$ *and* $K = G'H \subseteq N_G(H) = N$.

Lemma 4.13. *If* $G = \langle A, B \rangle$, *then* $A \cap Z(K) = B \cap Z(K) = Z(G) \cap K$ *is a normal abelian subgroup of* G.

Proof: Both $V = (A \cap K) \times H_0 = (B \cap K) \times H_0$ and $Z(K)$ are normal abelian subgroups of G. Thus $C = V \cap Z(K) = A \cap Z(K) = B \cap Z(K)$ is normal in G. By 2.2(iv), $C \subseteq Z(G) \cap K$. Conversely, $Z(G) \cap K \subseteq C$ by 2.2(ii). □

Lemma 4.14. $[(A \cup B) \cap K, G] \subseteq PH_0 = U$.

Proof: Let $a \in A \cap K$, $x \in G$, $x = bu$, $b \in B$, $u \in H$. Then $a^x = a^{bu} = (aw)^u = a^u w = a[a,u]w$, where $w = [a,b] \in H_0$ by 4.11(iv). Further, $[a,u] \in K' \subseteq PH_0$ by 4.11(viii), and hence $a^x \in aPH_0$ and $[a,x] \in PH_0$. □

Lemma 4.15. $N_G(aU) = N_G(aZ(K)) = N_G(bU) = N_G(bZ(K)) = G$ *for all* $a \in A \cap K$, $b \in B \cap K$.

Proof: If $x \in G$, then $(aU)^x = a^x U = aU$ by 4.14. The rest is clear. □

Corollary 4.16. *The subgroups* $\langle a, U \rangle, \langle a, Z(K) \rangle$ $\langle b, U \rangle$ *and* $\langle b, Z(K) \rangle$ *are normal abelian subgroups of* G, $a \in A \cap K$, $b \in B \cap K$.

5. Further results (b)

In this section, we assume that H is non-trivial finite abelian, $L = 1$ and $G = \langle A, B \rangle$. By 3.6(i), we have $Z(G) \cap K \neq 1$. By 3.7, K is finite. Now, let P be a non-trivial subgroup of $Z(G) \cap K$. We put $U = \mathrm{Core}_G(PH) = P \times H_0$, $H_0 = U \cap H$.

Proposition 5.1. (i) *$\overline{A} = AU/U$ and $\overline{B} = BU/U$ are $\overline{H}$-connected transversals to $\overline{H} = UH/U$ in $\overline{G} = G/U$.*

(ii) *$\overline{G} = \langle \overline{A}, \overline{B} \rangle$.*

(iii) *$\overline{H} \cong H/H_0$ and* $\mathrm{Core}_{\overline{G}}(\overline{H}) = 1$ *(notice that $UH = PH$).*

(iv) *$\overline{K} = K/U = (\overline{G})'\overline{H}$.*

(v) *$P \times H = PH \trianglelefteq K$ if and only if $\overline{K} \subseteq N_{\overline{G}}(\overline{H})$.*

(vi) *$[\overline{K} : \overline{H}] = [K : UH]$.*

Proof: Easy to check. □

Remark 5.2. Assume that $G' \not\subseteq N$ and $|K|$ is minimal with respect to this property. It follows easily from 5.1 that $(P \times H =)PH \trianglelefteq K$ for every non-trivial subgroup P of $Z(G) \cap K$ and consequently 4.11 takes place. Among others, $[A \cap K, B] \cup [B \cap K, A] \subseteq Z(K) \cap H$, both $A \cap K$ and $B \cap K$ are abelian subgroups of G, $\mathrm{Core}_K(H) = Z(K) \cap H$ and $G''' = 1$. Further, since H is not normal in K but $\cap PH$ is, we have $\cap P \neq 1$ and it means that $Z(G) \cap K$ is a cyclic p-group for a prime p.

Remark 5.3. Assume that there exists a prime q such that q divides $|H|$ but q does not divide $[G' : G' \cap H]$. We claim that $G' \subseteq N = N_G(H)$.

Again, let G be a counterexample with smallest $|K|$ and let P be a subgroup of $Z(G) \cap K$ such that $|P| = p$ is a prime. Then p divides $[K : H] = [G' : G' \cap H]$, and hence $p \neq q$. Now, if $Q \subseteq H_0$, Q being the Sylow q-subgroup of H, then Q is characteristic in $U = P \times H_0$, and then $Q \trianglelefteq G$, $Q \subseteq L = 1$, a contradiction. Thus $Q \not\subseteq H_0$ and q divides $|\overline{H}|$ (see 5.1). According to 5.1(vi), q does not divide $[\overline{K} : \overline{H}]$, and hence $PH \trianglelefteq K$ by 5.1(v) and the minimality of $|K|$. Now, $Q \subseteq H \subseteq PH \trianglelefteq K$, Q is a Sylow q-subgroup of K and Q is characteristic in $PH = P \times H$, since the latter group is abelian. Further, PH is normal in K and consequently Q is the only Sylow q-subgroup of K. Then $Q \trianglelefteq G$, a contradiction with $L = 1$.

Remark 5.4. Assume that $G' \not\subseteq N$ (equivalently, $N = N_G(H)$ is not normal in G).

(i) By 5.3, if p is a prime number dividing $|H|$, then p divides $[G' : G' \cap H] = [K : H] = |A \cap K| = |B \cap K|$.

(ii) If $|A \cap K|$ is a power of a prime p, then $K = G'H$ is a (finite) p-group.

(iii) If G is finite and $|A| = |B| = [G : H]$ is a power of a prime p, then G is a p-group.

Remark 5.5. Combining 1.1 and 5.4, we get the following result: Let Q be a finite loop such that $I(Q)$ is abelian, but Q is nilpotent of class 3 or more. Then every prime number dividing $|I(Q)|$ divides $|Q|$, too. In particular, if $|Q|$ is a power of a prime p, then $M(Q)$ is a p-group.

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