On multiplication groups of left conjugacy closed loops

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Abstract. A loop Q is said to be left conjugacy closed (LCC) if the set $\{L_x; x \in Q\}$ is closed under conjugation. Let Q be such a loop, let $\mathcal{L}$ and $\mathcal{R}$ be the left and right multiplication groups of Q, respectively, and let $\operatorname{Inn} Q$ be its inner mapping group. Then there exists a homomorphism $\mathcal{L} \to \operatorname{Inn} Q$ determined by $L_x \mapsto R_x^{-1} L_x$, and the orbits of $[\mathcal{L}, \mathcal{R}]$ coincide with the cosets of $A(Q)$, the associator subloop of Q. All LCC loops of prime order are abelian groups.

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The structure of left conjugacy closed loops (LCC loops) seems to be far less transparent than that of (both sided) conjugacy closed loops (the CC loops). Nevertheless, some basic facts transfer from CC to LCC easily, and we shall review them in Section 2. Section 3 extends the similarities to the connection of the associator subloop $A(Q)$ with the group $[\mathcal{L}, \mathcal{R}]$. We shall prove that the cosets modulo $A(Q)$ of an LCC loop Q coincide with the orbits of $[\mathcal{L}, \mathcal{R}]$. The associator subloop $A(Q)$ is the least normal subloop with $Q/A(Q)$ a group, and $\mathcal{L} = \langle L_x; x \in Q\rangle$ and $\mathcal{R} = \langle R_x; x \in Q\rangle$ are the *left* and *right multiplication groups* of Q, respectively.

In Section 4 we study LCC loops with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ (where $\operatorname{Mlt} Q = \langle \mathcal{L}, \mathcal{R}\rangle$ is the *multiplication group* of Q). We show that $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q \Leftrightarrow \mathcal{R}_1 \le \mathcal{L}_1$ and that for such loops $\mathcal{L}_1$ and $A(Q)$ are abelian groups ($\mathcal{L}_1 \le \mathcal{L}$ is the stabilizer of the unit element $1 \in Q$, i.e., the *left inner mapping group* of Q).

The main published sources on LCC loops seem to be [1] and [11], with [8] and [12] being also relevant. Basarab's paper [1] contains a proof that Q/N_λ is an abelian group if Q is a loop for which the LCC property is isotopically invariant (i.e., all principal isotopes of Q are LCC loops). Such loops are also known as *universal left conjugacy closed loops*, and they get a lot of attention in the paper of Strambach and P. Nagy [11] as well. However, that paper also contains examples of other LCC loops. In fact, a large part of [11] is concerned with constructions of various examples of LCC loops. Some of them are universal

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LCC loops, some of them are Bol loops, and some of them are differentiable. The algebraic theory developed in [11] mostly concerns properties of such special classes of LCC loops, in particular Bol loops. On general level it corresponds (roughly spoken) to Propositions 2.2 and 2.4 below, and to observations that commutative LCC loops are abelian groups and LCC loops with inverse property are extra loops. Another large part of [11] is concerned with geometry of LCC loops; here we shall consider only the algebraic aspects.

The meaning of N_λ is standard, i.e., $N_\lambda = \{a \in Q;\ a(xy) = (ax)y$ for all $x, y \in Q\}$ denotes the *left nucleus* of Q. We shall also work with N_μ and N_ρ, the *middle* and *right* nuclei of Q, respectively. In general, no two nuclei have to coincide, and none of them needs to be a normal subloop. In LCC loops N_λ is normal and coincides with N_μ (see Proposition 2.7). This is proved in [1] as the first step towards showing that Q/N_λ is an abelian group if Q is an universal LCC loop. Basarab's proof of the latter theorem is restated in [9] for CC-loops, and [5] contains another variant of the same idea. The arguments used in [5] are modified in Section 5 to show, among others, that either the isotope $x \circ y = x \cdot (f\backslash y)$ is never LCC, for all $f \in Q$, $f \neq 1$, or at least one of the nuclei N_λ and N_ρ is nontrivial.

Questions concerning the possible triviality of N_λ and N_ρ are discussed in Section 6, together with various suggestions for future research.

Section 1 contains auxiliary statements. Those without proof come from Sections 1 of [5] or [6]. These statements are both well known and simple enough to expect that most of the readers will supply their own proof rather than consult the given sources. However, their knowledge can be helpful for other reasons, since many results in this paper are generalizations of theorems from [5] and [6] (some of which come from the now classical paper [7] of Goodaire and Robinson).

The standard books for loop theory are [4] and [3].

Mappings are composed in this paper from the right to the left, and hence T_a means here $R_a^{-1}L_a$ (and not $L_a^{-1}R_a$, which is more usual and which is equal here to T_a^{-1}). The *inner mapping group* $(\operatorname{Mlt} Q)_1$ will be denoted by $\operatorname{Inn} Q$. We shall often use the fact that $\operatorname{Inn} Q$ is generated by $\mathcal{L}_1 \cup \mathcal{R}_1 \cup \{T_x;\ x \in Q\}$ (and the fact that $\mathcal{L}_1$ is generated by mappings of the form $L_{xy}^{-1}L_xL_y$, while $\mathcal{R}_1$ is generated by mappings of the form $R_{yx}^{-1}R_xR_y$).

Theorem 2.8 was formulated in [6] for CC loops. It was possible to carry over the proof in a direct way, since it uses only the left conjugacy closedness.

LCC Bol loops are called in [11] Burn loops. It is shown there that a left Bol loop Q is LCC if and only if $x^2 \in N_\lambda$ for all $x \in Q$. Left Bol loops of exponent 2 are hence always LCC loops, and that makes [8] an important source of examples of LCC loops. In [12] one shows that multiplicative loops of certain quasifields are universal LCC loops. I am indebted to Michael Kinyon for pointing out the relevance of [8] and [12].

Recent papers that concern CC loops and are known to the author are [2], [10],

[9], [5] and [6]. It seems that the concept of conjugacy closedness was formulated first by Soikis in [13], and then independently by Goodaire and Robinson in [7].

1. Loops and groups

Lemma 1.1. *Let G be a permutation group on Ω, let 1 and ω be elements of Ω, and let $g \in G$ map 1 to ω. Suppose that T is a normal subgroup of G. Then $T_\omega = gT_1g^{-1}$.*

Proof: We have $gT_1g^{-1} \le G_\omega \cap T = T_\omega$, and $g^{-1}T_\omega g \le G_1 \cap T = T_1$ gives $T_\omega \le gT_1g^{-1}$. □

Corollary 1.2. *Let G be a permutation group on Ω and let T be a normal subgroup of G. Then $\langle T_\omega;\, \omega \in \Omega\rangle$ is a normal subgroup of G as well.*

Proof: Indeed, from Lemma 1.1 we see that $gT_\omega g^{-1}$ is equal to some $T_{\omega'}$, $\omega' \in \Omega$, for every $g \in G$ and $\omega \in \Omega$. □

Corollary 1.3. *Let G be a transitive permutation group on Ω and let T be a normal subgroup of G. Define equivalence $\sim$ on Ω by $\alpha \sim \beta \Leftrightarrow T_\alpha = T_\beta$. The equivalence classes of $\sim$ form a set of conjugate blocks of G.*

Proof: Consider $g \in G$ and assume $\alpha \sim \beta$, where $\alpha, \beta \in \Omega$. Then $g(\alpha) \sim g(\beta)$, since $T_{g(\alpha)} = gT_\alpha g^{-1} = gT_\beta g^{-1} = T_{g(\beta)}$, by Lemma 1.1. □

Each congruence of a loop Q is determined by a *normal* subloop K. It is well known that a subloop K is normal if and only if $x(yK) = (xy)K$, $(Ky)x = K(yx)$ and $xK = Kx$ for all $x, y \in Q$. This list of conditions can be slightly modified:

Lemma 1.4. *Let K be a subloop of a loop Q. The subloop K is normal if and only if $x(yK) = (xy)K$, $x(Ky) = (xK)y$ and $xK = Kx$ for all $x, y \in Q$.*

Proof: The conditions of the lemma imply $(Ky)x = (yK)x = y(Kx) = y(xK) = (yx)K = K(yx)$, and so K is normal. If it is normal, then $x(Ky) = x(yK) = (xy)K = K(xy) = (Kx)y = (xK)y$. □

Another way how to read the conditions under which a subloop K is normal, is to observe that they express the requirement that the generators of $\operatorname{Inn} Q$ preserve K, which is the same as to say that $\operatorname{Inn} Q$ preserves K. From that one easily obtains that a subset K of Q containing 1 is a normal subloop if and only if it is a block of $\operatorname{Mlt} Q$ (this is a well known fact).

Note that $a \in Q$ belongs to N_ρ if and only if a is fixed by every generator $L_{xy}^{-1}L_xL_y$ of $\mathcal{L}_1$ (and thus by every $\varphi \in \mathcal{L}_1$). From Corollary 1.3 we hence obtain immediately:

Lemma 1.5. *Let Q be a loop and let $\mathcal{L}$ and $\mathcal{R}$ be its left and right multiplication groups, respectively. If $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then N_ρ is a normal subloop of Q. If $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$, then N_λ is a normal subloop of Q.*

Let x, y and z be elements of a loop Q. The associator (x, y, z) is defined as $(x \cdot yz)\backslash(xy \cdot z)$. The associator subloop $A(Q)$ is thus the least normal subloop of Q that contains all associators (x, y, z). The following lemma is inspired by the results of [9, Section 4].

Lemma 1.6. *Let Q be a loop and let $\mathcal{L}$ be its left multiplication group. Then $\varphi(xa) = \varphi(x)a$ for every $x \in Q$, $a \in N_\rho$ and $\varphi \in \mathcal{L}$. If $A(Q) \le N_\rho$, then $L_{xy}^{-1}L_xL_y(z) = z(x, y, z)^{-1}$ for all $x, y, z \in Q$. If $A(Q) \le N_\rho$ is abelian, then $\mathcal{L}_1$ is abelian as well.*

Proof: One has $L_y(xa) = y(xa) = (yx)a = L_y(x) \cdot a$ for all $x, y \in Q$ and $a \in N_\rho$. In such a situation xa equals $y((y\backslash x)a)$, and so $L_y^{-1}(xa) = L_y^{-1}(x) \cdot a$. If $\varphi(xa) = \varphi(x)a$ and $\psi(xa) = \psi(x)a$ for all $x \in Q$, then $\varphi\psi(xa) = \varphi(\psi(x) \cdot a) = \varphi\psi(x) \cdot a$.

For $x, y, u \in Q$ consider $a \in Q$ such that $L_{xy}^{-1}L_xL_y(ua) = u$. Then $x(y \cdot ua) = xy \cdot u$ and $a \in A(Q)$. Assume $A(Q) \le N_\rho$. Then $u = L_{xy}^{-1}(L_xL_y(u) \cdot a) = L_{xy}^{-1}L_xL_y(u) \cdot a$, by the preceding part of the proof, and $(x \cdot yu)a = xy \cdot u$ yields $a = (x \cdot yu)\backslash(xy \cdot u) = (x, y, u)$.

Let now $A(Q) \le N_\rho$ be, in addition, abelian. Set $\varphi = L_{xy}^{-1}L_xL_y$ and $\psi = L_{uv}^{-1}L_uL_v$, where $x, y, u, v \in Q$. Then $\varphi\psi(z) = \varphi(z \cdot (u, v, z)^{-1}) = \varphi(z) \cdot (u, v, z)^{-1} = z(x, y, z)^{-1}(u, v, z)^{-1} = z(u, v, z)^{-1}(x, y, z)^{-1} = \psi\varphi(z)$, and the rest is clear. $\square$

An *isotopism* of loops Q_1 and Q_2 is a triple (α, β, γ) of bijective mappings $Q_1 \to Q_2$, where

$$\alpha(x) \cdot \beta(y) = \gamma(x \cdot y) \quad \text{for all} \quad x, y \in Q_1.$$

When $Q_1 = Q_2$, then one speaks about an *autotopism*. All autotopisms of a loop Q clearly form a group.

Lemma 1.7. *Let Q be a loop and let α and β be permutations of Q such that (β, α, α) is an autotopism. If $\alpha(1) = 1$, then $\alpha = \beta$ and α is an automorphism.*

Lemma 1.8. *Let Q be a loop and a its element. Then:*

(i) $a \in N_\lambda \Leftrightarrow (L_a, \mathrm{id}_Q, L_a)$ *is an autotopism,*
(ii) $a \in N_\rho \Leftrightarrow (\mathrm{id}_Q, R_a, R_a)$ *is an autotopism, and*
(iii) $a \in N_\mu \Leftrightarrow (R_a^{-1}, L_a, \mathrm{id}_Q)$ *is an autotopism.*

Lemma 1.9. *Let Q be a loop and let $\mathcal{L}$ and $\mathcal{R}$ be its left and right multiplication groups, respectively. Put $G = \mathrm{Mlt}\, Q$. Then*

$$C_G(\mathcal{R}) = \{L_a;\ a \in N_\lambda\} \quad \textit{and} \quad C_G(\mathcal{L}) = \{R_a;\ a \in N_\rho\}.$$

Lemma 1.10. *Let Q be a loop and let $\mathcal{L}$ be its left multiplication group. Then*

$$\{R_a;\ a \in N_\rho\} \cap \mathcal{L} = Z(\mathcal{L}) \cong M_\rho = \{a \in Q;\ R_a \in Z(\mathcal{L})\} \leq Z(N_\rho),$$

and $a \mapsto R_a$ yields an isomorphism $M_\rho \cong Z(\mathcal{L})$.

Proposition 1.11. *Let Q be a loop and let $\mathcal{L}$ and $\mathcal{R}$ be its left and right multiplication groups, respectively. Put $B = \langle \mathcal{L}_u;\, u \in Q\rangle$, and $C = \langle \mathcal{R}_u;\, u \in Q\rangle$. Then the orbits of both B and C coincide with the cosets of the associator subloop $A(Q)$.*

Lemma 1.12. *Let Q be a loop and $\mathcal{L}$ be its left multiplication group. Let H be a normal subloop of Q. If $\{\varphi \in \mathcal{L};\, \varphi(H) = H\}$ is a normal subgroup of $\mathcal{L}$, then Q/H is a group.*

2. Basic properties

Say that a subset S of a group G is *closed under conjugation*, if $xyx^{-1} \in S$ and $x^{-1}yx \in S$ for all $x, y \in S$. In other words, S is closed under conjugation if and only if S is a normal subset of $\langle S\rangle$. A loop Q is *left conjugacy closed*, by definition, if the set $\{L_x;\ x \in Q\}$ is closed under conjugation. Seemingly, one could define a stronger notion by demanding this set to be a normal subset not only of $\mathcal{L}$, the left multiplication group of Q, but also of the full multiplication group $\mathrm{Mlt}\, Q$. We shall see in a while that in such a case Q has to be a group.

Lemma 2.1. *Let Q be a loop and let x, y, z be its elements.*
(i) *If $L_xL_yL_x^{-1} = L_z$, then $z = (xy)/x$; and*
(ii) *if $R_xL_yR_x^{-1} = L_z$, then $z = y$.*

Proof: To get (i), write $L_xL_yL_x^{-1} = L_z$ as $L_xL_y = L_zL_x$ and apply this equality to 1. Then $xy = zx$ and $z = (xy)/x$. Similarly, $R_xL_y = L_zR_x$ yields $yx = zx$, and $y = z$ follows. □

Now, if $\{L_x;\ x \in Q\}$ is a normal subset of $\mathrm{Mlt}\, Q$, then $\mathcal{L}$ centralizes $\mathcal{R}$. But that means $Q = N_\lambda = N_\rho$, by Lemma 1.9, and so Q is a group. The purported generalization of LCC thus brings nothing new.

We have $(xy)/x = T_x(y)$, and so the LCC loops can be described by the equality $L_xL_yL_x^{-1} = L_{T_x(y)}$, or, alternatively, by $L_x^{-1}L_yL_x = L_{T_x^{-1}(y)}$. Another description can be made by the means of autotopisms:

Proposition 2.2. *A loop Q is left conjugacy closed if and only if (T_x, L_x, L_x) is an autotopism for every $x \in Q$.*

Proof: Triples (T_x, L_x, L_x) yield an autotopism for all $x \in Q$ if and only if

$$((xy)/x) \cdot (xz) = x(yz) \quad \text{for all} \quad x, y, z \in Q.$$

This is the same as $L_{(xy)/x}L_x = L_xL_y$, and the rest is clear. □

Corollary 2.3. *Let Q be a left conjugacy closed loop. Then*

(i) *the nuclei N_λ and N_μ coincide, and*
(ii) *an element $x \in Q$ belongs to N_ρ if and only $T_x \in \operatorname{Aut} Q$.*

Proof: The product of $(R_a^{-1}, L_a, \mathrm{id}_Q)$ and $(L_a, \mathrm{id}_Q, L_a)$ is an autotopism, by Proposition 2.2. Hence if one of the triples is an autotopism, then the other one has to be an autotopism as well. We thus see that (i) follows from Lemma 1.8. Another combination of Lemma 1.8 and Proposition 2.2 implies that $a \in N_\rho$ if and only if $(\mathrm{id}_Q, R_a^{-1}, R_a^{-1})(T_a, L_a, L_a) = (T_a, T_a, T_a)$ is an autotopism. This proves (ii). □

Let Q be a loop and let $\mathcal{L}$ be its left multiplication group. The loop Q is said to be an *A_ℓ-loop* if $\mathcal{L}_1 \leq \operatorname{Aut} Q$. To verify that Q is an A_ℓ-loop it clearly suffices to verify that the generators $L_{xy}^{-1} L_x L_y$ of $\mathcal{L}_1$ are automorphisms.

Proposition 2.4. *Every left conjugacy closed loop Q is an A_ℓ-loop and satisfies $L_{xy}^{-1} L_x L_y = T_{xy}^{-1} T_x T_y$ for all $x, y \in Q$.*

Proof: From Proposition 2.2 we obtain that

$$(T_{xy}^{-1} T_x T_y,\ L_{xy}^{-1} L_x L_y,\ L_{xy}^{-1} L_x L_y)$$

is an autotopism. The rest follows from Lemma 1.7. □

Lemma 2.5. *Let Q be a loop. Consider any of the following six identities. The identity is satisfied by all $x, y, z \in Q$ if and only if Q is left conjugacy closed.*

$$\begin{array}{lll} L_x L_y L_x^{-1} = L_{(xy)/x}, & L_x R_y R_z^{-1} L_x^{-1} = R_{xy} R_{xz}^{-1}, & L_x R_y L_x^{-1} = R_{xy} R_x^{-1}, \\ R_{xy}^{-1} L_x R_y = T_x, & L_x^{-1} R_y R_z^{-1} L_x = R_{x\backslash y} R_{x\backslash z}^{-1}, \text{ and} & L_x^{-1} R_y L_x = R_{x\backslash y} R_{x\backslash 1}^{-1}. \end{array}$$

Proof: Write $L_x L_z = L_{(xz)/x} L_x$ as $x(zy) = ((xz)/x)(xy)$. We see that this identity can be written also as $L_x R_y = R_{xy} T_x$. This establishes equivalence of the identities in the first column. The identities of the second column are clearly equivalent, and this is true for the third column as well. Furthermore, the third column is obtained from the second column by setting $z = 1$. Replacing y by z in $R_{xy}^{-1} L_x R_y = T_x$ yields $R_{xy}^{-1} L_x R_y = T_x = R_{xz}^{-1} L_x R_z$, and $L_x R_y R_z^{-1} L_x^{-1} = R_{xy} R_{xz}^{-1}$ follows. The first column hence implies the second one. The identity $L_x R_y L_x^{-1} = R_{xy} R_x^{-1}$ of the third column can be written also as $R_{xy}^{-1} L_x R_y = R_x^{-1} L_x = T_x$, which gets us back to the first column. □

Corollary 2.6. *Let $\mathcal{R}$ be the right multiplication group of a left conjugacy closed loop Q. Then $\mathcal{R}$ is a normal subgroup of $\operatorname{Mlt} Q$.*

Proposition 2.7. *The left and middle nuclei of an LCC loop Q coincide and form a normal subloop.*

Proof: By point (i) of Corollary 2.3 we only need to prove the normality. However, that follows from Lemma 1.5, by Corollary 2.6. $\square$

Theorem 2.8. *Let Q be a left conjugacy closed loop. Denote by $\mathcal{L}$ its left multiplication group. Then there exists a unique homomorphism $\Lambda : \mathcal{L} \to \operatorname{Inn} Q$ that maps L_x to T_x for each $x \in Q$. This homomorphism is the identity on $\mathcal{L}_1$, and its kernel is equal to $Z(\mathcal{L}) = \{R_x;\, x \in Q\} \cap \mathcal{L}$. If $R_x \in Z(\mathcal{L})$, then $x \in N_\rho$.*

Proof: We need to show that $L_{x_1}^{\varepsilon_1} \dots L_{x_n}^{\varepsilon_n} = \mathrm{id}_Q$ implies $T_{x_1}^{\varepsilon_1} \dots T_{x_n}^{\varepsilon_n} = \mathrm{id}_Q$ for all $x_1, \dots, x_n \in Q$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$. The triples $(T_{x_i}^{\varepsilon_i}, L_{x_i}^{\varepsilon_i}, L_{x_i}^{\varepsilon_i})$ are autotopisms for every i, $1 \le i \le n$, by Proposition 2.2, and hence

$$(T_{x_1}^{\varepsilon_1} \dots T_{x_n}^{\varepsilon_n}, L_{x_1}^{\varepsilon_1} \dots L_{x_n}^{\varepsilon_n}, L_{x_1}^{\varepsilon_1} \dots L_{x_n}^{\varepsilon_n})$$

is an autotopism as well. Since the two right-hand members of this triple are assumed to equal id_Q, it follows that $T_{x_1}^{\varepsilon_1} \dots T_{x_n}^{\varepsilon_n} = \mathrm{id}_Q$ as well, by Lemma 1.7.

We have $\Lambda(\varphi) = \varphi$ whenever $\varphi = L_{xy}^{-1} L_x L_y$ for some $x, y \in Q$, by Proposition 2.4. Such mappings generate $\mathcal{L}_1$, and so $\Lambda(\varphi) = \varphi$ for all $\varphi \in \mathcal{L}_1$.

Every $\psi \in \mathcal{L}$ can be uniquely expressed as $L_a \varphi$, where $\varphi \in \mathcal{L}_1$ and $a \in Q$. Assume $\psi \in \operatorname{Ker} \Lambda$. Then $T_a = R_a^{-1} L_a = \varphi^{-1} \in \mathcal{L}_1$, which means $R_a \in \mathcal{L}$. We also have $T_a \in \operatorname{Aut} Q$, by Proposition 2.4, and hence $a \in N_\rho$, by point (ii) of Corollary 2.3. Now, $R_a \in Z(\mathcal{L})$, by Lemma 1.10, and $\psi = L_a \varphi = L_a L_a^{-1} R_a = R_a$. We have verified $\operatorname{Ker} \Lambda \le Z(\mathcal{L})$, and $Z(\mathcal{L}) \le \{R_a;\, a \in Q\} \cap \mathcal{L}$ is clear from Lemma 1.9. On the other hand, if $R_a \in \mathcal{L}$, then $R_a = L_a T_a^{-1} \in \mathcal{L}$ implies $T_a \in \mathcal{L}_1$, and so $\Lambda(R_a) = \Lambda(L_a)\Lambda(T_a^{-1}) = T_a T_a^{-1} = \mathrm{id}_Q$. $\square$

Corollary 2.9. *Let $\mathcal{L}$ the left multiplication group of a left conjugacy closed loop Q. Then $\mathcal{L}_1 \le \langle T_x;\, x \in Q \rangle$.*

Proof: Since $\mathcal{L}$ is generated by $\{L_x;\, x \in Q\}$, the image of Λ has to be generated by $\{T_x;\, x \in Q\}$. The image contains $\mathcal{L}_1$, by Theorem 2.8. $\square$

Corollary 2.10. *Let x and y be elements of a left conjugacy closed loop Q. Then*

$$T_x T_y T_x^{-1} = T_{T_x(y)}.$$

Proof: We have $\Lambda(L_x L_y L_x^{-1}) = \Lambda(L_{T_x(y)})$. $\square$

Theorem 2.11. *A left conjugacy closed loop of prime order is an abelian group.*

Proof: Let Q be an LCC loop with p elements, p a prime. Let $\Lambda : \mathcal{L} \to \operatorname{Inn} Q$ be the homomorphism of Theorem 2.8. Since p divides the order of $\mathcal{L}$, but does not divide the order of $\operatorname{Inn} Q$, p has to divide the order of $\operatorname{Ker} \Lambda$. From the description of $\operatorname{Ker} \Lambda$ in Theorem 2.8 it follows that N_ρ contains at least p elements. That means $N_\rho = Q$, and so Q has to be a group. $\square$

3. The associator subloop

Throughout this section Q will mean a left conjugacy closed loop, and $\mathcal{L}$ and $\mathcal{R}$ will be the left and right multiplication groups of Q, respectively.

Lemma 3.1. *If $x, y \in Q$, then*

$$[L_x, R_y] = L_x^{-1} R_y^{-1} L_x R_y = R_{x\backslash 1} R_{x\backslash y}^{-1} R_y \in \mathcal{R}_1, \quad \textit{and}$$
$$[L_x^{-1}, R_y^{-1}] = L_x R_y L_x^{-1} R_y^{-1} = R_{xy} R_x^{-1} R_y^{-1} \in \mathcal{R}_{xy}.$$

In particular, $[L_x, R_x] = R_{x\backslash 1} R_x$.

Proof: This follows directly from identities in the third column of Lemma 2.5. □

Proposition 3.2. *Let Q be a left conjugacy closed loop and $\mathcal{R}$ its right multiplication group. Then*

$$\mathcal{R}_1 = \langle [L_x, R_y];\ x, y \in Q \rangle \trianglelefteq \operatorname{Inn} Q.$$

Proof: The normality of $\mathcal{R}_1$ in $\operatorname{Inn} Q$ follows from the normality of $\mathcal{R}$ in $\operatorname{Mlt} Q$ (see Corollary 2.6). From Lemma 3.1 we obtain $[L_x, R_y] \in \mathcal{R}_1$, for all $x, y \in Q$. It remains to show that each $R_{xy}^{-1} R_y R_x$ can be expressed by a product of commutators. If y is replaced by $x\backslash y$, then $R_{xy}^{-1} R_y R_x$ turns to $R_y^{-1} R_{x\backslash y} R_x$ and from Lemma 3.1 we see that this is equal to

$$(R_{x\backslash 1} R_{x\backslash y}^{-1} R_y)^{-1} (R_{x\backslash 1} R_x) = [L_x, R_y]^{-1} [L_x, R_x].$$

□

Lemma 3.3. *The group $\langle \mathcal{R}_u;\ u \in Q \rangle$ is normal in $\operatorname{Mlt} Q$ and is contained in $[\mathcal{L}, \mathcal{R}]$.*

Proof: The normality follows from Corollary 1.2, as $\mathcal{R}$ is a normal subgroup of $\operatorname{Mlt} Q$, by Corollary 2.6. Fix $u \in Q$ and observe that $\mathcal{R}_u = L_u \mathcal{R}_1 L_u^{-1}$, by Lemma 1.1. This means that $\mathcal{R}_u$ is generated by mappings $L_u [L_x, R_y] L_u^{-1} = [L_u L_x L_u^{-1}, L_u R_y L_u^{-1}]$, where $x, y \in Q$, by Proposition 3.2. However, $L_u R_y L_u^{-1} \in \mathcal{R}$, by Corollary 2.6. Hence $\mathcal{R}_u \le [\mathcal{L}, \mathcal{R}]$, for every $u \in Q$. □

Lemma 3.4. *The group $\langle \mathcal{R}_u; u \in Q \rangle$ contains each of the commutators $[L_x, R_y]$, $[R_y, L_x]$, $[L_x^{-1}, R_y]$ and $[R_y, L_x^{-1}]$, for all $x, y \in Q$.*

Proof: One has $[u, v]^{-1} = [v, u]$ in every group G, for all $u, v \in G$. In view of Proposition 3.2 we thus need only to consider $[L_x^{-1}, R_y]$. That equals $L_x [R_y, L_x] L_x^{-1}$, which belongs to our group, as that is normal in $\operatorname{Mlt} Q$ and $[R_y, L_x]$ is in $\mathcal{R}_1$. □

Theorem 3.5. *Let Q be a left conjugacy closed loop and denote by $\mathcal{L}$ and $\mathcal{R}$ the left and right multiplication group of Q, respectively. Then*

$$[\mathcal{L},\mathcal{R}] = \langle \mathcal{R}_u;\ u \in Q\rangle \trianglelefteq \operatorname{Mlt} Q,$$

and the orbits of $[\mathcal{L},\mathcal{R}]$ coincide with the cosets modulo the associator subloop $A(Q)$.

Proof: Denote $\langle \mathcal{R}_u;\ u\in Q\rangle$ by B, and note that the orbits of B coincide with the cosets of $A(Q)$ by Proposition 1.11. From Lemma 3.3 we have $B \trianglelefteq \operatorname{Mlt} Q$ and $B \le [\mathcal{L},\mathcal{R}]$. It remains to show $[\alpha,\beta]\in B$ for all $\alpha\in\mathcal{L}$ and $\beta\in\mathcal{R}$. Write β as $R_y^{-1}\psi$, $\psi\in\mathcal{R}_1$, and α as $L_{x_1}^{\varepsilon_1}\dots L_{x_k}^{\varepsilon_k}$, where $x_i\in Q$ and $\varepsilon_i\in\{-1,1\}$, $1\le i\le k$. We shall proceed by induction on k. The case $k=0$ is trivial and so we assume $k\ge 1$. Then $\alpha = L_x^{\varepsilon}\gamma$, where $x=x_1$, $\varepsilon=\varepsilon_1$ and $\gamma = L_{x_2}^{\varepsilon_2}\dots L_{x_k}^{\varepsilon_k}$. Now, $[\alpha,\beta] = [L_x^{\varepsilon}\gamma, R_y^{-1}\psi] = \gamma^{-1}\mu\psi$, where $\mu = L_x^{-\varepsilon}\psi^{-1}R_yL_x^{\varepsilon}\gamma R_y^{-1}$. To prove $[\alpha,\beta]\in B$ is hence equivalent to proving $\gamma^{-1}\mu\in B$, which is equivalent to proving $\mu\gamma^{-1}\in B$, as $\psi\in\mathcal{R}_1\le B$ and $B\trianglelefteq \operatorname{Mlt} Q$. Now,

$$\mu\gamma^{-1} = (L_x^{-\varepsilon}\psi^{-1}L_x^{\varepsilon})(L_x^{-\varepsilon}R_yL_x^{\varepsilon}R_y^{-1})(R_y\gamma R_y^{-1}\gamma^{-1}),$$

$L_x^{-1}\psi^{-1}L_x \in \mathcal{R}_{x\backslash 1}\le B$, $L_x\psi^{-1}L_x^{-1}\in\mathcal{R}_x\le B$, $L_x^{-\varepsilon}R_yL_x^{\varepsilon}R_y^{-1}\in B$ by Lemma 3.4, and $R_y\gamma R_y^{-1}\gamma^{-1} = [\gamma^{-1},R_y^{-1}]^{-1}\in B$ by the induction assumption. □

4. Normality on the left

In this section we shall again assume that Q is a left conjugacy closed loop, with $\mathcal{L}$ and $\mathcal{R}$ being the left and right multiplication groups of Q, respectively. Furthermore, we shall denote by $\bar{\mathcal{L}}$ the least normal subgroup of $\operatorname{Mlt} Q$ that contains $\mathcal{L}$.

The equality $L_xR_yL_x^{-1} = R_{xy}R_x^{-1}$ (see Lemma 2.5) can be written as $R_y = L_x^{-1}R_{xy}R_x^{-1}L_x$ and so we can immediately state

Lemma 4.1. *All $x,y\in Q$ satisfy $R_{xy}^{-1}R_yR_x = (R_{xy}^{-1}L_x^{-1}R_{xy})(R_x^{-1}L_xR_x)$.*

Corollary 4.2. *The right inner multiplication group $\mathcal{R}_1$ is a subgroup of $\bar{\mathcal{L}}_1$.*

Proof: Indeed, by Lemma 4.1 the generators of $\mathcal{R}_1$ are products of conjugates of $L_x^{\pm 1}$, where x runs through Q. □

Proposition 4.3. *Let Q be a left conjugacy closed loop and let $\mathcal{L}$ and $\mathcal{R}$ be its left and right multiplication groups. Then $\mathcal{L}\trianglelefteq \operatorname{Mlt} Q$ if and only if $\mathcal{R}_1\le\mathcal{L}_1$.*

Proof: Assume $\mathcal{L}\trianglelefteq\operatorname{Mlt} Q$. Then $\mathcal{L}=\bar{\mathcal{L}}$, and $\mathcal{R}_1\le\mathcal{L}_1$ follows from Corollary 4.2. Assume now $\mathcal{R}_1\le\mathcal{L}_1$. We have $\operatorname{Mlt} Q = \mathcal{L}\mathcal{R}$, and so to get $\mathcal{L}\trianglelefteq\operatorname{Mlt} Q$ it suffices to show that the generators of $\mathcal{L}$ are normalized by the elements of $\mathcal{R}$.

This will be true if $R_y^{-1}L_xR_y \in \mathcal{L}$ and $R_yL_x^{-1}R_y^{-1} \in \mathcal{L}$ for all $x, y \in Q$, i.e., if $[L_x, R_y] \in \mathcal{L}$ and $[L_x^{-1}, R_y^{-1}] \in \mathcal{L}$ for all $x, y \in Q$. Now, $[L_x, R_y] \in \mathcal{R}_1$ and $[L_x^{-1}, R_y^{-1}] \in \mathcal{R}_{xy}$, by Lemma 3.1. However, $\mathcal{R}_1 \le \mathcal{L}$, which is assumed, implies $\mathcal{R}_{xy} \le \mathcal{L}$, as $\mathcal{R}_u = L_u\mathcal{R}_1L_u^{-1}$ for every $u \in Q$, by Lemma 1.1. □

Corollary 4.4. *If $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then $\mathcal{L} \cap \mathcal{R} = Z(\mathcal{L})\mathcal{R}_1$.*

Proof: If $R_a\varphi \in \mathcal{L} \cap \mathcal{R}$, where $\varphi \in \mathcal{R}_1$, then $R_a \in \mathcal{L}$, by Proposition 4.3, and $R_a \in Z(\mathcal{L})$, by Theorem 2.8. The converse inclusion follows from Theorem 2.8 and Proposition 4.3 immediately. □

Corollary 4.5. *If $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then the homomorphism $\Lambda : \mathcal{L} \to \operatorname{Inn} Q$, $L_x \mapsto T_x$ for all $x \in Q$, is surjective, and $\operatorname{Inn} Q$ is generated by the set $\{T_x;\ x \in Q\}$.*

Proof: The image of Λ always contains $\mathcal{L}_1$, by Theorem 2.8, and it contains all T_x, $x \in Q$, by the definition. If $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then it contains also $\mathcal{R}_1$, by Proposition 4.3, and the rest is clear. □

Set $M_\rho = \{a \in N_\rho;\ R_a \in Z(\mathcal{L})\}$, like in Lemma 1.10. By that lemma, $M_\rho \le Z(N_\rho)$. This is true for all loops. Since we assume that Q is left conjugacy closed, we can express M_ρ simply as $\{a \in Q;\ R_a \in \mathcal{L}\}$, by Theorem 2.8. Note also that $R_a \in \mathcal{L}$ if and only if $T_a \in \mathcal{L}_1$.

Proposition 4.6. *Let Q be a left conjugacy closed loop and let $\mathcal{L}$ be its left multiplication group. Suppose $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then M_ρ is a normal subloop of Q, and Q/M_ρ is a group. Furthermore, $M_\rho \le Z(N_\rho)$ is an abelian group and $\mathcal{L}_1$ is an abelian group as well. The right nucleus N_ρ is a normal subloop of Q.*

Proof: If $T_a \in \mathcal{L}_1$, then $T_xT_aT_x^{-1} \in \mathcal{L}_1$, as we assume $\mathcal{L}_1 \trianglelefteq \operatorname{Inn} Q$. However, $T_xT_aT_x^{-1}$ equals $T_{T_x(a)}$, by Lemma 2.10. In other words, $a \in M_\rho$ implies $T_x(a) \in M_\rho$. From Corollary 4.5 we now see that M_ρ is a normal subloop of Q.

Express a permutation $\psi \in \mathcal{L}$ as $L_x\varphi$, $\varphi \in \mathcal{L}_1$. Then $\Lambda(\psi) = T_x\varphi$ gets into $\mathcal{L}_1$ if and only if $T_x \in \mathcal{L}_1$, i.e., if and only if $x \in M_\rho$. We have proved that $\psi(M_\rho) = M_\rho$ if and only if $\psi \in \Lambda^{-1}(\mathcal{L}_1)$. This is true for every LCC loop, but under the assumption of $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, we know, in addition, that $\Lambda^{-1}(\mathcal{L}_1) \trianglelefteq \mathcal{L}$ and that M_ρ is a normal subloop of Q. Hence Lemma 1.12 can be used to see that Q/M_ρ is really a group. M_ρ is abelian by Lemma 1.10, $\mathcal{L}_1$ is abelian by Lemma 1.6 (since M_ρ contains $A(Q)$), and N_ρ is normal by Lemma 1.5. □

Proposition 4.7. *Let Q be a left conjugacy closed loop and let $\mathcal{L}$ be its left multiplication group. If $\mathcal{L} = \operatorname{Mlt} Q$, then Q is an abelian group.*

Proof: It suffices to show that Q has to be a group. Elements of $\mathcal{L}$ normalize the set of all left translations. Hence if $\mathcal{L}$ contains $\mathcal{R}$, then $L_xR_y = R_yL_x$ for all $x, y \in Q$, by point (ii) of Lemma 2.1. However, left and right translations commute if and only if Q is a group. □

Proposition 4.8. *Let Q be a left conjugacy closed loop and let $\mathcal{L}$ and $\mathcal{R}$ be its left and right multiplication groups, respectively. If $\mathcal{R} = \operatorname{Mlt} Q$, then there exists no proper normal subgroup of $\operatorname{Mlt} Q$ that contains $\mathcal{L}$.*

Proof: Let us assume $\mathcal{R} = \operatorname{Mlt} Q$. Then $\operatorname{Inn} Q = \mathcal{R}_1$, and so $\operatorname{Inn} Q \leq \bar{\mathcal{L}}_1$, by Lemma 4.2. However, that means $\bar{\mathcal{L}} = \operatorname{Mlt} Q$. □

5. Isotopes

Consider a loop Q. Its *principal isotopes* are the loops $Q(\circ)$ with $x \circ y = x/e \cdot f\backslash y$ for all $x, y \in Q$. Elements $e, f \in Q$ are parameters of $Q(\circ)$. Call $Q(\circ)$ a *left principal isotope* if $f = 1$ and *right principal isotope* if $e = 1$. If $Q(\circ)$ is a left principal isotope, then the set of left translations of $Q(\circ)$ coincides with the set of left translations of Q. Hence left principal isotopes of LCC loops are also LCC loops. In fact, we have more:

Proposition 5.1. *Let Q be a left conjugacy closed loop and let e be its element. Set $x \circ y = (x/e) \cdot y$, for all $x, y \in Q$. Then $x \mapsto ex$ constitutes an isomorphism $Q(\cdot) \cong Q(\circ)$.*

Proof: Indeed, $ex \circ ey = ((ex)/e) \cdot (ey) = e(xy)$, for all $x, y \in Q$. □

Proposition 5.1 appears in [11] as Remark 1.1.3. Each principal isotope can be obtained as right principal isotope of a left principal isotope. Hence we have:

Corollary 5.2. *An LCC loop Q is a universal LCC loop if and only if all right principal isotopes are left conjugacy closed.*

This was observed by Basarab [1] as well.

Lemma 5.3. *Let Q be a left conjugacy closed loop and let f be its element. The following conditions are equivalent:*

(i) *operation $x \circ y = x \cdot (f\backslash y)$ yields an LCC loop;*
(ii) $R_{xy}^{-1} L_{(fx)/f} R_y = R_x^{-1} L_{(fx)/f}$ *for all $x, y \in Q$;*
(iii) $x\backslash T_f(x) \in N_\lambda$ *for all $x \in Q$; and*
(iv) $T_f(x) \equiv x \bmod N_\lambda$ *for all $x \in Q$.*

If f satisfies these conditions, then $[L_x, L_f^{-1}] = L_u$, where $u = x\backslash T_f(x)$, for all $x \in Q$.

Proof: The operation $\circ$ is LCC if and only if it satisfies the law $x(z(fy)) = ((xz)/x)(x(fy))$, i.e., if

$$x(f\backslash(zy)) = ((x(f\backslash z))/(f\backslash x)) \cdot (f\backslash(xy))$$

for all $x, y, z \in Q$. This identity can be also expressed by

$$L_x L_f^{-1} R_y = R_{f\backslash(xy)} R_{f\backslash x}^{-1} L_x L_f^{-1}.$$

Since $R_{f\backslash xy}R_{f\backslash x}^{-1} = L_f^{-1}R_{xy}R_x^{-1}L_f$, by Lemma 2.5, we can convert the latter identity to

$$L_{(fx)/f}R_y = L_f L_x L_f^{-1} R_y = R_{xy}R_x^{-1}L_f L_x L_f^{-1} = R_{xy}R_x^{-1}L_{(fx)/f}.$$

This establishes the equivalence of (i) and (ii).

The identity of (ii) can be also expressed as $L_{(fx)/f}R_y L_{(fx)/f}^{-1} = R_{xy}R_x^{-1}$. However, $R_{xy}R_x^{-1}$ equals $L_x R_y L_x^{-1}$, by Lemma 2.5. Thus (ii) implies that $L_x^{-1}L_{(fx)/f}$ centralizes all R_y, $y \in Q$. That means, by Lemma 1.9, that $L_x L_{(fx)/f}$ is equal to some L_u, $u \in N_\lambda$. From $L_x L_u = L_{(fx)/f}$ we get $xu = T_f(x)$ and $u = x\backslash T_f(x)$. Hence (ii) implies (iii).

Assume now (iii) and put $u = x\backslash T_f(x)$. Then $u \in N_\mu$, by Proposition 2.7, and thus $L_x L_u = L_{xu} = L_{(fx)/f}$. Hence $L_x^{-1}L_{(fx)/f} = L_u$ centralizes all R_y, $y \in Q$, by Lemma 1.9, and so

$$L_{(fx)/f}R_y L_{(fx)/f}^{-1} = L_x R_y L_x^{-1} = R_{xy}R_x^{-1}$$

for all $x, y \in Q$. We see that (iii) implies (ii).

Point (iv) reformulates point (iii), since N_λ is a normal subloop.

If $L_x L_u = L_{(fx)/f}$, then $L_u = L_x^{-1}L_f L_x L_f^{-1} = [L_x, L_f^{-1}]$. □

To restate Basarab's theorem on universal LCC loops we need the following observation (see [11, Theorem 1.1.8]).

Lemma 5.4. *A commutative LCC loop is an abelian group.*

Proof: Let Q be a commutative LCC loop. Then the law $x(yz) = ((xy)/x)(xz)$ turns to $x(yz) = y(xz)$, and so $x(yz) = y(zx) = z(yx) = (xy)z$. □

Theorem 5.5 (Basarab)**.** *Let Q be a left conjugacy closed loop. Then Q is universally conjugacy closed if and only if Q/N_λ is an abelian group.*

Proof: If Q is a universal LCC loop, then $x \equiv (yx)/y \bmod N_\lambda$ for all $x, y \in Q$, by Lemma 5.3. Hence Q/N_λ is commutative, and Lemma 5.4 can be used. On the other hand, if Q/N_λ is an abelian group, then $T_f(x) \equiv x \bmod N_\lambda$ for all $x, f \in Q$. It follows that Q is universally LCC, by point (iv) of Lemma 5.3. □

Basarab [1] did not state his result as an equivalence, but as an implication. However, the converse statement is implicitly present already in his proof.

Let us now turn to the significance of Lemma 5.3 for nonuniversal LCC loops. For a loop Q put $C(Q) = \{a \in Q;\ ax = xa \text{ for all } x \in Q\}$.

Lemma 5.6. *Let Q be a left conjugacy closed loop. Then $C(Q) \subseteq N_\rho$.*

Proof: If $a \in C(Q)$, then $x(ay) = ((xa)/x)(xy)$ turns to $x(ay) = a(xy)$, and so $x(ya) = x(ay) = a(xy) = (xy)a$. □

Proposition 5.7. *Let Q be a left conjugacy closed loop. If both N_λ and $C(Q) \subseteq N_\rho$ are trivial, then there exists no $f \in Q$, $f \neq 1$, for which the operation $x \circ y = x(f \backslash y)$ yields a left conjugacy closed loop.*

PROOF: Let $x \circ y = x \cdot (f \backslash y)$ be an LCC loop. Then $x \backslash T_f(x) \in N_\lambda$ for all $x \in Q$, by Lemma 5.3. If $N_\lambda = 1$, then $x = T_f(x)$ for all $x \in Q$, which means $x \in C(Q)$. □

Proposition 5.7 was independently obtained by Piroska Csörgő (personal communication).

6. Questions and problems

The purpose of this paper has been to collect basic facts about general LCC loops as they can be derived from the present knowledge of CC loops. While writing the paper I was not aware of any nontrivial LCC loops with trivial left nucleus. When the first draft of this paper was circulated, such loops where readily supplied by M. Kinyon and J.D. Phillips, who used model builder `mace4` to find examples on 8, 9, 10 and 12 elements. All these examples contain $a \neq 1$ with $ax = xa$ for all x, all their right translations are even permutations and all of the examples contain a left translation which is an odd permutation. The right multiplication group is in the case of 8 elements equal to the alternating group, which means that the loop is simple. This example seems to suggest that there might exist LCC loops in which the right multiplication group equals the (full) multiplication group. However, no such loop seems to have been constructed yet.

M. Kinyon also noted that LCC loops Q with trivial N_λ can be obtained from [8], since that paper gives examples of Bol loops of exponent 2 on 16 elements that have trivial left nucleus. These loops are involutorial, and their left multiplication groups are 2-groups (and contain only even permutations). In one of the cases the right multiplication group $\mathcal{R}$ is an extension of $\mathrm{Alt}_7 \times \mathrm{Alt}_7$ by $\mathbb{Z}_4$, and is of index 2 in $\mathrm{Mlt}\, Q$ [8, Corollary 7]. These loops are solvable, but not nilpotent.

Nearly nothing seems to be known about nonassociative simple left conjugacy closed loops. For which orders do they exist?

Does there exist a nontrivial left conjugacy closed loop with trivial right nucleus?

Does there exist a nontrivial left conjugacy closed loop such that no nontrivial right principal isotope is left conjugacy closed (cf. Proposition 5.7)?

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