Comment.Math.Univ.Carolin. 45,2 (2004)265–273

Characters of finite quasigroups VII: permutation characters

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Abstract. Each homogeneous space of a quasigroup affords a representation of the Bose-Mesner algebra of the association scheme given by the action of the multiplication group. The homogeneous space is said to be faithful if the corresponding representation of the Bose-Mesner algebra is faithful. In the group case, this definition agrees with the usual concept of faithfulness for transitive permutation representations. A permutation character is associated with each quasigroup permutation representation, and specialises appropriately for groups. However, in the quasigroup case the character of the homogeneous space determined by a subquasigroup need not be obtained by induction from the trivial character on the subquasigroup. The number of orbits in a quasigroup permutation representation is shown to be equal to the multiplicity with which its character includes the trivial character.

Keywords: quasigroup, association scheme, permutation character

Classification: 20N05

1. Introduction

Earlier papers of this series ([3]–[8]) extended character theory from finite groups to finite quasigroups. The theory depends on the association scheme given by the multiplicity-free action on a finite quasigroup Q of its multiplication group G. Recently, the concept of a permutation representation has been extended from groups to quasigroups ([12]–[14]). The aim of the present paper is to investigate some connections between quasigroup character theory and permutation representations. Each homogeneous space of a quasigroup Q affords a representation of the Bose-Mesner algebra $\mathrm{End}_{\mathbb{C}G}\,\mathbb{C}Q$ of the association scheme given by the action of G on Q (Theorem 3.2). The homogeneous space is said to be faithful if the corresponding representation of the Bose-Mesner algebra is faithful (Definition 4.1). If Q is a group, this definition agrees with the usual concept of faithfulness for transitive permutation representations (Proposition 4.2). A permutation character is associated with each homogeneous space (Definition 5.1) or more general quasigroup permutation representation (Section 6). In the group case, this character is shown to correspond to the usual definition of the character of a permutation representation of a group (Proposition 5.2). However, an example (presented in Section 7) demonstrates that in the quasigroup case, the character of the homogeneous space determined by a subquasigroup need not be obtained by induction from the trivial character on the subquasigroup (using the quasigroup induction procedure described in [4]). The number of orbits in a quasigroup permutation representation is shown to be equal to the multiplicity with which its character includes the trivial or principal character (Theorem 6.1).

2. Quasigroup homogeneous spaces

The construction of a quasigroup homogeneous space for a finite quasigroup [12], [13] is analogous to the transitive permutation representation of a group Q (with stabiliser subgroup P) on the homogeneous space $P\backslash Q = \{Px \mid x \in Q\}$. Let P be a subquasigroup of a finite quasigroup Q. Let L be the relative left multiplication group of P in Q, the subgroup of the permutation group on the set Q generated by left multiplications by elements of P. Let $P\backslash Q$ be the set of orbits of the permutation group L on the set Q. Let A_P be the incidence matrix of the membership relation between the set Q and the set $P\backslash Q$ of subsets of Q. Thus the entry of A_P in the row labelled x and the column labelled yL, for elements x, y of Q, is 1 if $x \in yL$ and 0 otherwise. Let A_P^+ be the pseudoinverse of the matrix A_P [10]. Its entry in the row labelled yL and column labelled x is $|yL|^{-1}$ if $x \in yL$ and 0 otherwise [12, Theorem 2.1]. Note that $A_P^+ A_P$ is just the $|P\backslash Q| \times |P\backslash Q|$ identity matrix.

For each element q of Q, right multiplication in Q by q yields a permutation of Q. Let $R_Q(q)$ be the corresponding permutation matrix. Define a new matrix

$$R_{P\backslash Q}(q) = A_P^+ R_Q(q) A_P \,. \tag{2.1}$$

In the group case, the matrix (2.1) is just the permutation matrix given by the permutation $P\backslash Q \to P\backslash Q; Px \mapsto Pxq$. In the general quasigroup case, the matrix (2.1) is stochastic: each row consists of non-negative entries summing to 1. This algebraic fact may be interpreted probabilistically: in the homogeneous space of the quasigroup Q, each quasigroup element q yields a Markov chain on the state space $P\backslash Q$ with transition matrix $R_{P\backslash Q}(q)$ given by (2.1).

3. Representations of the centraliser ring

In this section, it is shown that each homogeneous space over a quasigroup Q with finite cardinality n affords a representation of the Bose-Mesner algebra of the association scheme given by the action of the multiplication group G of Q on Q. Recall that the *quasigroup conjugacy classes* are defined to be the orbitals of G on Q, i.e. the orbits $C_1, \ldots, C_s$ of G on the G-set $(Q, G)^2$. Let these orbits have respective cardinalities $nn_1, \ldots, nn_s$ and incidence matrices $\mathbf{A}_1, \ldots, \mathbf{A}_s$. Thus for elements x, y of Q, the entry of the $n \times n$ matrix $\mathbf{A}_i$ in the row indexed by x and the column indexed by y is 1 if $(x, y) \in C_i$, and 0 otherwise. The Bose-Mesner algebra is the linear span of the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_s$. For a finite set X, denote the complex vector space spanned by X as $\mathbb{C}X$, and identify the endomorphism ring $\mathrm{End}_{\mathbb{C}}\, \mathbb{C}X$ of this vector space with the algebra of matrices of the endomorphisms with respect to the basis X. Now the group algebra $\mathbb{C}G$ carries the multiplication extended from the multiplication of G by linearity and distributivity. The action $Q \times G \to Q$ of G on Q extends to an action $\mathbb{C}Q \times \mathbb{C}G \to \mathbb{C}Q$ making $\mathbb{C}Q$ a right $\mathbb{C}G$-module, with corresponding representation

$$\lambda : \mathbb{C}G \to \mathrm{End}_{\mathbb{C}}\, \mathbb{C}Q. \tag{3.1}$$

By Theorem 3.1 of [3], the Bose-Mesner algebra is the endomorphism ring $\mathrm{End}_{\mathbb{C}G}\, \mathbb{C}Q$ of this module. In Wielandt's terminology, it is the *centraliser ring* $V(G,Q)$ of G on Q.

Proposition 3.1 ([12, Theorem 4.2]). *For a subquasigroup P of Q with relative left multiplication group L in Q, one has*

$$A_P A_P^+ = |L|^{-1} \sum_{l \in L} \lambda(l). \tag{3.2}$$

In particular, $A_P A_P^+$ is an element of $\lambda(\mathbb{C}G)$.

PROOF: To simplify notation, drop the suffix P from A_P and A_P^+. For an element x of the basis Q of $\mathbb{C}Q$, it must be shown that the endomorphisms on each side of (3.2) have the same effect on x. Now $xAA^+ = \sum_{y \in Q} x(AA^+)_{xy} = x \sum_{X \in P\backslash Q} \sum_{y \in Q} A_{xX} A^+_{Xy} = x \sum_{y \in xL} A_{x,xL} A^+_{xL,y} = \sum_{y \in xL} (xL) A^+_{xL,y}$.

On the other hand,

$$|L|^{-1} \sum_{l \in L} xl = |L|^{-1} \cdot (|L|/|xL|) \sum_{y \in xL} y = |xL|^{-1} \sum_{y \in xL} y = \sum_{y \in xL} (xL) A^+_{xL,y}$$

as well. □

The map defined in (3.3) below is a restriction of the linear map

$$\rho_{P\backslash Q} : \mathrm{End}_{\mathbb{C}}\, \mathbb{C}Q \to \mathrm{End}_{\mathbb{C}}\, \mathbb{C}P\backslash Q; C \mapsto A_P^+ C A_P$$

of [12, (4.1)], [13, (1.2)].

Theorem 3.2. *Let P be a subquasigroup of a finite quasigroup Q. Then the map*

$$\rho_{P\backslash Q} : \mathrm{End}_{\mathbb{C}G}\, \mathbb{C}Q \to \mathrm{End}\, \mathbb{C}P\backslash Q\,;\ \mathbf{B} \mapsto A_P^+ \mathbf{B} A_P \tag{3.3}$$

is a homomorphism of $\mathbb{C}$-algebras.

PROOF: Consider two elements $\mathbf{B}_1$, $\mathbf{B}_2$ of the Bose-Mesner algebra. The definition (3.3) gives

$$\rho_{P\backslash Q}(\mathbf{B}_1)\rho_{P\backslash Q}(\mathbf{B}_2) = A_P^+\mathbf{B}_1 A_P A_P^+\mathbf{B}_2 A_P. \tag{3.4}$$

By Proposition 3.1, the central product $A_P A_P^+$ of the right hand side of (3.4) lies in $\lambda(\mathbb{C}G)$, and so commutes with elements of the Bose-Mesner algebra such as $\mathbf{B}_2$. Moreover, one has $A_P A_P^+ A_P = A_P$ as part of the specification of the pseudoinverse A_P^+ of A_P (cf. (2.2)(a) of [12]). The right hand side of (3.4) thus reduces to $\rho_{P\backslash Q}(\mathbf{B}_1\mathbf{B}_2)$, as required to show that (3.3) gives a monoid homomorphism. □

Remark 3.3. Both the domain and the codomain of $\rho_{P\backslash Q}$ in (3.3) carry C^*-algebra structure, with involution given by conjugate transposition of matrices with respect to the standard bases Q and $P\backslash Q$. However, the map $\rho_{P\backslash Q}$ is not necessarily a homomorphism of these C^*-algebra structures. In the example Q of Section 7 below, the incidence matrix of the conjugacy class C_3 is real and symmetric, but its image under $\rho_{0\backslash Q}$ is the asymmetric real matrix $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

4. Faithful homogeneous spaces

Definition 4.1. Let P be a subquasigroup of a finite quasigroup Q. The homogeneous space $P\backslash Q$ is said to be *faithful* if the corresponding map $\rho_{P\backslash Q}$ of (3.3) injects.

Proposition 4.2. *Let P be a subgroup of a finite group Q. Then the homogeneous space $P\backslash Q$ yields a faithful transitive permutation representation of Q if and only if the homogeneous space is faithful in the quasigroup sense of Definition* 4.1.

PROOF: Suppose that $P\backslash Q$ yields a transitive permutation representation which is not faithful. Let K be a non-identity group conjugacy class of Q contained in the kernel of the group permutation representation. Define the element

$$\mathbf{C} = \sum_{q\in K} R_Q(q)$$

of the Bose-Mesner algebra. (Cf. pp. 430–431 of [11] or the proof of Theorem 3.1 in [3].) Then $\rho_{P\backslash Q}(\mathbf{C})$ is a multiple of the identity in $\operatorname{End}\mathbb{C}P\backslash Q$, so that $\rho_{P\backslash Q}$ cannot inject.

On the other hand, if $P\backslash Q$ does yield a faithful transitive permutation representation, then the permutation matrices $A_P^+ R_Q(q) A_P$ of the elements q of Q afford a faithful linear representation of the complex group algebra of Q. In this case the map $\rho_{P\backslash Q}$, as the restriction of the linear representation to the centre of the group algebra, certainly injects. □

Remark 4.3. As noted in the proof of Proposition 4.2, a homogeneous space $P\backslash Q$ over a finite group Q is faithful (if and) only if the corresponding Markov matrices (2.1) of different group elements differ. In the non-associative quasigroup case, a homogeneous space $P\backslash Q$ may be faithful in the sense of Definition 4.1, and yet have $R_{P\backslash Q}(q_1) = R_{P\backslash Q}(q_2)$ for distinct elements q_1, q_2 of Q. For instance, in the example Q of Section 7 below, the homogeneous space $0\backslash Q$ is faithful, but $R_{0\backslash Q}(1) = R_{0\backslash Q}(3)$ according to (7.3).

5. Characters of homogeneous spaces

Definition 5.1. Let P be a subquasigroup of a quasigroup Q. Then the *permutation character* (or just *character*) of the homogeneous space $P\backslash Q$ is the class function $\pi_{P\backslash Q} : Q \times Q \to \mathbb{C}$ taking the value

$$n_i^{-1} \operatorname{Tr}(A_P^{+} \mathbf{A}_i A_P) \tag{5.1}$$

on each member of the i-th quasigroup conjugacy class C_i, for $1 \le i \le s$.

In order to verify the consistency of Definition 5.1 with the usual definition of a permutation character in the group case, recall that each quasigroup class function $\theta : Q \times Q \to \mathbb{C}$ over a group Q determines a corresponding group class function $\theta' : Q \to \mathbb{C}; q \mapsto \theta(1, q)$ (cf. p. 45 of [3]).

Proposition 5.2. *Let P be a subgroup of a finite group Q. Then $\pi'_{P\backslash Q}$ is the permutation character of the transitive permutation representation of Q on $P\backslash Q$.*

Proof: For each element q of Q, the value of the permutation character on q is the trace of the permutation matrix $A_P^{+} R_Q(q) A_P$. For each of the n_i elements of the i-th group conjugacy class of Q, these traces do not vary. Thus the value of the permutation character at an element q of the i-th group conjugacy class K_i may be written as

$$n_i^{-1} \sum_{q \in K_i} \operatorname{Tr}[A_P^{+} R_Q(q) A_P]. \tag{5.2}$$

By pp. 430–431 of [11] or the proof of Theorem 3.1 in [3], the quantity (5.2) agrees with (5.1). Finally, note that if q lies in the i-th group conjugacy class, then $(1, q)$ lies in the i-th quasigroup conjugacy class, as required to complete the proof of the proposition. □

Remark 5.3. In the context of Proposition 5.2, the permutation character of $P\backslash Q$ is obtained by inducing the principal character on the subgroup P up to the full group Q. For a subquasigroup P of a non-associative quasigroup Q, however, it need no longer be true that the permutation character of $P\backslash Q$ is obtained in this way (using the general quasigroup induction procedure discussed in [4]).

In the example Q of Section 7 below, for instance, the principal character on the subquasigroup $\{0\}$ induces up to the "regular" character π_4 on Q, and this of course differs from the permutation character π_3 of the homogeneous space $0\backslash Q$. On the other hand, the character π_2 of the homogeneous space $\{0,2\}\backslash Q$ *is* obtained by inducing up the principal character on the subquasigroup $\{0,2\}$.

6. General permutation characters

So far, the discussion has been restricted to the homogeneous spaces of a finite quasigroup Q. In [14], a definition of a general Q-set or permutation representation of Q was given. The natural definition lies within the context of coalgebras, but for present purposes one may use an equivalent, less canonical but more elementary definition. A finite set X equipped with a Markov chain action for each element of the quasigroup Q is called a Q-IFS (Section 3 of [14] — the name comes from fractal geometry [1]). Consider a function $f : X \to Y$ between two such objects, with incidence matrix F. Then f is defined to be a Q-IFS-homomorphism if for each element q of the quasigroup Q, its Markov matrices $R_X(q)$ on X and $R_Y(q)$ on Y are connected by the intertwining relation $R_X(q)F = FR_Y(q)$. The sum $X+Y$ of two Q-IFS X and Y is obtained by taking the disjoint union of the underlying sets, equipped with direct sum Markov matrices $R_X(q) \oplus R_Y(q)$. Then a general Q-set or permutation representation of Q may be defined as a direct sum of a finite number of homomorphic images of homogeneous spaces (Theorem 9.2 of [14]). The summands are known as the *orbits* of the permutation representation. Of course, this all specialises appropriately to the group case (Corollary 9.4 of [14]). In the group case, and in all the quasigroup cases studied so far, all the homomorphic images of homogeneous spaces observed are themselves homogeneous spaces. At any rate, for the image of a homogeneous space $P\backslash Q$ given by a surjective function with incidence matrix F, one may extend Definition 5.1 by assigning value $n_i^{-1}\,\mathrm{Tr}(F^+A_P^+\mathbf{A}_iA_PF)$ to each element of the i-th quasigroup conjugacy class C_i. The permutation character of a general permutation representation of Q is then defined to be the sum of the characters of its orbits. By Corollary 9.4 of [14] and Proposition 5.2 above, the definition is consistent with the usual definition for groups. The following result illustrates the use of these general quasigroup permutation characters.

Theorem 6.1. *Let X be a permutation representation of a finite quasigroup Q. Then the number of orbits of X is given by the multiplicity of the principal character ψ_1 of Q in the permutation character of X.*

Proof: It suffices to show that ψ_1 occurs with multiplicity 1 in the character π of the image of a homogeneous space $P\backslash Q$ under a surjective intertwining with incidence matrix F. Using the notation of [3], one has

$$\begin{aligned} n^2\langle \pi, \psi_1\rangle &= \pi * \psi_1(\widehat{Q}) \\ &= \sum_{x\in Q}\sum_{y\in Q} \pi(x,y)\psi_1(y,x) \\ &= \sum_{x\in Q}\sum_{y\in Q} \pi(x,y) \\ &= n\sum_{i=1}^{s} \mathrm{Tr}(F^+ A_P^+ \mathbf{A}_i A_P F) \\ &= n\,\mathrm{Tr}(F^+ A_P^+ \mathbf{J} A_P F) \\ &= n^2, \end{aligned}$$

the latter equation following by Theorem 10.2 of [14]. Thus $\langle \pi, \phi_1\rangle = 1$, as required. □

7. An example

Consider the quasigroup $Q = (\mathbb{Z}_4, -)$ of integers modulo 4 under subtraction. (This is isomorphic to the opposite of the example presented in Section 4 of [3], and has the same character table as that example.) Its subquasigroups Q, $\{0,2\}$, $\{0\}$ and $\varnothing$ yield homogeneous spaces with $1, 2, 3$ and 4 elements respectively. The 1-, 2-, and 4-element spaces are quite analogous to the homogeneous spaces of the group $\mathbb{Z}_4$: in particular, the corresponding Markov matrices (2.1) are all permutation matrices. On the other hand, the 3-element homogeneous space $0\backslash Q$ exhibits stochasticity. The orbits of the relative left multiplication group of $\{0\}$ in Q are $\{0\}$, $\{1,3\}$ and $\{2\}$, yielding

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{7.1}$$

as the corresponding incidence matrix. The pseudoinverse of A_0 is the matrix

$$A_0^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{7.2}$$

From (2.1), one then obtains

$$R_{0\backslash Q}(1) = R_{0\backslash Q}(3) = \begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } R_{0\backslash Q}(2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{7.3}$$

while $R_{0\backslash Q}(0)$ is the 3×3 identity matrix.

Table 1 shows the character table of $(\mathbb{Z}_4, -)$ and the permutation characters of its homogeneous spaces. The latter are indexed by the cardinalities of the spaces. These permutation characters are also exhibited as linear combinations of the basic characters ψ_1, ψ_2, ψ_3. Note that the principal character ψ_1 occurs exactly once in each permutation character, as described by Theorem 5.1. From Table 1, it may be seen that the inequivalent permutation representations $\{0,2\}\backslash Q + \varnothing\backslash Q$ and $0\backslash Q + 0\backslash Q$ have the same permutation character $\pi_2 + \pi_4 = 2\pi_3$.

Table 1. Basic and permutation characters of $(\mathbb{Z}_4, -)$.

$(\mathbb{Z}_4, -)$	C_1	C_2	C_3
ψ_1	1	1	1
ψ_2	1	1	-1
ψ_3	$\sqrt{2}$	$-\sqrt{2}$	0
$\psi_1 = \pi_1$	1	1	1
$\psi_1 + \psi_2 = \pi_2$	2	2	0
$\psi_1 + \psi_2 + \psi_3/\sqrt{2} = \pi_3$	3	1	0
$\psi_1 + \psi_2 + \psi_3 \cdot \sqrt{2} = \pi_4$	4	0	0

Remark 7.1. Consider the conformal field theory describing the scaling limit of the Ising model at the critical point (cf. Example 5.2.12 of [2] or [9]). This theory has three physical representations $\rho_0, \rho_1, \rho_{1/2}$, with respective statistical dimensions $1, 1, \sqrt{2}$ (Example 11.3.22 of [2] or (1.57) of [9]). These statistical dimensions are the dimensions of the basic characters ψ_1, ψ_2, and ψ_3 of $(\mathbb{Z}_4, -)$. Now the centraliser ring of $(\mathbb{Z}_4, -)$ yields the fusion rules of the conformal field theory under the assignments $\rho_0 \mapsto \mathbf{A}_1$, $\rho_1 \mapsto \mathbf{A}_2$, $\rho_{1/2} \mapsto \mathbf{A}_3/\sqrt{2}$. It is then of interest to note that the Markov matrices $R_{0\backslash Q}(1)$ and $R_{0\backslash Q}(3)$ in the faithful permutation representation $0\backslash Q$ of $Q = (\mathbb{Z}_4, -)$ have exactly the stochasticity of the sites of the Ising model: a uniform two-way split between "spin up" and "spin down".

References

[1] Barnsley M.F., *Fractals Everywhere*, Academic Press, San Diego, CA, 1988.

[2] Chari V., Pressley A.N., *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.

[3] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups*, Eur. J. Comb. **5** (1984), 43–50.

[4] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups II: induced characters*, Eur. J. Comb. **7** (1986), 131–137.

[5] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups III: quotients and fusion*, Eur. J. Comb. **10** (1989), 47–56.

[6] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups IV: products and superschemes*, Eur. J. Comb. **10** (1989), 257–263.

[7] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups V: linear characters*, Eur. J. Comb. **10** (1989), 449–456.
[8] Johnson K.W., Smith J.D.H., Song S.Y., *Characters of finite quasigroups VI: critical examples and doubletons*, Eur. J. Comb. **11** (1990), 267–275.
[9] Mack G., Schomerus V., *Conformal field algebras with quantum symmetry from the theory of superselection sectors*, Comm. Math. Phys. **134** (1990), 139–196.
[10] Penrose P., *A generalised inverse for matrices*, Proc. Cambridge. Phil. Soc. **51** (1955), 406–413.
[11] Smith J.D.H., *Centraliser rings of multiplication groups on quasigroups*, Math. Proc. Cambridge Phil. Soc. **79** (1976), 427–431.
[12] Smith J.D.H., *Quasigroup actions: Markov chains, pseudoinverses, and linear representations*, Southeast Asia Bull. Math. **23** (1999), 719–729.
[13] Smith J.D.H., *Quasigroup homogeneous spaces and linear representations*, J. Algebra **241** (2001), 193–203.
[14] Smith J.D.H., *A coalgebraic approach to quasigroup permutation representations*, Algebra Universalis **48** (2002), 427–438.

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(*Received* September 22, 2003, *revised* January 30, 2004)