Infinite simple Bol loops

Hubert Kiechle, Michael K. Kinyon

Abstract. If the left multiplication group of a loop is simple, then the loop is simple. We use this observation to give examples of infinite simple Bol loops.

Keywords: Bol loop, K-loop, Bruck loop

Classification: 20N05

Introduction

A loop satisfying the identity $x(y \cdot xz) = (x \cdot yx)z$ is a (left) *Bol loop.* It is generally agreed that the most significant open problem in loop theory today is the existence or nonexistence of finite, simple Bol loops which are not Moufang. In this note, we give examples of infinite simple Bol loops. The construction of the loops is well-known, and all we actually need to do is to compute the left multiplication group, which turns out to be simple. It follows from a lemma in §1 that the loop is simple, as well.

In §2 we recall the construction based on $\mathrm{SL}(n, F)$ over suitable fields F to obtain examples. In §3 we use a result of Rózga [7] on Lie groups of noncompact type to obtain even more examples.

The example from $\mathrm{SL}(2, \mathbb{R})$ has been treated in detail in [6, §22]. In particular, it is shown that the topological left multiplication group is $\mathrm{PSL}(2, \mathbb{R})$, a simple group. Our results imply that $\mathrm{PSL}(2, \mathbb{R})$ is also the (combinatorial) left multiplication group, which is what we need for our conclusion.

It turns out that all of the examples we provide satisfy the automorphic inverse property $(xy)^{-1} = x^{-1}y^{-1}$. Bol loops satisfying this property are known by various names: K-loops [5], Bruck loops [1], or gyrocommutative gyrogroups [3]. We do not know of any simple, proper Bol loops not satisfying the automorphic inverse property.

In the finite case, it is known that a simple K-loop, if such a loop exists, must consist only of elements of order a power of 2 [AKP]. By contrast, the infinite simple K-loops described here are uniquely 2-divisible, that is, the squaring map is a permutation.

Our notation conventions follow [5]. Mappings act on the left of their arguments. For a loop L, we denote left translations by $\lambda_x y := xy$. The left multiplication group is $\mathcal{M}_\ell(L) := \langle \lambda_x : x \in L \rangle$, while the left inner mapping group is

$\mathcal{D}(L) := \mathcal{M}_\ell(L)_1 = \{\alpha \in \mathcal{M}_\ell : \alpha(1) = 1\}$. $\mathcal{D}(L)$ is generated by all mappings $\delta_{x,y} := \lambda_{xy}^{-1}\lambda_x\lambda_y$, $x, y \in L$. The quasidirect product of L and $\mathcal{D}(L)$, denoted by $L \times_Q \mathcal{D}(L)$, is the set $L \times \mathcal{D}(L)$ with the product $(x,\alpha)(y,\beta) := (x\cdot\alpha(y), \delta_{x,\alpha(y)}\alpha\beta)$. $L \times_Q \mathcal{D}(L)$ is just an isomorphic copy of $\mathcal{M}_\ell$, but is more convenient for some purposes.

1. Left multiplication group

Let L be a loop. A subloop N of L is called *normal* if

$$ab \cdot N = a \cdot bN = aN \cdot b \text{ for all } a, b \in L.$$

Then there is a *factor loop* L/N and a canonical epimorphism $L \to L/N$ with kernel N. Conversely, the kernel of any homomorphism is a normal subloop, see [2, IV]. For each subloop N in L, define

$$\mathcal{L}(N) := \{\alpha \in \mathcal{M}_\ell : \alpha(x) \in Nx, \forall x \in L\}.$$

As in [2, IV.1] one shows

Lemma 1.1. *Let L, L' be loops and $\theta : L \to L'$ a homomorphism. Then*

(1) *the map $\theta_* : \mathcal{M}_\ell(L) \to \mathcal{M}_\ell(\theta(L))$, defined by $\theta_*(\alpha)(\theta(x)) = \theta\alpha(x)$ for all $x \in L$ is an epimorphism with kernel $\mathcal{L}(\ker\theta)$;*

(2) *if N is a normal subloop in L, then $\mathcal{L}(N)$ is a normal subgroup in $\mathcal{M}_\ell(L)$ and $\mathcal{M}_\ell(L/N) = \mathcal{M}_\ell(L)/\mathcal{L}(N)$. Moreover, N is a proper subloop of L if and only if $\mathcal{L}(N)$ is a proper subgroup of $\mathcal{M}_\ell(L)$.*

Indeed, one computes $\theta_*(\lambda_a) = \lambda_{\theta(a)}$, and the rest is straightforward. See also [6, 1.7].

A loop is called *simple* if every normal subloop is trivial. As an immediate corollary we get

Theorem 1.2. *If $\mathcal{M}_\ell$ is simple, then L is simple.*

2. Examples

We will now use Theorem 1.2 to show the simplicity of certain K-loops.

Let R be an ordered field, and let $K := R(i)$, where $i^2 = -1$. Then K has a unique nonidentity automorphism $z \mapsto \bar{z}$, which fixes R elementwise and maps i to $-i$. This will also be applied componentwise to matrices over K. Let $n \geq 2$ be a fixed integer and consider the set L of positive definite symmetric, resp. hermitian $n \times n$-matrices over R, resp., K. We assume that R is *n-real*, i.e., the characteristic polynomial of every matrix in L splits over K (in fact over R) into linear factors.

Remark 2.1. An ordered field is 2-real if and only if it is pythagorean. Therefore, every n-real field is pythagorean. Real closed fields are n-real for every n. See [5] for more on n-real fields.

Furthermore, let $G = \mathrm{SL}(n, F)$, and $\Omega = \mathrm{SO}(n, R)$, or $\Omega = \mathrm{SU}(n, K)$, where $F = R$, K, respectively. Finally, put $L_G := L \cap G$.

Then Ω is a subgroup of G and L_G is a transversal containing I_n, the $n \times n$ identity matrix. In particular, $G = L_G\Omega$. Then for all $A, B \in L_G$ there exist unique $A \circ B \in L_G$, $d_{A,B} \in \Omega$ such that $AB = (A \circ B)d_{A,B}$.

By [5, (9.2)] we have

Theorem 2.2. $(L_G, \circ)$ *is a K-loop, and therefore a Bol loop.*

Our aim is to show

Theorem 2.3. $(L_G, \circ)$ *is simple.*

Proof: Indeed, we will show that $\mathcal{M}_\ell = \mathrm{PSL}(n, F)$, which is a simple group. The desired result will then follow from Theorem 1.2.

Consider the map

$$\Phi : \begin{cases} G = L_G\Omega & \to \ L_G \times_Q \mathcal{D} \\ g = A\omega & \mapsto \ (A, \widehat{\omega}) \end{cases} \quad (A \in L_G,\ \omega \in \Omega),$$

where $\widehat{\omega}(A) = \omega A \omega^{-1}$ for all $A \in L_G$. Recall that $\widehat{\omega}$ acts on L_G, that is, $\widehat{\omega}(A) \in L_G$ for all $A \in L_G$. By [5, (9.3.1)] we have $\mathcal{D} = \{\widehat{\omega};\ \omega \in \Omega\}$, so Φ is well-defined and surjective.

Let $g = A\omega, g' = B\omega' \in G$ be decomposed as in the definition of Φ. Then

$$gg' = A\omega B\omega' = A\widehat{\omega}(B)\omega\omega' = (A \circ \widehat{\omega}(B))d_{A,\widehat{\omega}(B)}\omega\omega'.$$

Since $\widehat{d_{A,\widehat{\omega}(B)}} = \delta_{A,\widehat{\omega}(B)}$, we have

$$\Phi(gg') = \left(A \circ \widehat{\omega}(B), \delta_{A,\widehat{\omega}(B)}\widehat{\omega}\widehat{\omega}'\right) = (A, \widehat{\omega})(B, \widehat{\omega}') = \Phi(g)\Phi(g').$$

Thus Φ is an epimorphism. Then $N := \ker(\Phi) = \{\omega \in \Omega : \widehat{\omega}(A) = A,\ \forall A \in L_G\}$, that is, N is the centralizer of L_G inside Ω. By [5, (1.22)] N is the center of $\mathrm{SL}(n, F)$. Therefore,

$$\mathcal{M}_\ell = L_G \times_Q \mathcal{D} = G/N = \mathrm{PSL}(n, F)$$

is a simple group (see, e.g., [8, 10.8.4] or [4, II.6.13]). □

3. Examples from simple Lie groups

Let G be a noncompact Lie group which is simple as an abstract group. Then G has a simple Lie algebra. By [7, §6], there exists a subgroup $\Omega < G$ and a subset $L \subset G$ such that L is a transversal of G/Ω and L is a K-loop. Indeed, L and Ω come from the Cartan decomposition of the Lie algebra of G.

Theorem 3.1. *L is a simple K-loop.*

Proof: By [7, Proposition 24], L generates G. Then by [5, (2.8)], $\mathcal{M}_\ell(L)$ is a homomorphic image of G. Since G is simple, we have $\mathcal{M}_\ell(L) = G$. An application of Theorem 1.2 gives the result. □

This applies to the groups $\mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSL}(n,\mathbb{C})$, $\mathrm{PSL}(n,\mathbb{H})$ as well as $\mathrm{PSp}(2n,\mathbb{R})$, $\mathrm{PSp}(2n,\mathbb{C})$. Simplicity for these groups is proved for instance in [4, II.6.13, II.9.22].

References

[1] Aschbacher M., Kinyon M.K., Phillips J.D., *Finite Bruck loops*, submitted. Available at `http://www.arXiv.org/abs/math.GR/0401193`.

[2] Bruck R.H., *A Survey of Binary Systems*, Springer-Verlag, Berlin-Heidelberg-New York, 1971; MR 20#76, Zbl. 206:30301.

[3] Foguel T., Ungar A.A., *Gyrogroups and the decomposition of groups into twisted subgroups and subgroups*, Pacific J. Math. **197** (2001), 1–11; MR 2002e:20142, Zbl. pre01589578.

[4] Huppert B., *Endliche Gruppen* **I**, Springer-Verlag, Berlin-Heidelberg-New York, 1967; MR 37 #302, Zbl. 0217.07201.

[5] Kiechle H., *Theory of K-loops*, Lecture Notes in Math. **1778**, Springer-Verlag, Berlin-Heidelberg-New York, 2002; MR 2003d:20109, Zbl. 0997.20059.

[6] Nagy P.T., Strambach K., *Loops in Group Theory and Lie Theory*, de Gruyter Expositions in Mathematics **35**, Walter de Gruyter, Berlin-New York, 2003; MR 2003d:20110, Zbl. pre01732502.

[7] Rózga K., *On central extensions of gyrocommutative gyrogroups*, Pacific J. Math. **193** (2000), 201–218; MR 2001a:20115, Zbl. 1010.20055.

[8] Scott W.R., *Group Theory*, Dover, New York, 1987; MR 88d:20001, Zbl. 0641.20001.

Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55,
D-20146 Hamburg, Germany

E-mail: kiechle@math.uni-hamburg.de
`http://www.math.uni-hamburg.de/home/kiechle/kiechle.html`

Department of Mathematical Sciences, Indiana University South Bend,
South Bend, IN 46634, USA

E-mail: mkinyon@iusb.edu
`http://mypage.iusb.edu/~mkinyon`

(*Received* February 17, 2004, *revised* March 11, 2004)