Kikkawa loops and homogeneous loops

Michihiko Kikkawa

Abstract. In H. Kiechle's publication "Theory of K-loops" [3], the name Kikkawa loops is given to symmetric loops introduced by the author in 1973. This concept started from an analogical imagination of sum of vectors in Euclidean space brought up on a sphere. In 1975, this concept was extended by him to the more general concept of homogeneous loops, and it led us to a non-associative generalization of the theory of Lie groups.

In this article, the backstage of finding these concepts will be disclosed from the viewpoint how a new mathematical concept appears and grows up in imagination of a mathematician.

Keywords: loops, Lie triple algebras, symmetric spaces

Classification: 20N05, 05B07, 53C35

1. Geodesic local loops

It was a primitive question that occurred in the author's mind: How the concept of sum $x+y$ of vectors x and y in a Euclidean plane could be generalized into more general spaces which are not Euclidean?

To do this, some concepts corresponding to "parallelism" and "lines" should be valid for the general spaces. So, he considered that the space should have some concepts of "parallelism of tangent vectors" and "geodesics", that is, it should be a manifold with a linear connection ∇.

Let G be a linearly connected manifold. In some neighborhood of a fixed point e, let $c(t)$ be a geodesic curve starting from $e = c(0)$ to a certain point x.

For any point y joined by a geodesic curve from e tangent to a vector Y_e at e, by parallel displacement of Y_e to x along $c(t)$, we can define a local product $xy = \mu(x, y)$ of two points x and y in the following way [4]:

$$\mu(x, y) = \operatorname{Exp}_x \circ \tau_c(e, x) \circ \operatorname{Exp}_e^{-1} y,$$

where $\tau_c(e, x)$ denotes the parallel displacement of vectors from e to x along the curve $c(t)$.

The product $xy = \mu(x, y)$ of x and y is not always commutative nor associative. Nevertheless, it forms a local loop with the unit e, that is;

Presented at the conference LOOPS '03 Prague, on August 11, 2003.

$\mu(e,x) = \mu(x,e) = x$ for any x in a neighborhood of e.

Later, it used to be called a *geodesic local loop* or a *geodesic loop* on linearly connected manifolds.

It is easy to see that Lie groups with the $(-)$-connection of E. Cartan are associative examples of geodesic local loops, in which any one-parameter subgroup is a geodesic curve.

This concept of local loops appeared in his mind in 1963, and he told it to his supervisor, the late Professor K. Morinaga. Then, Professor Morinaga suggested him that non-associativity of the loop must be related to the curvature of the linear connection ∇ of the manifold.

Now, denote L_x the left translation by x, that is $L_x y = xy$. Then, each left translation is a local diffeomorphism.

Non-associativity of the local loop $\mu(x,y)$ can be checked by the difference between

$$x(yz) = L_x L_y z \quad \text{and} \quad (xy)z = L_{xy} z,$$

that is, the difference of $L_x L_y$ from L_{xy}.

We call a local diffeomorphism $L_{x,y} = L_{xy}^{-1} L_x L_y$ around the unit e a *left inner mapping*.

To certify the expectation of his supervisor, he tried several calculations on geodesic loops and found the following [4]:

Theorem 1.1. *The left inner mapping group generated by left inner mappings $\{L_{x,y}\}$ is coincident with the local holonomy group at e of the linear connection ∇.*

2. Geodesic loops on symmetric spaces

At that stage the author wanted to get:

(1) some significant examples of geodesic loops besides of Lie groups;
(2) axiomatic algebraic systems induced by some geodesic loops on some class of spaces.

On an autumn day in 1971, he had handed a book titled "Symmetric Spaces I" by O. Loos [15], in which he found an axiomatic definition of Symmetric Spaces.

The definition was presented by means of *reflections* of symmetric spaces [15]. A point brought by a reflection of a point y across another point x is denoted by $x * y$ in [15]. Here we denote $S_x y$ instead of $x * y$. Then Loos's definition of symmetric spaces is given by the following:

Definition 2.1. A differentiable manifold G is said to be a *symmetric space* if, for each $x \in G$, there is a diffeomorphism $S_x : G \to G$, called *the reflection at x*, such that

(R.1) $S_x x = x$,
(R.2) $S_x(S_x y) = y$,
(R.3) $S_x(S_y z) = S_{(S_x y)}(S_x z)$, and

(R.4) for each x, $S_x y = y$ implies $y = x$ in some neighborhood of x.

In this definition, assumption (R.3) means that all reflections are automorphisms of the reflection system $\{S_x | x \in G\}$ on G.

Indeed, the reflection S_x at x induces the reflection along geodesic curves across x, with respect to the canonical connection ∇ of the symmetric space.

The author recognized at once that the multiplication of the geodesic loop in any symmetric space might be expressed in terms of the reflections.

Definition 2.2. Let G be a symmetric space [5], [6]. In a neighborhood U of e, assume that there exists $\sqrt{x} \in U$ for $x \in U$ such that

$$S_{\sqrt{x}} x = e.$$

Then, for $y \in U$, the *multiplication* $\mu(x, y)$ of x and y can be defined as

$$\mu(x, y) := S_{\sqrt{x}} S_e y = S_{S_{\sqrt{x}} e} S_{\sqrt{x}} y.$$

The element $\sqrt{x}$ is the middle point of the geodesic curve joining e with x, hence it exists uniquely in a neighborhood U of e.

The local binary operation (U, μ) forms a local loop, and has some relations with the reflection, i.e.:

The left translation L_x by x is given by

$$L_x = S_{\sqrt{x}} S_e.$$

The inverse element is given by

$$x^{-1} = S_e x.$$

For the inverse element x^{-1}, we have

$$\sqrt{x^{-1}} = S_e \sqrt{x}.$$

By Loos's theory of symmetric space, we can show that the multiplication μ above is coincident with the geodesic local loop at e with respect to the canonical connection ∇ of the symmetric space.

3. Symmetric loops (Kikkawa loops)

By checking up geodesic local loops μ on symmetric spaces above, we can find the following algebraic properties [5], [6], [9]:

Theorem 3.1. *At any fixed point e in a symmetric space G, the geodesic local loop $(U,\mu), xy=\mu(x,y)$ on some neighborhood U of e, satisfies the following relations:*
(Left inverse property) *The inverse of a left translation L_x is a left translation $L_{x^{-1}}$, i.e.*

(L.I.) $\qquad L_x^{-1}=L_{x^{-1}}.$

(Automorphic inverse property) *The inversion $J=S_e: x\longmapsto x^{-1}$ is an automorphism of μ, i.e.*

(A.I.) $\qquad (xy)^{-1}=x^{-1}y^{-1}.$

(Left homogeneity property) *Any left inner mapping $L_{x,y}$ is an automorphism of μ;*

(L.H.) $\qquad L_{x,y}\mu(u,v)=\mu(L_{x,y}u,L_{x,y}v).$

Now we are in position to introduce an algebraic system defined by these axioms.

Definition 3.1. An abstract algebraic binary system (G,μ) with a unit element e is said to be a *symmetric loop* if it has *the left inverse property* (L.I.), *the automorphic inverse property* (A.I.) and *the left homogeneity property* (L.H.) ([7], [9]).

Remark 1. Recently, Prof. Hubert Kiechle published a book "Theory of K-Loops" [3], in which he called the symmetric loops by the name of *Kikkawa loops*, and set his school's *K-Loops* as an special kind of Kikkawa loops.

Example 1. Let $\mathbf{S_n}$ (resp. $\mathbf{H_n}$) be the space of all real symmetric (resp. Hermitian) matrices of order n. For any $X,Y\in\mathbf{S_n}$ (resp. $\mathbf{H_n}$), set the multiplication μ by:

$$\mu(X,Y):=\sqrt{X}\,Y\sqrt{X}.$$

Then μ forms a symmetric loop with the unit I_n.

Example 2. K-loops are Bol loops which satisfy the automorphic inverse property. By Theorem (6.7) in Kiechle's Book [3], any K-loop is a symmetric loop.

4. Homogeneous loops

Symmetric spaces are spaces with structures invariant under any reflection, so that any reflection is an automorphism of the space.

In this sense, the automorphic inverse property (A.I.) characterizes the "symmetry" in the definition of symmetric loops.

In fact, the inversion J is an reflection across the unit e, and the automorphic inverse property asserts that the inversion is an automorphism of the multiplication μ.

On the other hand, the left inverse property (L.I.) is an extensive property valid for all geodesic loops.

Now we consider the meaning of the left homogeneity property. This can be understood to be the property characterizing the "homogeneity" of the spaces, because it asserts that the two kind of translations L_{xy} and $L_x L_y$ differ by an automorphism of the multiplication μ wherever x, y in G.

Thus, in 1974 the author had arrived at the concept of "homogeneous loops" which are not always symmetric, that is:

Definition 4.1. A loop (G, μ) with the left inverse property (L.I.) and the left homogeneity property (L.H.) is called a *homogeneous loop* [7].

Example 3. Any group (G, μ) is a homogeneous loop with the trivial left inner mapping group, because any left inner mapping $L_{x,y}$ is the identity map on G.

Example 4. Any symmetric loop is a homogeneous loop. Any group (G, μ) which forms also a symmetric loop must be an Abelian group.

5. Homogeneous left Lie loops and tangent Lie triple algebras

Since the definition of homogeneous loops depends only on their left translations, to develop the theory, the underlying algebraic system should not only be loops but also *left loops*, that is to say, the right translations need not be bijective.

Definition 5.1. A *homogeneous left loop* is a *left* loop (G, μ) with the left inverse property (L.I.) and the left homogeneity property (L.H.).

Let (G, μ) be a homogeneous left loop whose underlying space G is a differentiable manifold and the multiplication μ is differentiable. Then we call (G, μ) a *homogeneous left Lie loop*.

The main results on homogeneous left Lie loops are the following [7], [14]:

Theorem 5.1. *Let (G, μ) be a connected homogeneous left Lie loop and K the left inner mapping group. Then, for the product space $A = G \times K$, G can be considered as a reductive homogeneous space A/K with the canonical connection ∇* (cf. [16])*, so that the torsion tensor S and the curvature tensor R satisfy $\nabla S = 0$ and $\nabla R = 0$.*

In this case, the tangent space $\mathfrak{G}$ at the unit e forms a Lie triple algebra under the two kinds of products;

$$[X, Y] := S_e(X, Y) \quad \textit{and}$$
$$[X, Y, Z] := R_e(X, Y)Z \quad \textit{for} \quad X, Y, Z \in \mathfrak{G}.$$

Example 5. Any connected Lie group (G, μ) is a typical example of a homogeneous left Lie loops. The canonical connection ∇ is the $(-)$-connection of Cartan so that $R = 0$ and the Lie triple algebra is reduced to the Lie algebra $(\mathfrak{G}, [X, Y])$.

Example 6. Let $(G, \mu), xy = \mu(x, y)$ be a Lie group. Consider a class of multiplications $\{\mu_p\}$ on G given by

$$\mu_p(x, y) := x^{p+1} y x^{-p}, \quad \text{for any} \quad x, y \in G.$$

Then each μ_p forms a homogeneous left Lie loop on G. We call them *Akivis left loops* on the Lie group G. This multiplication μ_p was given by the author in [11].

If $p = 0$ the multiplication μ_0 is reduced to the original Lie group multiplication μ.

The tangent Lie triple algebra $\mathfrak{G}_p$ of an Akivis left loop μ_p is given by

$$[X, Y]_p = (1 + 2p)[X, Y],$$
$$[X, Y, Z]_p = -p(1 + p)[[X, Y]Z],$$

where $[X, Y]$ is the bracket of the Lie algebra $\mathfrak{G}$ of the Lie group G (cf. [1], [12], [13], [18]).

Definition 5.2. A homogeneous left Lie loop (G, μ) is said to be *geodesic* if the multiplication μ coincides with the multiplication of the geodesic loop at the unit e, with respect to the canonical connection ∇.

Remark 2. Any Lie group is a geodesic homogeneous Lie loop, and any symmetric Lie loop is geodesic (cf. [7]).

6. Non-associative generalization of the theory of Lie groups

Non-associative generalization of the well-known theory of Lie groups and Lie algebras has been established consistently by the author. By means of the concept of *homogeneous systems* (the definition is not mentioned here, cf. [10]), the theory has been developed including the theory of subloops and subalgebras of the tangent algebras, as the *theory of geodesic homogeneous left Lie loops* [7].

Let (G, μ) be a homogeneous left Lie loop which is assumed to be geodesic, that is, the multiplication μ coincides with the geodesic local loop of the canonical connection, in some neighbourhood of the unit e.

For instance, the following results have been shown by the author ([7], [8], [14]):

Theorem 6.1. *Any homomorphism of geodesic homogeneous left Lie loops induces a homomorphism of their tangent Lie triple algebras.*

Two geodesic homogeneous left Lie loops are locally isomorphic if and only if their tangent Lie triple algebras are isomorphic.

Moreover, if the geodesic homogeneous left Lie loops are analytic and the underlying manifolds are connected and simply connected, then they are isomorphic if and only if their tangent Lie triple algebras are isomorphic.

Theorem 6.2. *Let H be an invariant left Lie subloop of a geodesic homogeneous left Lie loop G. Then, its tangent Lie triple algebra $\mathfrak{H}$ is an invariant Lie triple subsystem of the tangent Lie triple algebra $\mathfrak{G}$ of G.*

Conversely, any invariant subsystem $\mathfrak{H}$ of $\mathfrak{G}$ is the tangent Lie triple algebra of an invariant left Lie subloop H of G.

References

[1] Akivis M., *Local algebras of a multidimensional web* (in Russian), Sibir. Mat. Z. **17** (1976), 5–11.

[2] Chein O., Pflugfelder H.O., Smith J.D.H., *Quasigroups and Loops: Theory and Applications*, Heldermann Verlag, 1990.

[3] Kiechle H., *Theory of K-Loops*, Lecture Note in Mathematics 1778, Springer, 2002.

[4] Kikkawa M., *On local loops in affine manifolds*, J. Sci. Hiroshima Univ. **A-I:28** (1964), 199–207.

[5] Kikkawa M., *On some quasigroups of algebraic models of symmetric spaces*, Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **6** (1973), 9–13.

[6] Kikkawa M., *On some quasigroups of algebraic models of symmetric spaces II*, Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **7** (1974), 29–35.

[7] Kikkawa M., *Geometry of homogeneous Lie loops*, Hiroshima Math. J. **5** (1975), 141–179.

[8] Kikkawa M., *A note on subloops of a homogeneous Lie loop and subsystems of its Lie triple algebra*, Hiroshima Math. J. **5** (1975), 439–446.

[9] Kikkawa M., *On some quasigroups of algebraic models of symmetric spaces III*, Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **9** (1975), 7–12.

[10] Kikkawa M., *On the left translations of homogeneous loops*, Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **10** (1976), 19–25.

[11] Kikkawa M., *Projectivity of homogeneous left loops on Lie groups I*, Mem. Fac. Sci. Shimane Univ. **23** (1989), 17–22.

[12] Kikkawa M., *Projectivity of homogeneous left loops*, Proc. International Symposium on Non-associative Algebras and Related Topics, World Scientific, 1991, pp. 77–99.

[13] Kikkawa M., *Remarks on Akivis left loops*, Mem. Fac. Sci. Shimane Univ. **29** (1995), 1–9.

[14] Kikkawa M., *Geometry of homogeneous left Lie loops and tangent Lie triple algebras*, Mem. Fac. Sci. Engr. Shimane Univ. **32-B** (1999), 69–74.

[15] Loos O., *Symmetric Spaces I*, Benjamin, 1969.

[16] Nomizu K., *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954), 33–65.

[17] Sabinin L.V., *Geometry of loops* (in Russian), Mat. Zametki **12** (1972), 605–616.

[18] Sanami M., Kikkawa M., *A class of double Lie algebras on simple Lie algebras and projectivity of simple Li groups*, Mem. Fac. Sci. Shimane Univ. **25** (1991), 39–44.

current address: 366-29 Makata, Matsue City, 690-0024 Japan

E-mail: kicker@mx.miracle.ne.jp

(*Received* October 30, 2004)