Axioms for trimedial quasigroups

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Abstract. We give new equations that axiomatize the variety of trimedial quasigroups. We also improve a standard characterization by showing that right semimedial, left F-quasigroups are trimedial.

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1. Introduction

A quasigroup $\mathcal{Q} = (Q;\ \cdot, \backslash, /)$ is a set Q with three binary operations $\cdot, \backslash, / : Q \times Q \to Q$ satisfying the equations:

$$x\backslash(x \cdot y) = y \qquad (x \cdot y)/y = x$$
$$x \cdot (x\backslash y) = y \qquad (x/y) \cdot y = x.$$

Basic references for quasigroup theory are [1], [5], [6], [14].

A quasigroup is *medial* if it satisfies the identity

$$(M) \qquad xy \cdot uv = xu \cdot yv.$$

A quasigroup is *trimedial* if every subquasigroup generated by three elements is medial. Medial quasigroups have also been called abelian, entropic, and other names, while trimedial quasigroups have also been called triabelian, terentropic, etc. (See Chapter IV of [6], especially p. 120, for further background.) The classic Toyoda-Bruck theorem asserts that every medial quasigroup is isotopic to an abelian group [15], [4]. This result was generalized by Kepka to trimedial quasigroups: every trimedial quasigroup is isotopic to a commutative Moufang loop [7].

There are two distinct, but related, generalizations of trimedial quasigroups. The variety of *semimedial* quasigroups (also known as weakly abelian, weakly medial, etc.) is defined by the equations

$$(S_l) \qquad xx \cdot yz = xy \cdot xz$$
$$(S_r) \qquad zy \cdot xx = zx \cdot yx.$$

A quasigroup satisfying (S_l) (resp. (S_r)) is said to be *left* (resp. *right*) *semimedial*. Every semimedial quasigroup is isotopic to a commutative Moufang loop [7]. (In the trimedial case, the isotopy has a more restrictive form; see the cited references for details.)

The variety of *F-quasigroups* was introduced by Murdoch in [13], the same paper in which he introduced what we now call medial quasigroups. F-quasigroups are defined by the equations

$$(F_l) \qquad x \cdot yz = xy \cdot (x\backslash x)z$$

$$(F_r) \qquad zy \cdot x = z(x/x) \cdot yx.$$

A quasigroup satisfying (F_l) (resp. (F_r)) is said to be a *left* (resp. *right*) *F-quasigroup*. Murdoch did not actually name the variety of F-quasigroups. We thank one of the referees for suggesting that the earliest use of the name might be in [2].

One among many links between these two generalizations of trimedial quasigroups is the following ([9, Proposition 6.2]).

Proposition 1.1. *A quasigroup is trimedial if and only if it is a semimedial, left* (*or right*) *F-quasigroup.*

Together with Kepka, we have been investigating the structure of F-quasigroups, and have shown that every loop isotopic to an F-quasigroup is Moufang. Full details will appear elsewhere [10]. As part of that investigation, we were led to consider the following equations, which are similar in form to (F_l), (F_r):

$$(E_l) \qquad x \cdot yz = (x/x)y \cdot xz$$

$$(E_r) \qquad zy \cdot x = zx \cdot y(x\backslash x).$$

The main result of the present paper is the following.

Theorem 1.2. *A quasigroup is trimedial if and only if it satisfies* (E_l) *and* (E_r).

Kepka [7], [8] showed that the variety of trimedial quasigroups is axiomatized by the semimedial laws (S_l), (S_r), and by the equation $(x \cdot xx) \cdot uv = xu \cdot (xx \cdot v)$. Later [11] we showed that (S_l) is redundant. Theorem 1.2 offers a more symmetric alternative.

As an auxiliary result, we will also use (E_l) and (E_r) to obtain the following improvement of Proposition 1.1.

Theorem 1.3. *Let Q be a quasigroup. The following are equivalent:*

1. *Q is trimedial;*
2. *Q is a right semimedial, left F-quasigroup;*
3. *Q is a left semimedial, right F-quasigroup.*

Our investigations were aided by the equational reasoning tool OTTER developed by McCune [12]. We thank T. Kepka for suggesting that (E_l), (E_r) might axiomatize an interesting variety of quasigroups; he was certainly correct.

2. Proofs

Our strategy for proving Theorem 1.2 is to use Proposition 1.1: we will show that a quasigroup satisfying (E_l), (E_r) is a semimedial, F-quasigroup. First we introduce some notation for local right and left unit elements in a quasigroup:

$$e(x) := x\backslash x \qquad f(x) := x/x.$$

If $\mathcal{Q} = (Q; \cdot, \backslash, /)$ is a quasigroup, then so are the *left parastrophe* $\mathcal{Q}_l := (Q; \backslash, \cdot, /_{\mathrm{op}})$, the *right parastrophe* $\mathcal{Q}_r := (Q; /, \cdot, \backslash_{\mathrm{op}})$, and the *opposite parastrophe* $\mathcal{Q}_{\mathrm{op}} := (Q; \cdot_{\mathrm{op}}, /, \backslash)$, where for a binary operation $*$, we use $*_{\mathrm{op}}$ to denote the opposite operation. Note that $(\mathcal{Q}_l)_l = \mathcal{Q}$, $(\mathcal{Q}_r)_r = \mathcal{Q}$, $(\mathcal{Q}_{\mathrm{op}})_{\mathrm{op}} = \mathcal{Q}$, $(\mathcal{Q}_{\mathrm{op}})_l = (\mathcal{Q}_r)_{\mathrm{op}}$, and $(\mathcal{Q}_{\mathrm{op}})_r = (\mathcal{Q}_l)_{\mathrm{op}}$. For a more complete discussion of parastrophes, including alternative notation conventions, see [1], [14].

We state the following obvious result formally for later ease of reference.

Lemma 2.1. *Let $\mathcal{Q} = (Q; \cdot, \backslash, /)$ be a quasigroup.*

1. *$\mathcal{Q}$ satisfies (F_l) if and only if $\mathcal{Q}_{\mathrm{op}}$ satisfies (F_r).*
2. *$\mathcal{Q}$ satisfies (S_l) if and only if $\mathcal{Q}_{\mathrm{op}}$ satisfies (S_r).*
3. *$\mathcal{Q}$ satisfies (E_l) if and only if $\mathcal{Q}_{\mathrm{op}}$ satisfies (E_r).*

Parts (1) and (2) of the following lemma are well-known, although the authors have not been able to find a specific reference.

Lemma 2.2. *Let $\mathcal{Q} = (Q; \cdot, \backslash, /)$ be a quasigroup.*

1. *$\mathcal{Q}$ is a left F-quasigroup if and only if $\mathcal{Q}_l$ is left semimedial.*
2. *$\mathcal{Q}$ is a right F-quasigroup if and only if $\mathcal{Q}_r$ is right semimedial.*
3. *$\mathcal{Q}$ satisfies (E_l) if and only if $\mathcal{Q}_l$ satisfies (E_l).*
4. *$\mathcal{Q}$ satisfies (E_r) if and only if $\mathcal{Q}_r$ satisfies (E_r).*

PROOF: For (1): In $\mathcal{Q}_l$, (S_l) is $e(x)\backslash(y\backslash z) = (x\backslash y)\backslash(x\backslash z)$. Multiply on the left by $e(x)$, replace y with xy, and z with xz to get $(xy)\backslash(xz) = e(x)(y\backslash z)$. Now replace z with yz and multiply on the left by xy to get $x \cdot yz = xy \cdot e(x)z$, which is (F_l) in $\mathcal{Q}$. Since $(\mathcal{Q}_l)_l = \mathcal{Q}$, the converse also holds.

For (2): By part (1), (F_l) holds in $\mathcal{Q}_{\mathrm{op}}$ iff (S_l) holds in $(\mathcal{Q}_{\mathrm{op}})_l = (\mathcal{Q}_l)_{\mathrm{op}}$. The desired result now follows from Lemma 2.1.

For (3): In $\mathcal{Q}_l$, (E_l) is $((x/_{\mathrm{op}}x)\backslash y)\backslash(x\backslash z) = x\backslash(y\backslash z)$. Multiply on the left by $f(x)\backslash y$, and replace z with yz to get $x\backslash(yz) = (f(x)\backslash y)(x\backslash z)$. Multiply on the left by x, replace y with $f(x)y$, and z with xz to get $f(x)y \cdot xz = x \cdot yz$, which is (E_l) in $\mathcal{Q}$. Since $(\mathcal{Q}_l)_l = \mathcal{Q}$, the converse also holds.

For (4): By part (3), (E_l) holds in $\mathcal{Q}_{\mathrm{op}}$ iff (E_l) holds in $(\mathcal{Q}_{\mathrm{op}})_l = (\mathcal{Q}_r)_{\mathrm{op}}$. The desired result now follows from Lemma 2.1. $\square$

Lemma 2.3. *Let $\mathcal{Q}$ be a quasigroup.*

1. *If $\mathcal{Q}$ satisfies (E_l), then $f : Q \to Q$ is an endomorphism of $\mathcal{Q}$.*
2. *If $\mathcal{Q}$ satisfies (E_r), then $e : Q \to Q$ is an endomorphism of $\mathcal{Q}$.*
3. *If $\mathcal{Q}$ is a left F-quasigroup, then $e : Q \to Q$ is an endomorphism of $\mathcal{Q}$.*
4. *If $\mathcal{Q}$ is a right F-quasigroup, then $f : Q \to Q$ is an endomorphism of $\mathcal{Q}$.*

Proof: In each case, it is enough to show that the multiplication is preserved.

For (1): $f(x)f(y) \cdot xy = f(x)x \cdot f(y)y = xy$, and so $f(x)f(y) = (xy)/(xy) = f(xy)$.

For (2): Since $f : Q \to Q$ is an endomorphism of $\mathcal{Q}$ iff $e : Q \to Q$ is an endomorphism of $\mathcal{Q}_{\mathrm{op}}$, this follows from part (1) and Lemma 2.1(3).

For (3): if (F_l) holds, then $xy \cdot e(x)e(y) = x \cdot ye(y) = xy$, and so $e(x)e(y) = (xy)\backslash(xy) = e(xy)$ ([2, p. 38, equation (32)], [9, Lemma 4.2], [3]).

For (4): This follows from part (3) and Lemma 2.1(1). □

Lemma 2.4. *Let $\mathcal{Q}$ be a quasigroup. If $e : Q \to Q$ or $f : Q \to Q$ is an endomorphism of $\mathcal{Q}$, then $f(e(x)) = e(f(x))$ for all $x \in Q$.*

Proof: If f is an endomorphism, then $f(e(x)) = f(x)\backslash f(x) = e(f(x))$, and the case where e is an endomorphism is similar. □

Lemma 2.5. *Let $\mathcal{Q}$ be a quasigroup.*

1. *If $\mathcal{Q}$ satisfies (E_l), then $\mathcal{Q}$ is a left F-quasigroup if and only if it is left semimedial.*
2. *If $\mathcal{Q}$ satisfies (E_r), then $\mathcal{Q}$ is a right F-quasigroup if and only if it is right semimedial.*

Proof: For (1): Assume $\mathcal{Q}$ satisfies (E_l). By Lemma 2.2(3), (E_l) holds in $\mathcal{Q}_l$. We will prove the implication $(S_l) \implies (F_l)$. Since this will also hold in $\mathcal{Q}_l$, it will follow from Lemma 2.2(1) that the implication $(F_l) \implies (S_l)$ will hold in $\mathcal{Q}$. Now if $\mathcal{Q}$ is left semimedial, then

$$\begin{aligned}
xx \cdot yz &= xy \cdot xz && \text{by } (S_l)\\
&= f(xy)x \cdot (xy \cdot z) && \text{by } (E_l)\\
&= (f(x)f(y) \cdot x) \cdot (xy \cdot z) && \text{by Lemma 2.3}\\
&= (x \cdot f(y)e(x)) \cdot (xy \cdot z) && \text{by } (E_l)\\
&= xx \cdot (f(y)e(x) \cdot (x\backslash(xy \cdot z))) && \text{by } (S_l).
\end{aligned}$$

Cancelling and replacing z with $e(x)z$, we have

$$\begin{aligned}
f(y)e(x) \cdot (x\backslash(xy \cdot e(x)z)) &= y \cdot e(x)z\\
&= f(y)e(x) \cdot yz \text{ by } (E_l).
\end{aligned}$$

Cancelling, we obtain $yz = x\backslash(xy \cdot e(x)z)$ or $x \cdot yz = xy \cdot e(x)z$, which is (F_l).

For (2): If $\mathcal{Q}$ satisfies (E_r), then $\mathcal{Q}_{\rm op}$ satisfies (E_l) by Lemma 2.1(3). By part (1), (F_l) and (S_l) are equivalent in $\mathcal{Q}_{\rm op}$, and so (F_r) and (S_r) are equivalent in $\mathcal{Q}$ by Lemma 2.1(1), (2). □

Lemma 2.6. *A quasigroup satisfying* (E_l), (E_r) *is an F-quasigroup.*

Proof: We will show $(E_l), (E_r) \implies (F_l)$. Since this implication will also hold in the opposite parastrophe (by Lemma 2.1(3)), (F_r) will follow from Lemma 2.1(1). Thus we compute

$$\begin{aligned}
x \cdot yz &= f(x)y \cdot xz && \text{by } (E_l)\\
&= [f(x)e(f(x)) \cdot x(x\backslash y)] \cdot xz &&\\
&= [x \cdot e(f(x))(x\backslash y)] \cdot xz && \text{by } (E_l)\\
&= [x \cdot xz] \cdot [e(f(x))(x\backslash y) \cdot e(xz)] && \text{by } (E_r)\\
&= [x \cdot xz] \cdot [f(e(x))(x\backslash y) \cdot e(x)e(z)] && \text{by Lemmas 2.3 and 2.4}\\
&= [x \cdot xz] \cdot [e(x) \cdot (x\backslash y)e(z)] && \text{by } (E_l).
\end{aligned}$$

Now $xz \cdot (x\backslash y)e(z) = yz$ by (E_r), and so

$$x \cdot yz = [x \cdot xz] \cdot e(x)[(xz)\backslash(yz)].$$

Now replace y with $(xz \cdot y)/z$ to obtain

$$x(xz \cdot y) = [x \cdot xz] \cdot e(x)y.$$

Finally replace z with $x\backslash z$ to get

$$x \cdot zy = xz \cdot e(x)y,$$

which is (F_l). □

We can now prove our main result.

Proof of Theorem 1.2: Suppose $\mathcal{Q}$ is trimedial. Since each of (E_l) and (E_r) is a special case in three variables of the medial identity (M), these identities will hold in $\mathcal{Q}$. Indeed, fix $a, b, c \in Q$. Then the subquasigroup $\langle a, b, c\rangle$ generated by $\{a, b, c\}$ is medial. Taking $x = f(a)$, $y = a$, $u = b$, $v = c$ in (M), we obtain $a \cdot bc = f(a)b \cdot ac$, while taking $x = c$, $y = b$, $u = a$, $v = e(a)$ in (M) gives $cb \cdot a = ca \cdot (b \cdot e(a))$. Since $a, b, c \in Q$ were chosen arbitrarily, we have (E_l) and (E_r).

Conversely, if $\mathcal{Q}$ satisfies (E_l) and (E_r), then by Lemma 2.6, $\mathcal{Q}$ is an F-quasigroup, and by Lemma 2.5, $\mathcal{Q}$ is semimedial. Proposition 1.1 completes the proof. □

Lemma 2.7. 1. *A right semimedial, left F-quasigroup satisfies* (E_r).
2. *A left semimedial, right F-quasigroup satisfies* (E_l).

Proof: For (1): Suppose $\mathcal{Q}$ satisfies (S_r) and (F_l). Then

$$\begin{aligned}(xz \cdot y) \cdot e(xz)z &= xz \cdot yz && \text{by } (F_l) \\ &= xy \cdot zz && \text{by } (S_r) \\ &= ((xy)/e(xz))e(xz) \cdot zz \\ &= ((xy)/e(xz))z \cdot e(xz)z && \text{by } (S_r).\end{aligned}$$

Cancelling and using Lemma 2.3(3), we have $xz \cdot y = ((xy)/e(x)e(z))z$. Now $x(y/e(z)) \cdot e(x)e(z) = xy$ by (F_l), and so $xz \cdot y = x(y/e(z)) \cdot z$. Replacing y with $ye(z)$, we obtain $xz \cdot ye(z) = xy \cdot z$, which is (E_r).

For (2): If (S_l) and (F_r) hold in $\mathcal{Q}$, then (S_r) and (F_l) hold in $\mathcal{Q}_{\mathrm{op}}$ (Lemma 2.1(1), (2)), and so (E_r) holds in $\mathcal{Q}_{\mathrm{op}}$ by part (1). By Lemma 2.1(3), (E_l) holds in $\mathcal{Q}$. □

We now turn to our auxiliary result.

Proof of Theorem 1.3: Let $\mathcal{Q}$ be a right semimedial, left F-quasigroup. By Lemma 2.7(1), (E_r) holds. By Lemma 2.5(2), $\mathcal{Q}$ is an F-quasigroup. We will now show that (S_l) holds. First, using (S_r), (F_l), and (S_r) again, we have

$$((xy)/z)e(x) \cdot z^2 = xy \cdot e(x)z = x \cdot yz = (x/z)y \cdot z^2.$$

Cancelling and dividing on the right by $e(x)$, we obtain

$$(xy)/z = ((x/z)y)/e(x). \tag{*}$$

Next we use (S_r), (E_r), and (S_r) again to compute

$$((xy)/e(z))z \cdot e(z)^2 = xy \cdot z = xz \cdot ye(z) = ((xz)/e(z))y \cdot e(z)^2.$$

Cancelling, we obtain

$$((xy)/e(z))z = ((xz)/e(z))y. \tag{**}$$

Finally, we verify (S_l) as follows:

$$\begin{aligned}xy \cdot xz &= ((xy)/z)x \cdot z^2 && \text{by } (S_r) \\ &= [((x/z)y)/e(x)]x \cdot z^2 && \text{by } (*) \\ &= [((x/z)x)/e(x)]y \cdot z^2 && \text{by } (**) \\ &= (x^2/z)y \cdot z^2 && \text{by } (*) \\ &= x^2 \cdot yz && \text{by } (S_r).\end{aligned}$$

Since we have shown that $\mathcal{Q}$ is a semimedial, F-quasigroup, it follows from Proposition 1.1 that $\mathcal{Q}$ is trimedial.

On the other hand, if $\mathcal{Q}$ is a right semimedial, left F-quasigroup, then $\mathcal{Q}_{\text{op}}$ satisfies (S_l) and (F_r) by Lemma 2.1(1), (2). By the preceding argument, $\mathcal{Q}_{\text{op}}$ is a semimedial, F-quasigroup, and thus so is $\mathcal{Q}$ by Lemma 2.1(1), (2). Once again, Proposition 1.1 completes the proof. □

In closing, we note that further investigations suggest themselves. For example, it would be of interest to determine the structure of quasigroups that are only assumed to satisfy (E_l), or, in view of Lemma 2.5, those satisfying (E_l), (S_l) and (F_l). In this line we pose a couple of problems.

Problem 2.8. 1. *Characterize the loop isotopes of quasigroups satisfying* (E_l).

2. *Characterize the loop isotopes of quasigroups satisfying* (E_l), (S_l), *and* (F_l).

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