Subloops of sedenions

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Abstract. This note investigates sedenion multiplication from the standpoint of loop theory. New two-sided loops are obtained within the version of the sedenions introduced by the second author. Conditions are given for the satisfaction of standard loop-theoretical identities within these loops.

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1. Introduction

One of the most mature parts of loop theory is the theory of Moufang loops. Moufang loops trace their origins back to the primal examples given by various loops of non-zero or unit-norm octonions (Cayley numbers). The octonions furnish algebra structure on 8-dimensional Euclidean space, the Euclidean norm $|x|$ being *multiplicative* in the sense that

$$|xy| = |x|.|y| \tag{1.1}$$

under octonion multiplication. For n-dimensional Euclidean space, multiplicativity of the norm reduces to a question that has long been of interest to many mathematicians: "Can the product of two sums of n squares be expressed as a sum of n squares?" In other words:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) = A_1^2 + A_2^2 + \cdots + A_n^2. \tag{1.2}$$

A famous 1898 theorem of Hurwitz [2], [3] shows that an algebra (i.e. with both distributivities) has a multiplicative Euclidean norm only for $n = 1, 2, 4$ and 8. In particular, the usual Cayley-Dickson process for starting from the reals $\mathbb{R}$ and successively generating the complex numbers $\mathbb{C}$, Hamilton's quaternions $\mathbb{H}$, and the octonions $\mathbb{K}$, each with multiplicative Euclidean norm, does not make the Euclidean norm on 16-dimensional space multiplicative. In 1967 Pfister [5] proved that for $n = 2^k$, a product may be defined on Euclidean n-space such that (1.2) holds. However, Pfister was not concerned with algebraic properties of the product. Subsequently, there have been various multiplications defined on 16-dimensional real space with the intention of rendering the Euclidean norm multiplicative (cf. [2, §6.11], [10]).

In [7, p. 132], the second author introduced the formula

$$(a+bf)(c+df) = \textbf{if } b=0 \textbf{ then } ac+da.f \quad \textbf{else } (ab.cb^{-1}-b\overline{d})+(b\overline{c}+db^{-1}.ab)f \tag{1.3}$$

for multiplication on 16-dimensional Euclidean space $\mathbb{K}\oplus\mathbb{K}f$. This multiplication is not right distributive, but it is left distributive, and does make the Euclidean norm multiplicative. Elements of the semi-algebra $\mathbb{S}$ obtained are called *sedenions*. The octonions $\mathbb{K}$ form a subalgebra of $\mathbb{S}$. The non-zero sedenions form a left loop under multiplication (in the strong sense [8, Chapter I, §4.3] that includes a two-sided identity).

The intention of the present note is to consider the sedenions from the point of view of loop theory. Although the sedenions do not directly yield loops in the way that Moufang loops are obtained from octonions, the left loop of non-zero sedenions does contain new two-sided subloops defined in Section 2 below as sedenion extensions. These loops are constructed abstractly as extensions of subloops L of the octonions. The paper examines the satisfaction by these extensions of various standard loop-theoretical identities. For example, it turns out that they are all flexible (Proposition 2.1) and power-associative (Corollary 4.2), even though the full left loop of all non-zero sedenions is not itself flexible or power-associative. Given the current lack of a good conceptual approach to the multiplication in the sedenion extensions, the verifications of the identities are worked out in careful detail.

2. Sedenion extensions

Let L be a multiplicative subloop of the non-zero octonions. Then its *sedenion extension* $L \rtimes S^0$ is the disjoint union $L \cup Lf$ within the sedenions. Elements of this union are encoded as pairs (a, ε), with $\varepsilon \in S^0 = \{1, -1\}$, by

$$a \mapsto (a,1), \quad af \mapsto (a,-1)$$

[7, (5.4)]. Specializing (1.3), the multiplication in the full sedenion extension $\mathbb{K}^* \rtimes S^0$ of the loop $\mathbb{K}^*$ of all non-zero octonions is given by:

$$(x,1)(y,1) = (xy,1); \tag{2.1}$$

$$(x,1)(y,-1) = (yx,-1); \tag{2.2}$$

$$(x,-1)(y,1) = (x\overline{y},-1); \tag{2.3}$$

$$(x,-1)(y,-1) = (-x\overline{y},1) \tag{2.4}$$

[7, Proposition 5.2].[1] The sedenion extensions are certainly two-sided loops. Their respective left and right divisions are given by:

$$(x,1)\backslash(y,1)=(x\backslash y,1),\qquad (x,1)/(y,1)=(x/y,1); \tag{2.5}$$
$$(x,1)\backslash(y,-1)=(y/x,-1),\qquad (x,1)/(y,-1)=(-x/\overline{y},-1); \tag{2.6}$$
$$(x,-1)\backslash(y,1)=(-\overline{y}/\overline{x},-1),\qquad (x,-1)/(y,1)=(x/\overline{y},-1); \tag{2.7}$$
$$(x,-1)\backslash(y,-1)=(\overline{y}/\overline{x},1),\qquad (x,-1)/(y,-1)=(y\backslash x,1). \tag{2.8}$$

Each sedenion extension $L\rtimes S^0$ appears in an exact sequence

$$1\longrightarrow L\xrightarrow{j}L\rtimes S^0\xrightarrow{p}S^0\longrightarrow 1 \tag{2.9}$$

of loops with $j:x\mapsto(x,1)$ and $p:(x,\varepsilon)\mapsto\varepsilon$.

Suppose that the identity

$$a\backslash 1=1/a \tag{2.10}$$

holds in a two-sided loop L. Then the *inverse* a^{-1} of an element a of L is defined to be the common value $a^{-1}=a\backslash 1=1/a$. A loop L is said to have the *anti-automorphic inverse property* if (2.10) holds, and then

$$(ab)^{-1}=b^{-1}a^{-1} \tag{2.11}$$

for all elements a,b of L.

Proposition 2.1 ([4]). *Let L be a multiplicative subloop of the octonions. Then:*

(1) *$L\rtimes S^0$ satisfies the flexible law $a(ba)=(ab)a$;*
(2) *$L\rtimes S^0$ possesses the anti-automorphic inverse property.*

Proof: (1) Since the image of the map j of (2.9) is diassociative, there are just three cases to check:

$$\begin{aligned}[(x,1)(y,-1)](x,1)&=(yx,-1)(x,1)=(yx\overline{x},-1)\\&=(y\overline{x}x,-1)=(x,1)(y\overline{x},-1)=(x,1)[(y,-1)(x,1)];\end{aligned} \tag{2.12}$$

$$\begin{aligned}[(x,-1)(y,1)](x,-1)&=(x\overline{y},-1)(x,-1)=(-x\overline{y}\,\overline{x},1)\\&=(-x(\overline{xy}),1)=(x,-1)(xy,-1)=(x,-1)[(y,1)(x,-1)];\end{aligned} \tag{2.13}$$

[1]Note that (2.2) in [7] contains a typographical error: the second factor should read $(y,1)$. To correct a misleading impression created by the wording of [7, Proposition 5.2], we would also like to point out that the sequence (2.9) does not split.

$$
\begin{aligned}
(2.14)\qquad [(x,-1)(y,-1)](x,-1) &= (-x\overline{y},1)(x,-1) = (-xx\overline{y},-1)\\
&= (-x(\overline{y\overline{x}}),-1) = (x,-1)(-y\overline{x},1) = (x,-1)[(y,-1)(x,-1)].
\end{aligned}
$$

(2) In a flexible loop $a = 1a = a(a\backslash 1).a = a.(a\backslash 1)a \Rightarrow 1 = (a\backslash 1)a$, so Part (1) shows that the identity (2.10) holds in the sedenion extension $L \rtimes S^0$. As above, there are then three non-trivial cases of (2.11) to check, using (2.6) and (2.7):

$$
\begin{aligned}
(2.15)\qquad ((x,-1)(y,1))^{-1} &= (x\overline{y},-1)^{-1} = (-x\overline{y}/|xy|^2,-1)\\
&= (\overline{y}/|y|^2,1)(-x/|x|^2,-1) = (y,1)^{-1}(x,-1)^{-1};
\end{aligned}
$$

$$
\begin{aligned}
(2.16)\qquad ((x,1)(y,-1))^{-1} &= (yx,-1)^{-1} = (-yx/|xy|^2,1)\\
&= (-y/|y|^2,-1)(\overline{x}/|x|^2,1) = (y,-1)^{-1}(x,1)^{-1};
\end{aligned}
$$

$$
\begin{aligned}
(2.17)\qquad ((x,-1)(y,-1))^{-1} &= (-x\overline{y},1)^{-1} = (-y\overline{x}/|xy|^2,1)\\
&= (-y/|y|^2,-1)(-x/|x|^2,-1) = (y,-1)^{-1}(x,-1)^{-1}.
\end{aligned}
$$

□

3. The main theorem

We begin with a preliminary observation required for the proof of the main theorem. Recall that a loop is said to be *abelian* if it is both commutative and associative, i.e. an abelian group.

Lemma 3.1. *Each commutative multiplicative subloop of the octonions is associative.*

Proof: A commutative multiplicative subloop L of the octonions spans a commutative subalgebra R of the octonions. This subalgebra R is a commutative, alternative division ring. As such, it is associative [1, Lemma 6, p. 133]. Thus its subreduct L is also associative. □

Theorem 3.2. *Let L be a multiplicative subloop of the octonions. Then the following statements are equivalent:*

(1) $L \rtimes S^0$ *is a group;*
(2) $L \rtimes S^0$ *has the right inverse property;*
(3) $L \rtimes S^0$ *has the left inverse property;*
(4) $L \rtimes S^0$ *satisfies the left alternative law* $a.ab = a^2b$*;*
(5) $L \rtimes S^0$ *satisfies the right alternative law* $ba.a = ba^2$*;*
(6) L *is commutative;*
(7) L *is abelian.*

Proof: The implications (1) $\Rightarrow$ (2), (4), (5) are immediate.

- (2) $\Rightarrow$ (3) is a standard consequence of Proposition 2.1(2):

$$(a^{-1}b^{-1})b = a^{-1} \Rightarrow a = [(a^{-1}b^{-1})b]^{-1} = b^{-1}(a^{-1}b^{-1})^{-1} = b^{-1}(ba).$$

- (3) $\Rightarrow$ (6): Consider an element x of L. Recall that

$$(x,-1)\backslash(1,1) = (-x/|x|^2,-1). \tag{3.1}$$

Then for a further element y of L,

$$\begin{aligned}(-x/|x|^2,-1)[(x,-1)(y,1)] &= (-x/|x|^2,-1)(x\overline{y},-1)\\ &= (xy\overline{x}/|x|^2,1) = (xyx^{-1},1).\end{aligned} \tag{3.2}$$

If the left inverse property holds in $L \rtimes S^0$, the L-component of the final term in (3.2) has to equal y. Thus L is commutative.

- (4) $\Rightarrow$ (6): Consider elements x, y of L. Then

$$(x,-1)[(x,-1)(y,1)] = (x,-1)(x\overline{y},-1) = (-xy\overline{x},1), \tag{3.3}$$

while

$$[(x,-1)(x,-1)](y,1) = (-|x|^2,1)(y,1) = (-|x|^2y,1). \tag{3.4}$$

If the left alternative law holds in $L \rtimes S^0$, the L-components of the final terms of (3.3) and (3.4) have to agree for all x, y in L. Now $xy\overline{x} = |x|^2y \Rightarrow xyx^{-1} = y$, so that L is commutative.

- (5) $\Rightarrow$ (6): This is similar to the proof that (4) $\Rightarrow$ (6).
- (6) $\Rightarrow$ (7): Apply Lemma 3.1.
- (7) $\Rightarrow$ (1): Suppose that L is abelian. There are seven non-trivial cases of associativity to consider in $L \rtimes S^0$, arising whenever an S^0-component ε, ζ, η in the following equation is negative:

$$(x,\varepsilon)[(y,\zeta)(z,\eta)] = [(x,\varepsilon)(y,\zeta)](z,\eta).$$

The computations for these cases are as follows:

$$\begin{aligned}[(x,1)(y,1)](z,-1) &= (xy,1)(z,-1) = (zxy,-1)\\ &= (zyx,-1) = (x,1)(zy,-1) = (x,1)[(y,1)(z,-1)];\end{aligned} \tag{3.5}$$

$$\begin{aligned}[(x,1)(y,-1)](z,1) &= (yx,-1)(z,1) = (yx\overline{z},-1) \\ &= (y\overline{z}x,-1) = (x,1)(y\overline{z},-1) = (x,1)[(y,-1)(z,1)];\end{aligned} \tag{3.6}$$

$$\begin{aligned}[(x,1)(y,-1)](z,-1) &= (yx,-1)(z,-1) = (-yx\overline{z},1) \\ &= (-xy\overline{z},1) = (x,1)(-y\overline{z},1) = (x,1)[(y,-1)(z,-1)];\end{aligned} \tag{3.7}$$

$$\begin{aligned}[(x,-1)(y,1)](z,1) &= (x\overline{y},-1)(z,1) = (x\overline{y}\ \overline{z},-1) \\ &= (x(\overline{yz}),-1) = (x,-1)(yz,1) = (x,-1)[(y,1)(z,1)];\end{aligned} \tag{3.8}$$

$$\begin{aligned}[(x,-1)(y,1)](z,-1) &= (x\overline{y},-1)(z,-1) = (-x\overline{y}\ \overline{z},1) \\ &= (-x(\overline{zy}),1) = (x,-1)(zy,-1) = (x,-1)[(y,1)(z,-1)];\end{aligned} \tag{3.9}$$

$$\begin{aligned}[(x,-1)(y,-1)](z,1) &= (-x\overline{y},1)(z,1) = (-x\overline{y}\ z,1) \\ &= (-x(\overline{y\overline{z}}),1) = (x,-1)(y\overline{z},-1) = (x,-1)[(y,-1)(z,1)];\end{aligned} \tag{3.10}$$

$$\begin{aligned}[(x,-1)(y,-1)](z,-1) &= (-x\overline{y},1)(z,-1) = (-zx\overline{y},-1) \\ &= (-x(\overline{y\overline{z}}),-1) = (x,-1)(-y\overline{z},1) = (x,-1)[(y,-1)(z,-1)].\end{aligned} \tag{3.11}$$

□

4. Consequences of the theorem

The first consequence of Theorem 3.2 is negative.

Corollary 4.1. *No sedenion extension loop can be a proper Moufang or Bol loop.*

Proof: Moufang loops satisfy both inverse properties, while left or right Bol loops satisfy the respective left or right inverse property [8, Chapter I, §§4.1–2]. Thus a sedenion extension satisfying a Moufang or Bol law would be a group. □

On the positive side, the following result contrasts with the observation that the full left loop of non-zero sedenions under (1.3) is not power-associative [7, Proposition 5.1(i)].

Corollary 4.2. *Each sedenion extension loop is power-associative.*

Proof: Each element (x,ε) of a sedenion extension loop is an element of the sedenion extension of the abelian group $\langle x\rangle$. By Theorem 3.2, this extension is associative. □

Since each sedenion extension loop $L \rtimes S^0$ is power-associative, it possesses an *exponent* defined in the classical group-theoretical way as either 0 or else the smallest positive integer n for which $L \rtimes S^0$ satisfies the identity $a^n = 1$. (Recall that a related concept of exponent for a general quasigroup in a variety of quasigroups was defined in [6, Definition 5.2].) The next result shows how the exponent of a multiplicative subloop of the octonions determines the exponent of its sedenion extension.

Proposition 4.3. *Let L be a multiplicative subloop of the octonions.*

(1) *Suppose L has positive exponent n. Then the sedenion extension $L \rtimes S^0$ has a positive exponent m which is the least common multiple* $\operatorname{lcm}\{4, n\}$ *of* 4 *and n.*

(2) *Suppose L has exponent* 0. *Then the sedenion extension $L \rtimes S^0$ also has exponent* 0.

Proof: (1) Suppose $L \rtimes S^0$ satisfies $a^k = 1$. First note $(1, -1)^2 = (-1, 1) \neq (1, 1)$ and $(1, -1)^4 = (1, 1)$, so $4 \mid k$. Also $n \mid k$, since L is a subloop of $L \rtimes S^0$. In other words, the exponent of $L \rtimes S^0$ is either zero or a multiple of m. Conversely, consider $x \in L$. Then $x^n = 1 \Rightarrow x^m = 1 \Rightarrow |x|^m = 1$. Now $(x, 1)^m = (x^m, 1) = (1, 1)$. Also $(x, -1)^m = (|x|^m, 1)$ since $4 \mid m$, so $(x, -1)^m = (1, 1)$. Thus $L \rtimes S^0$ does satisfy $a^m = 1$.

(2) If L has elements of infinite order, then so does $L \rtimes S^0$, since it contains L as a subloop. □

Remark 4.4. Note that since a sedenion extension $L \rtimes S^0$ always contains an element $(1, -1)$ of exponent 4, it is never torsion-free. This contrasts with the conjecture of [2, §7.4] that "the loop generated by n generic real octonions is in fact the free Moufang loop on n generators". (A related question of W. Taylor [9] asks for the position in the interpretability lattice of the clone of continuous operations on the 7-sphere. The corresponding conjecture would locate this clone at the theory of Moufang loops.)

The final considerations concern commutativity of sedenion extensions.

Proposition 4.5. *Let L be a multiplicative subloop of the octonions. Then the sedenion extension $L \rtimes S^0$ is commutative if and only if L is a multiplicative subgroup of the reals.*

Proof: If x and y are elements of a multiplicative subgroup L of the reals, then

$$(x, 1)(y, -1) = (yx, -1) = (y\overline{x}, -1) = (y, -1)(x, 1)$$

and

$$(x, -1)(y, -1) = (-x\overline{y}, 1) = (-xy, 1) = (-y\overline{x}, 1) = (y, -1)(x, -1),$$

so the sedenion extension is commutative. Conversely, if x is an element of a commutative sedenion extension $L \rtimes S^0$, then

$$(-\overline{x}, 1) = (1, -1)(x, -1) = (x, -1)(1, -1) = (-x, -1),$$

so x is real. □

To conclude, we obtain an analogue of Lemma 3.1.

Corollary 4.6. *A commutative sedenion extension loop is associative.*

Proof: Suppose that a sedenion extension loop $L \rtimes S^0$ is commutative. By Proposition 4.5, L is a subgroup of the reals, and so abelian. The extension $L \rtimes S^0$ then satisfies the equivalent conditions (1)–(5) of Theorem 3.2. In particular, it is associative. $\square$

Remark 4.7. Corollary 4.6 would be a more complete analogue of Lemma 3.1 if it were known that the only two-sided subloops of the multiplicative left loop of non-zero sedenions appear either as subloops of the octonions, or as sedenion extensions in the sense of this paper. At present, however, this problem is open.

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References

[1] Albert A.A, ed., *Studies in Modern Algebra*, Prentice-Hall, Englewood Cliffs, NJ, 1963.
[2] Conway J.H., Smith D.A., *On Quaternions and Octonions: Their Geometry and Symmetry*, A.K. Peters Ltd., Natick, MA, 2003.
[3] Ebbinghaus H.-D. *et al.*, *Numbers*, Springer Verlag, New York, NY, 1991.
[4] Kinyon M.K., *private communication.*
[5] Pfister A., *Zur Darstellung definiter Funktionen als Summe von Quadraten*, Inv. Math. **4** (1967), 229–237.
[6] Smith J.D.H., *Quasigroup representation theory*, pp. 195–207 in "Universal Algebra and Quasigroup Theory", (eds. A. Romanowska and J.D.H. Smith), Heldermann, Berlin, 1992.
[7] Smith J.D.H., *A left loop on the* 15*-sphere*, J. Algebra **176** (1995), 128–138.
[8] Smith J.D.H., Romanowska A.B., *Post-Modern Algebra*, Wiley, New York, NY, 1999.
[9] Taylor W., *Spaces and equations*, Fund. Math. **164** (2000), 193–240.
[10] Zassenhaus H., Eichhorn W., *Herleitung von Acht- und Sechzehn-Quadrate-Identitäten mit Hilfe von Eigenschaften der verallgemeinerten Quaternionen und der Cayley-Dicksonschen Zahlen*, Arch. Math. **17** (1966), 492–496.

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