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Reflection loops of spaces with congruence and hyperbolic incidence structure

Alexander Kreuzer

Abstract. In an absolute space $(P, \mathfrak{L}, \equiv, \alpha)$ with congruence there are line reflections and point reflections. With the help of point reflections one can define in a natural way an addition $+$ of points which is only associative if the product of three point reflection is a point reflection again. In general, for example for the case that $(P, \mathfrak{L}, \alpha)$ is a linear space with hyperbolic incidence structure, the addition is not associative. $(P, +)$ is a K-loop or a Bruck loop.

Keywords: ordered space with congruence, point reflection, Bol loop, K-loop

Classification: 51D, 20N05

1. Introduction

Reflections are a powerful tool in order to prove geometric theorems. They have been used by J. Hjelmslev, G. Thomsen, A. Schmidt, F. Bachmann, E. Sperner and many others as a basic concept for an axiomatization of plane absolute geometry. There are two types of reflections, line reflections and point reflections. For characterizations of absolute planes one has to consider products of reflections. One central theorem of absolute planes is the well known three reflections theorem. Let A, B, C be lines, either through a common point or perpendicular to a common line. Then the product of the line reflections about A, B, C is a line reflection again. For an absolute space it is convenient to consider the product of three point reflections. If the points are on a line we get a point reflection again. In the general case we may interpret the product of three point reflections as the product of a reflection and a rotation about the same point. In the euclidian case the rotation is always the identity, but not in the hyperbolic case.
If one fixes a point 0 with its point reflection $\widetilde{0}$, one can attach to any two point reflections α, β exactly that point reflection, which is determined by the product of $\alpha\,\widetilde{0}\,\beta$, and one gets a binary operation. Since we may identify points with point reflections, we may consider the operation as a binary operation of the point set. We recall that the product of three point reflections is not a point reflection itself. Because of that, it turns out that the operation is not associative. We prove that we get a Bol loop with the automorphic inverse property (cf. Theorem 5.3).

In Section 2 we introduce a space with hyperbolic incidence structure. The hyperbolic incidence structure is defined using only order without congruence.

This concept is introduced in the papers [3], [4], [5], [7]. In Section 3 a short definition of a space with congruence is given, introduced by K. Sörensen in [13]. The results in [13], [15] are proved for a plane geometry, using the powerful characterization in [13, (1.12)] of line reflections, which is not true for absolute spaces. Distinct to [15], [2] we use a weaker formulation of the axiom (WK) without using order, and we use this axiom for the proof that the restriction of line reflections to planes are motions. On the other hand, in Section 3 we do not use any assumption on the incidence structure of $(P, \mathfrak{L})$, like the assumption (E) or (F), respectively, of [15], [2]. We generalize the results of [13], [15], [6], [2] to our assumptions. The proofs of the first results in Section 3 can be found in [13], but for the convenience of the reader we repeat the short proofs.
In Section 4 we assume in addition an ordered space. Now we can define point reflections and we show that point and line reflections are motions. We do not need that any two points have a midpoint, but for completeness, we show that this is true in an absolute space. We give a short proof here and remark that one can get this result by combining results of [13], [15], [6], [2]. In Section 5 we define the addition of points and, using the results of Section 3 and 4, we get the main result Theorem 5.3.
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2. Ordered spaces

Let $(P, \mathfrak{L})$ denote a *linear space* or *incidence space* with the point set P, the line set $\mathfrak{L}$ and at least three points on every line, i.e.,
- for any two points there is exactly one line containing it and
- for any line $L \in \mathfrak{L}$ we have $|L| \geq 3$.

A *subspace* is a subset $U \subset P$ such that for all distinct points $x, y \in U$ the unique line passing through x and y, denoted by $\overline{x, y}$, is contained in U. Let $\mathfrak{U}$ denote the set of all subspaces. For every subset $X \subset P$ we define the following *closure operation*

$$(1) \qquad \overline{} : \mathfrak{P}(P) \to \mathfrak{U},\ X \mapsto \overline{X} \qquad \text{by} \qquad \overline{X} := \bigcap_{\substack{U \in \mathfrak{U} \\ X \subset U}} U.$$

For $U \in \mathfrak{U}$ we call $\dim U := \inf\{|X| - 1 : X \subset U \text{ and } \overline{X} = U\}$ the *dimension* of U. A subspace of dimension two is a *plane*.

Now let $P^{(3)} := \{(a, b, c) \in P^3 : a, b, c \text{ collinear and } a \neq b, c\}$. We call

$$\alpha : P^{(3)} \to \{1, -1\};\ (a, b, c) \to (a|b, c)$$

a betweenness function and $(P,\mathfrak{L},\alpha)$ an ordered space if the following axioms are satisfied:

(A1) For a,b,c,d collinear with $a\neq b,c,d:(a|b,c)(a|c,d)=(a|b,d)$.

(A2) For distinct and collinear points $a,b,c\in P$, exactly one of the values $(a|b,c),(b|c,a)$ or $(c|a,b)$ equals -1.

(A3) For all distinct points $a,b\in P$ there exists a point $c\in P$ with $(a|b,c)=-1$.

We denote by $]a,b[:=\{x\in\overline{a,b}:(x|a,b)=-1\}$.

(PA) (Axiom of Pasch). Let a,b,c be non-collinear points and $G\subset\overline{a,b,c}$ be a line of the plane generated by a,b,c with $G\cap\,]a,b[\neq\emptyset$. Then $G\cap\,]a,c[\neq\emptyset$ or $G\cap\,]b,c[\neq\emptyset$.

By [14], [8] it holds:

Lemma 2.1. *Every ordered space $(P,\mathfrak{L},\alpha)$ satisfies the* exchange condition: *For $S\subset P$ and $x,y\in P$ with $x\in\overline{S\cup\{y\}}\setminus\overline{S}$, it follows that $y\in\overline{S\cup\{x\}}$.*

An ordered space is called *desarguesian*, if the following axiom (D) is satisfied.

(D) For $z\in P$ let $G_1,G_2,G_3\in\mathfrak{L}$ be distinct lines through z with distinct points $x_i,y_i\in G_i\setminus\{z\}$, $i\in\{1,2,3\}$, such that $p_1:=\overline{x_2,x_3}\cap\overline{y_2,y_3}$, $p_2:=\overline{x_3,x_1}\cap\overline{y_3,y_1}$, $p_3:=\overline{x_1,x_2}\cap\overline{y_1,y_2}$ exist. Then p_1,p_2,p_3 are collinear.

We recall that for $\dim P\geq 3$, every ordered space $(P,\mathfrak{L},\alpha)$ is a desarguesian space (cf. [6]).

For two distinct points $a,b\in P$ let denote
$\overrightarrow{a,b}:=\{x\in\overline{a,b}\setminus\{a\}:(a|b,x)=1\}$ the *half line* starting with a and containing b.

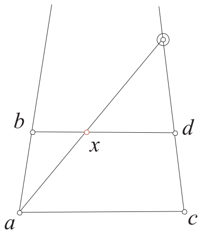
Lemma 2.2. *For points $a\neq b$ let $b'\in\overrightarrow{a,b}$. Then*

$$\overrightarrow{a,b}=\overrightarrow{a,b'}.$$

Proof: Since $(a|b',b)=1$, it follows $(a|b,x)=(a|b',b)(a|b,x)=(a|b',x)$. □

For $a,b,c,d\in P$ with $a\neq b,c\neq d$ we call the half line $\overrightarrow{a,b}$ *h-parallel* to $\overrightarrow{c,d}$, denoted by $\overrightarrow{a,b}\nparallel\overrightarrow{c,d}$, if

(i) $\overline{a,b}\cap\overline{c,d}=\emptyset$,
(ii) $]a,c[\,\cap\,]b,d[=\emptyset$,
(iii) $\forall x\in\,]b,d[$ it holds $\overline{a,x}\cap\overrightarrow{c,d}\neq\emptyset$.



First we show some easy properties (cf. [4]):

Lemma 2.3. *Let $a, b, c, d \in P$ with $a \neq b$, $c \neq d$ and $\overline{a,b} \cap \overline{c,d} = \emptyset$. Then there are equivalent*

(α) $\overline{a,c} \cap \,]b,d[\, \neq \emptyset$;

(β) $]a,c[\, \cap \overline{b,d} \neq \emptyset$;

(γ) $]a,c[\, \cap \,]b,d[\, \neq \emptyset$.

Proof: It suffices to show that (α) implies (γ). Let $x = \overline{a,c} \cap \,]b,d[$. By (A2) we have $(b|x,d) = (d|x,b) = 1$. Consider the three points (x,c,d) and the line $\overline{a,b}$. Now by axiom (PA) $\overline{a,b} \cap \overline{c,d} = \emptyset$ and $(b|x,d) = 1$ imply $(a|x,c) = 1$. In the same way $(c|a,x) = 1$, hence $(x|a,c) = -1$ by (A2). $\square$

Lemma 2.4. *Let $a, b, c, d \in P$ with $a \neq b$, $c \neq d$ and $\overline{a,b} \cap \overline{c,d} = \emptyset$.*

(a) *For $b' \in \overrightarrow{a,b}$, $d' \in \overrightarrow{c,d}$ we have $\overrightarrow{a,b'} \nparallel \overrightarrow{c,d'}$ if and only if $\overrightarrow{a,b} \nparallel \overrightarrow{c,d}$.*

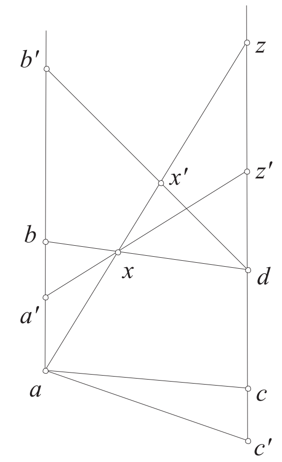
(b) *For $c' \in \overline{c,d}$ with $(d|c,c') = 1$ we have $\overrightarrow{a,b} \nparallel \overrightarrow{c',d}$ if and only if $\overrightarrow{a,b} \nparallel \overrightarrow{c,d}$.*

(c) *Let $a' \in \,]a,b[$ and assume $\overrightarrow{a,b} \nparallel \overrightarrow{c,d}$, then $\overrightarrow{a',b} \nparallel \overrightarrow{c,d}$.*

Proof: (a) We have $(a|b,b') = 1$, hence by (PA) for $x \in \,]b,d[$ the point $x' = \overline{a,x} \cap \,]b',d[$ exists and vice versa. Hence $\overline{a,x} \cap \overrightarrow{c,d} = \overline{a,x'} \cap \overrightarrow{c,d}$. Consider the three points b, b', d. By (PA) $y := \overline{a,c} \cap \,]b,d[\, \neq \emptyset$ if and only if $y' := \overline{a,c} \cap \,]b',d[\, \neq \emptyset$.

By Lemma 2.3, $\overline{a,c} \cap \,]b,d[\, = \,]a,c[\, \cap \,]b,d[$ and $\overline{a,c} \cap \,]b',d[\, = \,]a,c[\, \cap \,]b',d[$, therefore $]a,c[\, \cap \,]b,d[\, = \emptyset$ if and only if $]a,c[\, \cap \,]b',d[\, = \emptyset$ and $\overrightarrow{a,b} \nparallel \overrightarrow{c,d}$ if and only if $\overrightarrow{a,b'} \nparallel \overrightarrow{c,d}$. In the same way $\overrightarrow{a,b'} \nparallel \overrightarrow{c,d}$ if and only if $\overrightarrow{a,b'} \nparallel \overrightarrow{c,d'}$.

(b) Since $(d|c,c') = 1$, by (PA) we have $\overline{b,d} \cap \,]a,c[\, \neq \emptyset$ if and only if $\overline{b,d} \cap \,]a,c'[\, \neq \emptyset$. By Lemma 2.3 $]b,d[\, \cap \,]a,c[\, \neq \emptyset$ if and only if $]b,d[\, \cap \,]a,c'[\, \neq \emptyset$. Assume that for $x \in \,]b,d[$ the point $z = \overline{a,x} \cap \overline{c,d}$ exists. Since $(b|x,d) = (d|x,b) = 1$ and $\overline{a,b} \cap \overline{c,d} = \emptyset$, we have $(a|x,z) = (z|x,a) = 1$ by (PA). By (A2) it follows $(x|a,z) = -1$. Assume $\overline{b,d} \cap \,]a,c[\, = \overline{b,d} \cap \,]a,c'[\, = \emptyset$, then this implies by (PA) $(d|z,c) = -1$ and $(d|z,c') = -1$, respectively (consider the points x, d, x and a, b, x, respectively), hence $(c|d,z) = (c'|d,z) = 1$. This shows $\overline{a,x} \cap \overrightarrow{c,d} \neq \emptyset$ if and only if $\overline{a,x} \cap \overrightarrow{c',d} \neq \emptyset$.



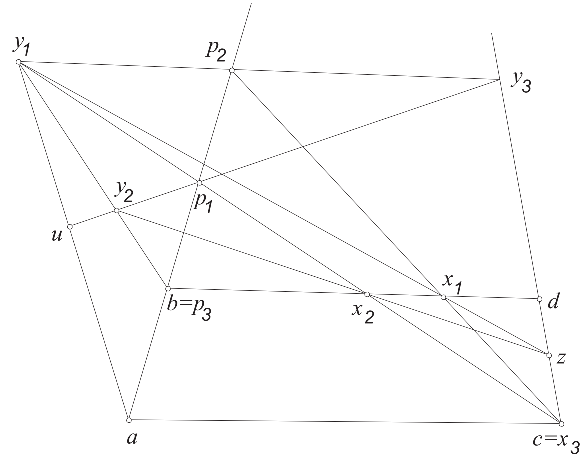
(c) For any $x \in \,]b,d[$ let $z = \overline{a,x} \cap \overrightarrow{c,d}$. Since $(a'|a,b) = -1 = (x|b,d)$ it follows $\overline{a',x} \cap \,]a,d[= \emptyset$ by (PA). Consider now the three points a,z,d and the line $\overline{a',x}$. Then by (PA) $z' := \overline{a',x} \cap \,]z,d[$ exists and $(z'|z,d) = -1$. Now $(c|d,z) = 1$ implies $(c|d,z') = 1$, i.e., $z' \in \overrightarrow{c,d}$ and $\overline{a',x} \cap \overrightarrow{c,d} \neq \emptyset$. (By (A1) the assumption $(c|d,z') = -1$ implies $(c|z,z') = -1$ and by (A2) $(z'|z,c) = 1 = (z'|c,d)$, hence $(z'|z,d) = 1$ by (A1), a contradiction to $(z'|z,d) = -1$.) $\square$

Lemma 2.5. *Let a,b,c,d with $\overrightarrow{a,b} \mathrel{|\!\!-\!\!|} \overrightarrow{c,d}$ and $G \subset \overline{a,b,c}$ with $a,c \notin G$ and $p = G \cap \overrightarrow{a,b} \neq \emptyset$. Then $G \cap \,]a,c[\neq \emptyset$ or $G \cap \overrightarrow{c,d} \neq \emptyset$.*

Proof: Let $p' \in \overline{a,b} = \overline{a,p}$ with $(p|a,p') = -1$, hence $(a|p,p') = 1$ and by Lemma 2.4(a), (c) $\overrightarrow{a,p'} \mathrel{|\!\!-\!\!|} \overrightarrow{c,d}$ and $\overrightarrow{p,p'} \mathrel{|\!\!-\!\!|} \overrightarrow{c,d}$. Assume $G \cap \,]a,c[= \emptyset$, then by (PA) $v := G \cap \,]p',c[$ exists. Now we consider the points c,d,p' and by (PA) and $G \cap \,]p',c[\neq \emptyset$ it follows $\emptyset \neq G \cap \,]c,d[\in \overrightarrow{c,d}$ or $x = G \cap \,]p',d[$. For the second case $\overrightarrow{p,p'} \mathrel{|\!\!-\!\!|} \overrightarrow{c,d}$ implies $G \cap \overrightarrow{c,d} = \overline{p,x} \cap \overrightarrow{c,d} \neq \emptyset$. $\square$

Theorem 2.6. *Let $(P,\mathfrak{L},\alpha)$ be a desarguesian ordered space. Let $a,b,c,d \in P$ with $\overrightarrow{a,b} \mathrel{|\!\!-\!\!|} \overrightarrow{c,d}$. Then we have $\overrightarrow{c,d} \mathrel{|\!\!-\!\!|} \overrightarrow{a,b}$.*

Proof: We assume that there exists $x_1 \in \,]b,d[$ with $\overline{c,x_1} \cap \overrightarrow{a,b} = \emptyset$. Let $p_1 \in \overline{a,b}$ with $(b|a,p_1) = -1$. Since $\overline{b,d} \cap \,]a,c[= \emptyset$ by Lemma 2.2, by (PA) the point $x_2 = \overline{b,d} \cap \,]p_1,c[$ exists. If $(x_1|x_2,b) = -1$, then (PA) would imply $\emptyset \neq \overline{c,x_1} \cap \,]b,p_1[\subset \overline{c,x_1} \cap \overrightarrow{a,b}$, a contradiction, hence $(x_1|x_2,b) = 1$. It follows $(x_1|x_2,d) = (x_1|b,d)(x_1|x_2,b) = (-1)\cdot 1 = -1$ and therefore $(x_2|x_1,d) = 1$. Consider the points x_2,c,d and x_2,b,p_1, respectively, the lines $\overline{a,b}$ and $\overline{c,d}$, respectively. Since $\overline{a,b} \cap \overline{c,d} = \emptyset$ and $(p_1|x_2,c) = (c|x_2,p_1) = 1$ (because $x_2 \in \,]p_1,c[\,$), by (PA) we have $(b|x_2,d) = (d|x_2,b) = 1$, hence $(x_2|b,d) = -1$. It follows $(x_2|b,x_1) = (x_2|b,d)(x_2|x_1,d) = (-1)\cdot 1 = -1$.



Let $y_1 \in \overline{c,p_1}$ with $(p_1|y_1,x_2) = -1$, hence $(x_2|y_1,c) = (x_2|p_1,c)(x_2|p_1,y_1) = (-1)\cdot 1 = -1$ and $(y_1|x_2,c) = (c|y_1,x_2) = 1$. Since $(x_1|x_2,d) = -1$ and $(y_1|x_2,c) = 1$ by (PA) the point $z = \overline{x_1,y_1} \cap \,]c,d[$ exists. Consider the points c,z,y_1 and the line $\overline{x_1,x_2}$. Since $(d|z,c) = 1$ (because $z \in \,]c,d[\,$) and $(x_2|c,y_1) = -1$ it follows by (PA) $(x_1|z,y_1) = -1$ and $(z|x_1,y_1) = 1$. Since $(z|x_1,y_1) = 1$ and $(x_2|b,x_1) = -1$ by (PA) $y_2 = \overline{z,x_2} \cap \,]b,y_1[$ exists. Since $(p_1|a,b) = 1$ and $(y_2|b,y_1) = -1$ by (PA) $u = \overline{p_1,y_2} \cap \,]a,y_1[$ exists. Consider the points a,c,y_1 and the line $\overline{u,p_1}$. Since $(u|a,y_1) = -1$ and $(p_1|y_1,c) = (p_1|y_1,x_2)(p_1|x_2,c) = (-1)\cdot 1 = -1$ (because $x_2 \in \,]p_1,c[\,$) by (PA) we have $\overline{u,p_1} \cap \,]a,c[= \emptyset$ and since $p_1 \in \overrightarrow{a,b}$ by Lemma 2.5, $y_3 = \overline{u,p_1} \cap \overrightarrow{c,d}$ exists.

Because $y_3 \in \overrightarrow{c,d}$, we have $(c|y_3,d) = 1$ and $(c|y_3,z) = (c|y_3,d)(c|d,z) = 1\cdot 1 = 1$ (because $(z|d,c) = -1$). Since $(c|y_3,z) = 1$ and $(x_1|z,y_1) = -1$ by (PA) the point $p_2 = \overline{c,x_1} \cap \,]y_1,y_3[$ exists. Now we set $x_3 = c$ and $p_3 = b$. Then with $G_i = \overline{x_i,y_i}$, $i = 1,2,3$ and (D) the points p_1,p_2,p_3 are collinear, i.e., $p_2 \in \overline{p_1,p_3} = \overline{a,b}$.

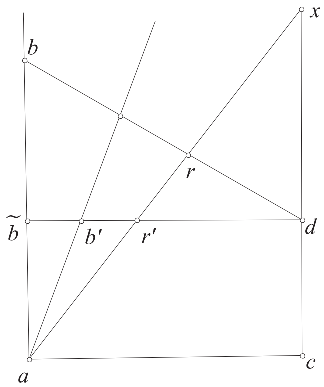
We consider the points b,p_2,x_1 and the line $\overline{d,c}$. Since $\overline{b,p_2} \cap \overline{c,d} = \overline{a,b} \cap \overline{c,d} = \emptyset$ and $(d|b,x_1) = 1$, by (PA) it follows $(c|p_2,x_1) = 1$. Now with the line $\overline{a,c}$, $(c|p_2,x_1) = 1$ implies $(a|b,p_2) = 1$ since $\overline{a,c} \cap \,]b,x_1[\,\subset\, \overline{a,c} \cap \,]b,d[= \emptyset$ by Lemma 2.3. Hence $p_2 \in \overline{c,x_1} \cap \overrightarrow{a,b}$, a contradiction to $\overline{c,x_1} \cap \overrightarrow{a,b} = \emptyset$. Therefore $\overrightarrow{c,d} \nparallel \overrightarrow{a,b}$. $\square$

Lemma 2.7. *Let $a,b,b',c,d \in P$ with $\overrightarrow{a,b} \nparallel \overrightarrow{c,d}$ and $\overrightarrow{a,b'} \nparallel \overrightarrow{c,d}$. Then* $\overline{a,b} =$

$\overline{a,b'}$.

Proof: Assume $\overline{a,b} \neq \overline{a,b'}$. Let $x \in \overline{c,d}$ with $(d|c,x) = -1$. Since with Lemma 2.2 $\overline{b,d} \cap\,]a,c[\, = \emptyset = \overline{b',d} \cap\,]a,c[$, by (PA) $r = \overline{b,d} \cap\,]a,x[$ and $r' = \overline{b',d} \cap\,]a,x[$ exist. It holds $(x|r,r') = (x|r,a)(x|r',a) = 1 \cdot 1 = 1$. Hence we may assume $(r'|r,x) = -1$ (or we change b and b'). Since $\overline{d,x} \cap \overline{a,b} = \overline{c,d} \cap \overline{a,b} = \emptyset$ and $(x|r,a) = 1$, by (PA) it follows $(d|r,b) = 1$. Because $(r'|r,x) = -1$ therefore with (PA) the point $\widetilde{b} = \,]a,b[\, \cap \overline{d,r'}$ exists. We have $(d|\widetilde{b},r') = 1$ by (PA), since $(b|\widetilde{b},a) = (r|r',a) = 1$, and therefore $(d|\widetilde{b},b') = (d|\widetilde{b},r')(d|b',r') = 1 \cdot 1 = 1$. It follows $(b'|\widetilde{b},d) = -1$ or $(\widetilde{b}|b',d) = -1$.

In the first case by $\overrightarrow{a,b} = \overrightarrow{a,\widetilde{b}} \mathrel{\text{⫲}} \overrightarrow{c,d}$ it holds $\overline{a,b'} \cap \overrightarrow{c,d} \neq \emptyset$, a contradiction to $\overline{a,b'} \cap \overline{c,d} = \emptyset$. In the second case $\overline{a,b} \cap \overrightarrow{c,d} \neq \emptyset$, a contradiction to $\overline{a,b} \cap \overline{c,d} = \emptyset$. □



We call a half line $\overrightarrow{a,b}$ *h-parallel* to a line G, denoted by $\overrightarrow{a,b} \mathrel{\text{⫲}} G$, if there are points $c,d \in G$ with $\overrightarrow{a,b} \mathrel{\text{⫲}} \overrightarrow{c,d}$.

We call two lines G,H *h-parallel*, denoted by $G \mathrel{\text{⫲}} H$, if there are distinct points $a,b \in H, c,d \in G$ with $\overrightarrow{a,b} \mathrel{\text{⫲}} \overrightarrow{c,d}$.

Theorem 2.8. *Let $G \in \mathfrak{L}$ and $a \in P \setminus G$. Then there are at most two distinct lines H_1, H_2 through a which are h-parallel to G.*

Proof: Let $c,d \in G$ be distinct. By Lemma 2.7 there is at most one b with $\overrightarrow{a,b} \mathrel{\text{⫲}} \overrightarrow{c,d}$ and at most one b' with $\overrightarrow{a,b'} \mathrel{\text{⫲}} \overrightarrow{d,c}$. With Lemma 2.4 the assumption now follows. □

We call a line set $\mathfrak{b} \subset \mathfrak{L}$ an *end*,

(1) if for any two lines $G, H \in \mathfrak{b}$ it holds $G \nparallel H$, and

(2) if $\bigcup_{G \in \mathfrak{b}} G = P$

An ordered space $(P, \mathfrak{L}, \alpha)$ is called an *ordered space with hyperbolic incidence structure* if the following holds

(H) for every line G and every point $a \in P \setminus G$ there are two distinct h-parallel lines H_1, H_2 through a;

(E) for any distinct $a, b \in P$ the set $\overrightarrow{(a, b)} := \{G \in \mathfrak{L} : \overrightarrow{a, b} \nparallel G\} \cup \{\overline{a, b}\}$ is an end.

Remarks. 1. For $\dim P \geq 3$, the axiom (H) implies the axiom (E) (cf. [5, (3.11)]).

2. Konrad has shown in [7] that, if we add a congruence relation $\equiv$ to an ordered space $(P, \mathfrak{L}, \alpha)$, we have a hyperbolic space $(P, \mathfrak{L}, \alpha, \equiv)$ if and only if $(P, \mathfrak{L}, \alpha)$ is an ordered space with hyperbolic incidence structure.

3. Spaces with congruence

In this section we introduce the concept of a space $(P, \mathfrak{L}, \equiv)$ with congruence (cf. [13]). We assume in this section a linear space $(P, \mathfrak{L})$ whose planes satisfy the exchange condition. Let $\equiv$ be a *congruence relation* on $P \times P$, i.e.

$\equiv$ is a equivalence relation with
$(a, b) \equiv (b, a)$ and
$(a, a) \equiv (b, c)$ if and only if $b = c$.

We use the notation $(x_1, x_2, x_3) \equiv (y_1, y_2, y_3)$ if and only if $(x_i, x_j) \equiv (y_i, y_j)$ for $i, j \in \{1, 2, 3\}$. $(P, \mathfrak{L}, \equiv)$ is a *space with congruence* if the axioms (W1), (W2) and (W3) are satisfied.

(W1) Let $a, b, c \in P$ be distinct and collinear, and let $a', b' \in P$ with $(a, b) \equiv (a', b')$. Then there exists exactly one $c' \in \overline{a', b'}$ with $(a, b, c) \equiv (a', b', c')$.

(W2) Let $a, b, x \in P$ be non-collinear and let $a', b', x' \in P$ with $(a, b, x) \equiv (a', b', x')$. For any $c \in \overline{a, b}$ and $c' \in \overline{a', b'}$ with $(a, b, c) \equiv (a', b', c')$ it holds $(x, c) \equiv (x', c')$.

(W3) For $a, b, x \in P$ non-collinear there exists exactly one $x' \in \overline{\{a, b, x\}} \setminus \{x\}$ with $(a, b, x) \equiv (a, b, x')$.

Let $(P, \mathfrak{L}, \equiv)$ be a space with congruence. Then $(P, \mathfrak{L}, \equiv)$ is called a *regular space with congruence*, if in addition (WK) is satisfied:

(WK) for $a, b, c \in P$ non-collinear and $c' \in \overline{\{a, b, c\}} \setminus \{c\}$ with $(a, b, c) \equiv (a, b, c')$ it holds $\overline{a, b} \cap \overline{c, c'} \neq \varnothing$.

We remark that we use here a weaker formulation of this axiom then used in [15], [2], [10]. In the following let $(P, \mathfrak{L}, \equiv)$ be a regular space with congruence. We call a bijective mapping $\phi : P \to P$ a *motion*, if $(x, y) \equiv (\phi(x), \phi(y))$ for all $x, y \in P$. It is well known:

Lemma 3.1. (i) *If a, b, c are collinear points and $a', b', c' \in P$ with $(a, b, c) \equiv (a', b', c')$, then a', b', c' are collinear.*

(ii) *Any motion ϕ is a collineation.*

Proof: (i) By (W1) the point $c'' \in \overline{a', b'}$ exists with $(a, b, c) \equiv (a', b', c'')$. If $c' \notin \overline{a', b'}$, then by (W3) it would follow $(c, c) \equiv (c', c'')$, hence $c' = c'' \in \overline{a', b'}$.

(ii) By (i), ϕ and ϕ^{-1} maps collinear points onto collinear points. □

Lemma 3.2. *Any distinct points b, b' have at most one midpoint $m \in \overline{b, b'}$ with $(b, m) \equiv (b', m)$.*

Proof: For $m, m' \in \overline{b, b'}$ with $(b, m) \equiv (b', m)$ and $(b, m') \equiv (b', m')$, we have $(m, m', b) \equiv (m, m', b')$. If $m \neq m'$, (W1) would imply $b = b'$. □

In (1.5) of [13] the following lemma is shown. Since we have distinct assumptions, we give a proof.

Lemma 3.3. *Let $a, b, x \in P$ be non-collinear points and let $x' \in \overline{a, b, x} \setminus \{x\}$ with $(a, b, x) \equiv (a, b, x')$. Then for any $c \in \overline{a, b, x}$ with $(x, c) \equiv (x'c)$ it holds $c \in \overline{a, b}$.*

Proof: We may assume $c \notin \overline{a, x}$ and $b \neq c$. Since $\overline{a, b, x}$ is an exchange plane, $\overline{a, b, x} = \overline{a, c, x}$, and $(a, c, x) \equiv (a, c, x')$ implies by (WK), $p = \overline{a, c} \cap \overline{x, x'}$. Also by (WK), $m = \overline{a, b} \cap \overline{x, x'}$. By (W2), m and p are midpoints of $\overline{x, x'}$, hence $m = p$ by Lemma 3.2 and $a, m \in \overline{a, b} \cap \overline{a, c}$. Therefore, if $a \neq m$, $\overline{a, b} = \overline{a, m} = \overline{a, c}$. If $a = m$, we show $m \in \overline{b, c}$ and $\overline{a, b} = \overline{b, m} = \overline{b, c}$. □

Lemma 3.4. *Let $E = \overline{a, b, c}$ be a plane and $\phi|_E$ be a motion.*

(i) *If a, b are fixed points of ϕ then any point $x \in \overline{a, b}$ is a fixed point.*

(ii) *If a, b, c are fixed points of ϕ then any point $x \in \overline{a, b, c}$ is a fixed point.*

Proof: We have $(a, b, x) \equiv (a, b, \phi(x))$ for any point $x \in \overline{a, b}$, hence by (W1), $\phi(x) = x$. Now let $x \in \overline{a, b, c} \setminus \overline{a, b}$. If $x \neq \phi(x)$, then $(a, b, x) \equiv (a, b, \phi(x))$ and $(c, x) \equiv (c, \phi(x))$ implies $c \in \overline{a, b}$ by Lemma 3.3, a contradiction to a, b, c non-collinear. Hence $x = \phi(x)$. □

Lemma 3.5. *Let E be a plane, $\phi|_E$ be a motion, $L \subset E$ be a line and let $z \in L$ with $\phi(z) = z$.*

(i) *If $\phi(L) = L$, then $\phi^2_{|L} = \mathrm{id}_{|L}$.*

(ii) *If $\phi(L) = L$ and $\phi(E) = E$, then $\phi^2_{|E} = \mathrm{id}_{|E}$.*

Proof: For $x \in L \setminus \{z\}$, $(z, x) \equiv (z, \phi(x))$ and $(z, \phi(x), x) \equiv (z, x, \phi(x)) \equiv (z, \phi(x), \phi^2(x))$, hence $x = \phi^2(x)$ by (W1). For $y \in E \setminus L$ it follows $(z, x, y) \equiv (z, \phi^2(x), \phi^2(y)) \equiv (z, x, \phi^2(y))$ and $(y, \phi(y)) \equiv (\phi(y), \phi^2(y))$. If $y \neq \phi^2(y)$, Lemma 3.3 implies $\phi(y) \in \overline{z, x} = L$, a contradiction to $y \notin L = \phi(L)$. □

For a line $L \in \mathfrak{L}$, $x \in P \setminus L$ and $a, b \in L$ with $a \neq b$ there exists by (W3) the unique point $x' \in \overline{L \cup \{x\}} \setminus \{x\}$ with $(a, b, x) \equiv (a, b, x')$. By (W2) x' is independent of the choice of $a, b \in L$, hence we may denote $x' = L(x)$.

We call the following mapping *line reflection*

$$\widetilde{L} : P \to P; \qquad x \to \begin{cases} x & \text{if } x \in L, \\ L(x) & \text{if } x \notin L. \end{cases}$$

Clearly $\widetilde{L}$ is an involutory bijection with $z = \widetilde{L}(z)$ if and only if $z \in L$.

Lemma 3.6. *Let L be a line of a plane E and let $p' := \widetilde{L}(p)$ for $p \in E$.*

(i) *If $(p, q) \equiv (p', q')$ for $p, q \in E$, then $(p, q, x) \equiv (p', q', x')$ for every $x \in \overline{p, q}$.*

(ii) *Let $a, b, c, d \in E$ be points with $(p, q) \equiv (p', q')$ for $p, q \in \{a, b, c, d\}$. Then $(x, y) \equiv (x', y')$ for $x \in \overline{a, b}$ and $y \in \overline{c, d}$.*

Proof: (i) By (W1) there exists a point $x_1 \in \overline{p', q'}$ with $(p, q, x) \equiv (p', q', x_1)$. If $x = x_1$, then by Lemma 3.3 $x \in L$. For any point $u \in L$, $(p, q, u) \equiv (p', q', u)$, hence by (W2) $(u, x) \equiv (u, x_1)$ and therefore $x_1 = x'$.

(ii) By (i) we have $(a, b, x) \equiv (a', b', x')$ and $(c, d, y) \equiv (c', d', y')$. If $c \in \overline{a, b}$, then $(c, x) \equiv (c', x')$ by (i). For $c \notin \overline{a, b}$, by the assumptions $(a, b, c) \equiv (a', b', c')$ and (W2) implies $(c, x) \equiv (c', x')$. Also $(d, x) \equiv (d', x')$. If $x \in \overline{c, d}$, then by (i), $(x, y) \equiv (x', y')$. If $x \notin \overline{c, d}$ with $(c, d, x) \equiv (c', d', x')$ it follows by (W2) $(x, y) \equiv (x', y')$. □

For the proof that the restriction of a line reflection to a plane is a motion, we have to consider the closure of three points. We define for a subset $X \subset P$

$$[X] := \bigcup_{x, y \in P} \overline{x, y}, \quad [X]_1 := X \quad \text{and} \quad [X]_{n+1} := [[X]_n] \quad \text{for} \quad n \in \mathbb{N}.$$

Clearly $[X]_n \subset \overline{X}$ and since $\bigcup_{n \in \mathbb{N}} [X]_n$ is a linear space, $\overline{X} = \bigcup_{n \in \mathbb{N}} [X]_n$.

Theorem 3.7. *Let L be a line of a plane E. Then $\widetilde{L}|_E$ is an involutory motion with $z = \widetilde{L}(z)$ if and only if $z \in L$.*

Proof: Let u, v, w be points with $L = \overline{u, v}$ and $E = \overline{u, v, w}$. We prove the theorem by induction. For $x, y \in [u, v, w]_1 = \{u, v, w\}$ clearly $(x, y) \equiv (x', y')$. Assume this property for $[u, v, w]_n$ and let $x, y \in [u, v, w]_{n+1}$. Then $a, b, c, d \in [u, v, w]_n$ exist with $x \in \overline{a, b}$ and $y \in \overline{c, d}$. By assumption a, b, c, d satisfy the conditions of Lemma 3.6(ii) and we have $(x, y) \equiv (x', y')$. With $\overline{X} = \bigcup_{n \in \mathbb{N}} [X]_n$ it follows $(x, y) \equiv (x', y')$ for any $x, y \in E$. □

Lemma 3.8. *Let a, b, b' be non-collinear points with $(a,b) \equiv (a,b')$. Then there exists exactly one line $L \subset \overline{a,b,b'}$ through a with $b' = L(b)$, and b, b' have a midpoint m.*

Proof: By (W3) there exists the point $a' \in \overline{a,b,b'} \setminus \{a\}$ with $(b,b',a) \equiv (b,b',a')$, since $(a,b) \equiv (a,b')$ also $(a,a',b) \equiv (a,a',b')$. For $L := \overline{a,a'}$ it follows $L \subset \overline{a,b,b'}$ and $L(b) = b'$. By Lemma 3.3, L is uniquely determined. By (WK), $m = L \cap \overline{b,b'}$ exists with $(m,b) \equiv (m,b')$. $\square$

We define for lines $A, B \in \mathfrak{L}$:

$$A \perp B \qquad \Longleftrightarrow \qquad \widetilde{A}(B) = B \;\text{ and }\; A \neq B.$$

Lemma 3.9. (i) *If $A \perp B$, then $A \cap B \neq \emptyset$ and $B \perp A$.*

(ii) *For lines A, B of a plane E, we have $A \perp B$ if and only if $\big(\widetilde{A}\widetilde{B}\big)^2_{|E} = \mathrm{id}_{|E}$.*

Proof: (i) Let $a_0, a_1 \in A \setminus B$ be distinct points, $b \in B \setminus A$ and $b' = A(b) \in B$. Then $(a_0, a_1, b) \equiv (a_0, a_1, b')$ and by (WK) $z = A \cap B$ exists. For $a_2 = B(a_1)$ we have $(b', b, a_1) \equiv (b', b, a_2)$, and by Lemma 3.3, $(b, a_2) \equiv (b, a_1) \equiv (b', a_1) \equiv (b', a_2)$ implies $a_2 \in A = \overline{a_1, a_0}$, hence by Theorem 3.7, $\widetilde{B}(A) = A$.

(ii) Let $A \perp B$ and $z = A \cap B$. For $\phi := \widetilde{A}\widetilde{B}$ and $E := \overline{A \cup B}$ we have $\phi(E) = E$, $\phi(B) = B$ and $\phi(z) = z$. By Lemma 3.5, $\phi^2_{|E} = \mathrm{id}_{|E}$.

Assume now $\phi^2_{|E} = \mathrm{id}_{|E}$, hence $\widetilde{A}\widetilde{B}\widetilde{A}\widetilde{B}(b) = b$ for any point $b \in B$, i.e., $\widetilde{B}(\widetilde{A}(b)) = \widetilde{A}(b)$. By Theorem 3.7, $\widetilde{A}(b) \in B$, i.e., $\widetilde{A}(B) = B$. $\square$

The following two lemmas one can find in Sections 1 and 4 of [13]. Since there $(P, \mathfrak{L})$ is assumed as a plane, we have to give proofs here.

Lemma 3.10. *For distinct points a, b, there exists a point $b' \in \overline{a,b} \setminus \{b\}$ with $(a,b) \equiv (a,b')$.*

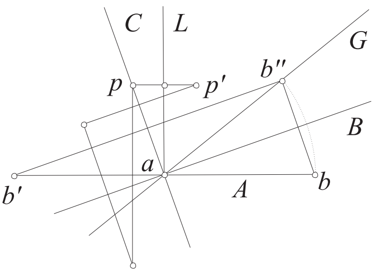
Proof: Let c be a point not on $A := \overline{a,b}$, and let $c' \in \overline{a,b,c} \setminus \{c\}$ with $(a,b,c) \equiv (a,b,c')$. Let $C := \overline{c,c'}$. By (WK), $m := C \cap \overline{a,b}$ exists. By Lemma 3.3 $a' = C(a) \in A$, and by (W2) $(a,m) \equiv (a',m)$. Now we use two times (W1) and get the points $m', b' \in A$ with $(m,a,a') \equiv (a,m,m')$ and $(a,m,b) \equiv (a,m',b')$. Therefore $(a,m) \equiv (a,m')$ and $(a,b) \equiv (a,b')$. $\square$

Theorem 3.11. *For a plane E and distinct points $a, b, b' \in E$ with $(a,b) \equiv (a,b')$, there exists a line $L \subset E$ with $a \in L$ and $L(b) = b'$.*

Proof: Let a, b, b' be collinear and $A := \overline{a,b}$. First we show that there exists $b'' \in E \setminus L$ with $(a,b) \equiv (a,b'')$. Let $q \in E \setminus L$, hence $\overline{a,b,q} = E$. For $a' = \overline{q,b}\,(a)$, we have $B := \overline{a,a'} \perp \overline{q,b}$ and therefore $b'' = B(b) \in \overline{q,b} \setminus \{b\}$, in particular

$b'' \notin A$. Clearly $(a,b) \equiv (a,b'')$. By Lemma 3.5, the line $C \subset E$ through a with $C(b') = b''$ exists. For $\phi := \widetilde{C}\widetilde{B}\widetilde{A}$ we have $\phi(b) = \widetilde{C}\widetilde{B}\widetilde{A}(b) = \widetilde{C}(b'') = b'$, $\phi(A) = A$ and $\phi(a) = a$. Also $\widetilde{C}\widetilde{B}(A) = A$ and $\widetilde{C}\widetilde{B}(a) = a$. By Lemma 3.5, $\phi^2_{|E} = \mathrm{id}_{|E}$ and $\big(\widetilde{C}\widetilde{B}\big)^2_{|E} = \mathrm{id}_{|E}$. Now Lemma 3.9 shows $\widetilde{B}(C) = C$. Since $A \neq G = \overline{a, b''}$, also $\widetilde{C}(A) = G \neq A$ and by Lemma 3.9 $\widetilde{A}(C) \neq A$ and therefore $\phi(C) \neq C$.

Hence for $p \in C \setminus \{a\}$ we have $a \notin \overline{p, \phi(p)}$ and $(a,p) \equiv (a, \phi(p))$. By Lemma 3.8 a line L through a exists with $L(p) = \phi(p) = p'$. Now by Lemma 3.4, $\phi\widetilde{L}(a) = a$, $\phi\widetilde{L}(p) = p$ and $\phi\widetilde{L}(p') = p'$ imply $\phi\widetilde{L}_{|E} = \mathrm{id}_{|E}$ and $L(b) = \phi(b) = b'$. $\square$



4. Absolute spaces

Let $(P, \mathfrak{L}, \alpha)$ be an ordered space and let $(P, \mathfrak{L}, \equiv)$ be a regular space with congruence. Then we call $(P, \mathfrak{L}, \equiv, \alpha)$ a *regular ordered space with congruence*. $(P, \mathfrak{L}, \equiv, \alpha)$ is called an *absolute space* if in addition (WF) is satisfied (cf. [15], [2]):

(WF) Let $a, b, c \in P$ be non-collinear points. Then there exists $d \in \overline{a, c}$ with $(a,b) \equiv (a,d)$.

Now we assume first that $(P, \mathfrak{L}, \equiv, \alpha)$ is a regular ordered space with congruence.

Theorem 4.1. *For $a, b, c \in P$ non-collinear and $c' \in \overline{\{a,b,c\}} \setminus \{c\}$ with $(a,b,c) \equiv (a,b,c')$ let $m := \overline{a,b} \cap \overline{c,c'}$. Then $(m|c,c') = -1$.*

Proof: For $L := \overline{a,b}$, by definition $c' = L(c)$. We assume $(c|m,c') = -1$, hence $(c'|m,c) = (m|c,c') = 1$. Let $x \in L$ with $(x|a,m) = -1$. By the axiom of Pasch (PA), there exists $y := \overline{c',x} \cap \,]a,c[$ and for $y' = L(y) = \overline{c,x} \cap \overline{a,c'}$ it follows $(y'|a,c') = 1$ by (PA), since $(c|m,c') = -1$. Therefore we have by (PA) $p = \overline{y,y'} \cap \,]c,c'[\, \neq \emptyset$, in particular $p \neq m$. Since $\widetilde{L}(\overline{y,y'}) = \overline{y',y}$, $\widetilde{L}(\overline{c,c'}) = \overline{c',c}$ it follows $L(p) = p \in L$ by Theorem 3.7, a contradiction to $p \neq m = \overline{c,c'} \cap L$. It follows $(c|m,c') = 1$ and also $(c'|m,c) = 1$, hence by (A2), $(m|c,c') = -1$. $\square$

Lemma 4.2. (i) *Let a, b, b' be collinear points with $(a, b) \equiv (a, b')$. Then $(a|b, b') = -1$.*

(ii) *For distinct points a, b there exists exactly one point $b' \in \overline{a, b} \setminus \{b\}$ with $(a, b) \equiv (a, b')$.*

(iii) *For a plane E, a line $L \subset E$ and $a \in L$, there exists exactly one line $G \subset E$ with $a \in G$ and $L \perp G$.*

Proof: (i) By Lemma 3.11 a line L exists with $a \in L$ and $L(b) = b'$. For any point $x \in L \setminus \{a\}$ we have $(x, a, b) \equiv (x, a, b')$ and by (WK), $a = \overline{b, b'} \cap \overline{a, x}$, i.e., $(a|b, b') = -1$ by 4.1.

(ii) By Lemma 3.10, b' exists. If there are $b', b'' \in \overline{a, b} \setminus \{b\}$ with $(a, b') \equiv (a, b) \equiv (a, b'')$, it follows by (i), $(a|b, b') = (a|b, b'') = (a|b', b'') = -1$, a contradiction to axiom (A1).

(iii) For $b \in L \setminus \{a\}$ we have $L \perp G$ if and only if $b \neq b' := G(b) \in \overline{a, b}$. Now (ii) and Lemma 3.11 show (iii). □

For distinct points $a, x \in P$ we denote by $a(x)$ the unique point $a(x) \in \overline{a, x} \setminus \{x\}$ with $(a, x) \equiv (a, a(x))$. We call the following mapping *point reflection*:

$$\widetilde{a} : P \to P; \qquad x \to \begin{cases} x & \text{if } x = a, \\ a(x) & \text{if } x \neq a. \end{cases}$$

Theorem 4.3. *Every point reflection $\widetilde{a}$ is an involutory motion with $x = \widetilde{a}(x)$ if and only if $x = a$.*

Proof: By 4.2(ii), $\widetilde{a}$ is an involutory bijection and $\widetilde{a}(x) = x$ only if $x = a$. Let $x, y \in P$, $x' = a(x)$ and $y' = a(y)$. We show $(x, y) \equiv (x', y')$. If x, y, a are non-collinear then $X := \overline{a, x} \neq Y := \overline{a, y}$. For $y'' = X(y)$ by 3.8 a line L through a exists with $L(y'') = y'$. (If $y' = y''$, $L = Y$.) Hence for $\phi := \widetilde{L}\widetilde{X}$ it holds $\phi(y) = y'$ and $\phi(Y) = Y$. For $E := \overline{a, x, y}$ also $\phi(E) = E$ and $\phi(a) = a$. Therefore by Lemma 3.5, $\phi^2|_E = \mathrm{id}$ and by Lemma 3.9, $\widetilde{L}(X) = X$. Now $L(x) \in X \setminus \{x\}$ implies by Lemma 4.2(ii) $L(x) = x'$, hence $\phi(x) = x'$. Since by Theorem 3.7, $\phi|_E$ is a motion, $(x, y) \equiv (x'.y')$. If x, y, a are collinear, by (W1) and Lemma 4.2(ii), $(a, x, y) \equiv (a, x', y')$. □

Theorem 4.4. *Every line reflection $\widetilde{L}$ is a motion.*

Proof: For any point a, we denote $a' := \widetilde{L}(a)$. Let $x, y \in P \setminus L$ and $z := \overline{x, x'} \cap L$. By Theorem 4.2(iii), the line $G \subset \overline{L \cup \{y\}}$ through z exists with $\widetilde{L}(G) = G$. Let $p \in G \setminus \{z\}$. We have $x' = z(x)$ and $p' = z(p)$, in particular $(x, p) \equiv (x', p')$ by Theorem 4.3. By Lemma 4.1, $(z|p, p') = -1$, hence with the axiom of Pasch (PA) we may assume that the point $q =]p, y[\cap L$ exists (else $]p', y[\cap L \neq \emptyset$). Since $q \in L$ and p, q, y are collinear, by Theorem 3.7 $(p, q, y) \equiv (p'q'y')$. Also since $q \in L$, $(p, q, x) \equiv (p', q', x')$. With (W2) it follows $(x, y) \equiv (x', y')$. □

Lemma 4.5. *Let ϕ be a motion with exactly one fixed point z. If $\phi^2 = \mathrm{id}$, then $\phi = \widetilde{z}$.*

Proof: For any point $x \neq z$ and $x' := \phi(x)$ we have $(z, x, x') \equiv (z, x', \phi^2(x)) \equiv (z, x', x)$, in particular $(z, x) \equiv (z, x')$. For $z' = \overline{x, x'}\,(z)$ we have $(z, z', x) \equiv (z, z', x')$ and by (WK) $m = \overline{x, x'} \cap \overline{z, z'}$ exists, and by (W2) m is the midpoint of x, x'. Let $m' := \phi(m) \in \overline{x', \phi^2(x)} = \overline{x', x}$. Then $(x, x', m) \equiv (x', \phi^2(x), m') \equiv (x', x, m')$ implies $(x, m') \equiv (x', m')$, i.e. $m = m'$ by Lemma 3.2. Since z is the only fixed point, we have $m = z$ and $x' = z(x)$. $\square$

Theorem 4.6. *Let a, b, c be collinear points with $a \neq b$ and assume that the midpoint m of c and $\widetilde{a}\widetilde{b}(c)$ exists.*
Then $\widetilde{a}\widetilde{b}\widetilde{c}$ has exactly the fixed point m and $\widetilde{a}\widetilde{b}\widetilde{c} = \widetilde{m}$.

Proof: For $L = \overline{a, b}$, $\widetilde{a}\widetilde{b}(c), m \in L$. By the definition of m, $\widetilde{m}\widetilde{a}\widetilde{b}(c) = c$. Let $\phi := \widetilde{m}\widetilde{a}\widetilde{b}$. Assume that ϕ has another fixed point on L, then by Lemma 3.4, $\phi_{|L} = \mathrm{id}_{|L}$ and $\phi(m) = m$ implies $\widetilde{a}(m) = \widetilde{b}(m)$, i.e. $b = a$, a contradiction. Assume now that there is a point $y \in P \setminus L$ with $\phi(y) = y$. Then $y' = a(y)$ and $y'' = b(a(y))$ are not collinear and a and b and m, respectively, are the midpoints of y, y' and y', y'', and y'', y, respectively. By Lemma 4.2(i), $(a|y, y') = (b|y', y'') = (m|y'', y) = -1$, by the axiom of Pasch (PA) a contradiction to m, a, b collinear. Therefore c is the only fixed point of ϕ.
Now let E be any plane with $L \subset E$. By definition of point reflections $\phi(E) = E$ and clearly $\phi(L) = L$. Now Lemma 3.5 shows $\phi^2_{|E} = \mathrm{id}_{|E}$ for any plane E containing L, and therefore $\phi^2 = \mathrm{id}$. Now $\phi = \widetilde{c}$ and $\widetilde{a}\widetilde{b}\widetilde{c} = \widetilde{m}$ by Lemma 4.5. $\square$

Lemma 4.7. *Let a, b be points. Then $\widetilde{a}\widetilde{b}\widetilde{a} = \widetilde{a(b)}$ is a point reflection.*

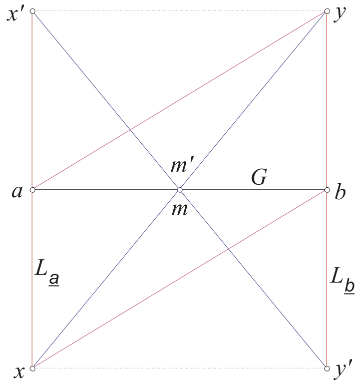
Proof: Because $\widetilde{a}$ is a motion, $(b, a, b(a)) \equiv (a(b), a, a(b(a)))$, and by the definition of a point reflection, $(b, a) \equiv (b, b(a))$. Therefore $(a(b), a) \equiv (a(b), a(b(a)))$, and $a(b)$ is the midpoint of $(a, a(b(a)))$. Now for $c = a$ and $m = a(b)$ by Theorem 4.6, $\widetilde{a}\widetilde{b}\widetilde{a} = \widetilde{m}$. $\square$

In the following we assume an absolute space $(P, \mathfrak{L}, \equiv, \alpha)$, i.e., we assume in addition the property (WF). In (16.11) of [6] with different assumption it is proved that any two points have a midpoint. In (5.4) of [13] and [15] it is shown for a plane that the axioms given here imply the assumptions of [6]. We give here a complete proof for an absolute space $(P, \mathfrak{L}, \equiv, \alpha)$.

Lemma 4.8. *For distinct points a, b, for a line L and for $x \in L$, there exists $y \in L$ with $(a, b) \equiv (x, y)$.*

Proof: We may assume $x \notin \overline{a, b}$ and $L \neq X := \overline{a, x}$. By (WF) there exists $c \in X$ with $(a, b) \equiv (a, c)$. By (W1), $d \in X$ exists with $(a, x, c) \equiv (x, a, d)$, and again (WF) shows the existence of $y \in L$ with $(x, d) \equiv (x, y)$. $\square$

Theorem 4.9. *Any two distinct points $a, b \in P$ have exactly one midpoint $m \in \overline{a, b}$.*



Proof: Let L_a and L_b be coplanar lines with $a \in L_a$, $b \in L_b$ and $L_a, L_b \perp G := \overline{a, b}$ (cf. 4.2(iii)). For $x \in L_a \setminus \{a\}$ let $x' = G(x) \in L_a$, $y \in L_b$ with $(a, x) \equiv (b, y)$ (cf. 4.8) and let $y' = G(y)$, in particular $(x, y') \equiv (x', y)$ and $(x, y) \equiv (x', y')$.

By (W1), $y'' \in L_b$ with $(a, x, x') \equiv (b, y, y'')$ exists and 4.2(ii) implies $y' = y''$, hence $(x, x') \equiv (y, y')$. Now from $(a, x, x') \equiv (b, y, y')$ and $(y, y', x) \equiv (x, x', y)$ it follows by (W2) also $(a, y) \equiv (b, x)$.

By 4.2(i), $(b|y, y') = -1$ and by axiom (PA), $G \cap \overline{x, y} \neq \emptyset$ or $G \cap \overline{x, y'} \neq \emptyset$. We may assume $m := G \cap \overline{x, y}$. By (W1) there exists $m' \in G$ with $(a, b, m) \equiv (b, a, m')$ and since $(a, b, y) \equiv (b, a, x)$ we have by (W2) $(x, m) \equiv (y, m')$ and $(x, m') \equiv (y, m)$. Therefore by Lemma 3.1(i), $(x, y, m) \equiv (x, y, m')$ implies $m = m'$, i.e., $(a, m) \equiv (b, m)$. By Lemma 3.2 there exists only one midpoint. $\square$

Since in an absolute space any two points have a midpoint, the point reflections act regularly on the point set P and by Theorem 4.6 the product of the point reflections of any three collinear points is a point reflection again.

5. Addition of points

In this section we assume an absolute space $(P, \mathfrak{L}, \alpha, \equiv)$. Using point reflections we introduce an addition of points. For an euclidian space this addition is exactly the vector addition of the corresponding vector space, but in general, for example for an ordered space with hyperbolic incidence structure, this addition is not associative.

First we recall some definitions. A set Q with a binary operation $\cdot$ is a *loop*, if for $a, b, c \in Q$ there are elements $x, y \in Q$ with $a \cdot x = c$ and $y \cdot b = c$, and if there is an neutral element $e \in Q$ with $x \cdot e = x = e \cdot x$ for all $x \in Q$.
A loop $(Q, \cdot)$ is a *Bol loop*, if for $a, b, c \in Q$ the (left) *Bol identity*

$$a(b \cdot ac) = (a \cdot ba)c$$

is satisfied. In a Bol loop every element $a \in Q$ has a unique inverse a^{-1} with $a \cdot a^{-1} = a^{-1} \cdot a = e$. Usually a Bol loop is called a *Bruck loop* or a *K-loop* (cf. [11], [9]), if the *inverse automorphic property*

$$(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$$

is satisfied. For the structure of a Bruck loop A. Ungar also introduced the name *gyrogroup* (cf. [12], [16], [17]).

For an absolute space $(P, \mathfrak{L}, \alpha, \equiv)$ we fix now a point $0 \in P$ and denote for any point $a \in P \setminus \{0\}$ the unique midpoint of 0 and a by $a/2$ (cf. Theorem 4.9). We denote $0 = 0/2$. Then for the point reflection corresponding to $a/2$ we have

$$\widetilde{a/2}(0) = a \quad \text{and} \quad \widetilde{a/2}(a) = 0.$$

We define for points a, b the addition $+$ on the point set P by

$$a + b := \widetilde{a/2} \circ \widetilde{0}(b) = \widetilde{a/2}\ \widetilde{0}(b).$$

Theorem 5.1. *$(P, +)$ is a loop with the neutral element 0. The point $-a := \widetilde{0}(a)$ is the inverse of $a \in P$.*
$(P, +)$ is associative if and only if for three points $a, b, c \in P$ the product $\widetilde{a}\,\widetilde{b}\,\widetilde{c}$ is a point reflection, too.

PROOF: Let $a, b, c \in P$ be given. Since $\widetilde{a/2}$ and $\widetilde{0}$ are bijections, there exists an unique point x with $\widetilde{a/2}\ \widetilde{0}(x) = c$, i.e., $a + x = c$. Let y' be the unique midpoint of $-b = \widetilde{0}(b)$ and c, i.e., $\widetilde{y'}(-b) = c$. For $y := \widetilde{y'}(0)$, i.e., $y' = y/2$ we have the unique solution y with $y + b = \widetilde{y/2}\,\widetilde{0}(b) = c$.
Clearly we have $a + 0 = \widetilde{a/2}(0) = a$ and $0 + a = \widetilde{0}\ \widetilde{0}(a) = a$. Also $a + (-a) = \widetilde{a/2}\ \widetilde{0}\ \widetilde{0}(a) = 0$ and $(-a) + a = \widetilde{-a/2}\ \widetilde{0}(a) = \widetilde{-a/2}(-a) = 0$, since $-a/2$ is the midpoint of $-a$ and 0.
Now we compute $a+(b+c) = \widetilde{a/2}\ \widetilde{0}\ \widetilde{b/2}\ \widetilde{0}(c)$ and $(a+b)+c = \widetilde{(a+b)/2}\ \widetilde{0}(c)$, hence the addition is associative if $\widetilde{a/2}\ \widetilde{0}\ \widetilde{b/2} = \widetilde{(a+b)/2}$. This is satisfied if and only if $\phi := \widetilde{a/2}\ \widetilde{0}\ \widetilde{b/2}$ is a point reflection, since then $\phi(0) = a + b$ implies $\phi = \widetilde{(a+b)/2}$. $\square$

We remark that in general for three non-collinear points $a, b, c \in P$ the product of the point reflections $\widetilde{a}\widetilde{b}\widetilde{c}$ is not a point reflection. For example if $(P, \mathfrak{L}, \alpha)$ is an ordered space with hyperbolic incidence structure. This is easy to see, if one uses the Klein model of a hyperbolic plane with the polar reflections as line reflections.

Lemma 5.2. *For points $a, b \in P$ it holds $\widetilde{0}\,\widetilde{a/2}\,\widetilde{0} = \widetilde{-a/2}$ and $\widetilde{a/2}\,\widetilde{0}\,\widetilde{b/2}\,\widetilde{0}\,\widetilde{a/2} = (a + \widetilde{(b+a)})/2$.*

PROOF: Obviously, the points $a/2, 0, a/2$ are collinear and by Lemma 4.7, $\phi = \widetilde{0}\,\widetilde{a/2}\,\widetilde{0}$ is a point reflection. Since $\phi(0) = -a$ it follows $\phi = \widetilde{-a/2}$. We compute $\psi(0) := \widetilde{a/2}\,\widetilde{0}\,\widetilde{b/2}\,\widetilde{0}\,\widetilde{a/2}(0) = \widetilde{a/2}\,\widetilde{0}(b+a) = a+(b+a)$, hence $\psi = (a + \widetilde{(b+a)})/2$, since $\beta = \widetilde{0}\,\widetilde{b/2}\,\widetilde{0}$ and $\phi = \widetilde{a/2}\,\beta\,\widetilde{a/2}$ are point reflections. □

Theorem 5.3. *$(P, +)$ is a Bruck loop and a K-loop.*

PROOF: Using Lemma 5.2 we compute $(-a) + (-b) = \widetilde{-a/2}\,\widetilde{0}(-b) = \widetilde{-a/2}(b) = \widetilde{0}\,\widetilde{a/2}\,\widetilde{0}(b) = \widetilde{0}(a+b) = -(a+b)$.
Also $a+(b+(a+c)) = \widetilde{a/2}\,\widetilde{0}\,\widetilde{b/2}\,\widetilde{0}\,\widetilde{a/2}(-c) = (a + \widetilde{(b+a)})/2(-c) = (a+(b+a))+c$. □

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Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, D–20146 Hamburg, Germany

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