On finite loops and their inner mapping groups

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Abstract. In this paper we consider finite loops and discuss the following problem: Which groups are (are not) isomorphic to inner mapping groups of loops? We recall some known results on this problem and as a new result we show that direct products of dihedral 2-groups and nontrivial cyclic groups of odd order are not isomorphic to inner mapping groups of finite loops.

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1. Introduction

If Q is a groupoid, then we say that Q is a loop if Q has a neutral element e and each of the two equations $ax = b$ and $ya = b$ has a unique solution for any $a, b \in Q$. For each $a \in Q$ we have two permutations L_a and R_a on Q defined by $L_a(x) = ax$ (left translation) and $R_a(x) = xa$ (right translation). The permutation group $M(Q)$ which is generated by all left and right translations is called the **multiplication group** of Q. Clearly, $M(Q)$ is a transitive permutation group on Q. The stabilizer of the neutral element e is denoted by $I(Q)$ and $I(Q)$ is called the **inner mapping group** of Q. These two notions link loop theory with group theory and what the author of this article is interested in, is the structures of $M(Q)$ and $I(Q)$ and their relation to the structure of Q. Thus, for example, we may ask which groups are isomorphic to multiplication groups of loops? Or, we may ask, under which conditions imposed on $M(Q)$ does it follow that Q is a solvable loop? In order to answer these questions (even partially) we have to look at the subgroups of $M(Q)$ and during our investigations it has become clear that the role of the subgroup $I(Q)$ is crucial.

Thus it is very important to know which groups can be (or cannot be) in the role of $I(Q)$. It is easy to see that $I(Q) = 1$ if and only if Q is an abelian group and in [6], [12] Kepka and Niemenmaa managed to show that $I(Q)$ is cyclic if and only if Q is an abelian group. We continued our investigations in [5], [9] and showed that $I(Q)$ cannot be isomorphic to $C \times D$, where C is a nontrivial finite cyclic group, D is a finite abelian group and $\gcd.(|C|, |D|) = 1$. In [10] it was shown by Niemenmaa that $I(Q)$ is never isomorphic to the direct product $C_{p^k} \times C_p$, where p is an odd prime number and $k \geq 2$. A neat little argument based on permutation group theory allows Drápal [3] to draw the conclusion that the inner mapping group of a loop is never a generalized group of quaternions.

The purpose of this paper is to investigate further the structure of $I(Q)$ and we now consider nonabelian groups which are direct products of dihedral 2-groups and nontrivial cyclic groups of odd order and we show that such groups can never be in the role of the inner mapping group of a finite loop.

As shown by Kepka and Niemenmaa [12], many properties of loops and their multiplication groups can be reduced to the properties of connected transversals in groups. Thus in Section 2 of this paper we introduce these transversals and some basic results on their properties. In Theorem 2.1 the reader is given a purely group theoretical characterization of multiplication groups of loops by using connected transversals. Section 2 also contains results on the solvability of finite groups, which are needed later in the proofs. In Section 3 we prove our main result: first in a purely group theoretic form (Theorem 3.4) and after that by using Theorem 2.1 we give a loop theoretic interpretation (Theorem 3.5). Finally, in Section 4, we discuss some open problems concerning the structure of abelian and nonabelian inner mapping groups of loops.

Our notation is standard and follows [4] and [12]. For those who are interested in the relation between finite solvable loops and their multiplication and inner mapping groups, we recommend the articles by Csőrgő et al. [2], Drápal [3], Myllylä [7], Niemenmaa [11] and Vesanen [14].

2. Loops and connected transversals

Now we assume that Q is a loop. We write $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$. Then the commutator subgroup $[A, B] \leq I(Q)$ and A and B are left transversals to $I(Q)$ in $M(Q)$. If $1 < K \leq I(Q)$, then K is not a normal subgroup of $M(Q)$. Finally, $M(Q) = \langle A, B\rangle$.

We then consider the situation in groups: Let H be a subgroup of G and let A and B be two left transversals to H in G. We say that A and B are H-connected if $[A, B] \leq H$. In fact, H-connected transversals are both left and right transversals ([12, Lemmas 2.1 and 2.2]). By H_G we denote the core of H in G, i.e. the largest normal subgroup of G contained in H. The relation between multiplication groups of loops and connected transversals is given by

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

For the proof, see [12, Theorem 4.1].

In the following two lemmas we assume that H is a subgroup of G and A and B are H-connected transversals in G.

Lemma 2.2. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

Lemma 2.3. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.*

For the proofs, see [12, Lemma 2.5 and Proposition 2.7]. We wish to point out that from Lemma 2.3 it follows that $K = K_G H$. In the following four lemmas we further assume that $G = \langle A, B\rangle$.

Lemma 2.4. *If H is a cyclic subgroup of G, then $G' \leq H$.*

Lemma 2.5. *Assume that G is a finite group and $H \cong C \times D$, where C is a nontrivial cyclic group and D is an abelian group. If $\gcd.(|C|, |D|) = 1$, then $H_G > 1$.*

Lemma 2.6. *If G is a finite group and H is abelian, then H is subnormal in G.*

Lemma 2.7. *If G is a finite group and H is a dihedral 2-group, then H is subnormal in G.*

For the proofs, see [6, Theorem 2.2], [9, Theorem 2.3], [13, Proposition 6.3] and [8, Theorem 4.1].

The following two lemmas show that in some cases the structure of H and the existence of H-connected transversals A and B force G to be a solvable group (the reader should observe that the condition $G = \langle A, B\rangle$ is not needed here).

Lemma 2.8. *If H is finite and abelian, then G is solvable.*

Lemma 2.9. *If H is a dihedral 2-group, then G is solvable.*

For the proofs, see [13, Theorem 4.1] and [8, Theorem 3.3].

We still need the following group theoretical result.

Lemma 2.10. *Let G be a finite group and H a nilpotent Hall subgroup of G and assume that H is not a Sylow subgroup of G. Assume further that for every prime p dividing $|H|$ and the respective Sylow p-subgroup P of H, $N_G(P) = H$ holds. Then there exists a normal subgroup N of G such that $G = NH$ and $N \cap H = 1$.*

For the proof, see [4, Theorem 7.3, p. 444–445].

3. Main theorems

Throughout this section we assume that G is a finite group, H is a subgroup of G and there exist H-connected transversals A and B in G.

Theorem 3.1. *If $H \cong D \times E$, where D is a dihedral 2-group and E is a nontrivial abelian group of odd order, then G is solvable.*

Proof: We argue by induction on the order of G. By using Lemmas 2.8 and 2.9 (or by induction), we may conclude that $H_G = 1$. If there exists a subgroup M of G such that $H < M < G$, then $M_G > 1$, by Lemma 2.3. As G/M_G is solvable and M is solvable by induction, the solvability of G follows. Thus we may assume that H is a maximal subgroup of G.

Now H is a nilpotent Hall subgroup of G and H is not a Sylow subgroup of G. If P is any Sylow subgroup of H, then $N_G(P) = H$. From Lemma 2.10 it follows that there is a normal subgroup N of G such that $G = NH$ and $N \cap H = 1$. Then let $x \in E$ be of prime order p and consider the group $T = N\langle x\rangle$. As $H_G = 1$, T is a Frobenius group with Frobenius complement $\langle x\rangle$. Thus T is solvable (see [4], p. 499), whence N and G are solvable. □

In the remaining group theoretical results of this section we assume that A and B are H-connected transversals in G and G is generated by $A \cup B$, i.e. $G = \langle A, B\rangle$.

Lemma 3.2. *If $H \cong D \times E$, where D and E are as in the previous theorem, then H is subnormal in G.*

Proof: Let G be a counterexample of smallest possible order. From this and from Lemmas 2.6 and 2.7 it follows that $H_G = 1$. By [12, p. 113], $1 \in A \cap B$. We now divide the proof into two parts.

1) Assume that H is a maximal subgroup of G. As G is solvable, by Theorem 3.1, G has a minimal normal subgroup N such that N is an elementary abelian p-group. Of course, p does not divide $|H|$, $G = NH$ and $N \cap H = 1$. Clearly, N is a Sylow subgroup of G and E is a Hall (or Sylow) subgroup of G.

If $1 \neq a \in A$, then $a = nh$, where $1 \neq n \in N$ and $h \in H$. If $h \notin D$, then $h = xy$, where $x \in D$ and $1 \neq y \in E$. It follows that $L = \langle a\rangle \cap E^g > 1$ for some $g \in G$. As $\langle a\rangle$ contains an element $1 \neq m \in N$ and L is thus normal in $\langle m, H^g\rangle = G$, we conclude that $H_G > 1$, a contradiction. It follows that $h \in D$ and thus $A \subseteq ND$. Similarly, we can show that $B \subseteq ND$, hence $G = \langle A, B\rangle \leq ND$, which is not possible.

2) Then assume that H is not a maximal subgroup of G. Let F be a proper subgroup of G such that H is a maximal subgroup of F. By Lemma 2.3, $F_G > 1$ and by induction (or by using Lemmas 2.6 and 2.7), $HF_G/F_G = F/F_G$ is subnormal in G/F_G and thus it follows that F is subnormal in G. If $V \neq F$ and H is a maximal subgroup of V, then also V is subnormal in G and $H = V \cap F$ is subnormal in G.

Thus we may assume that F is the only subgroup of G which has H as a maximal subgroup. Clearly, we can also assume that H is not normal in F, hence $N_G(H) = H$.

Let p be a prime divisor of $|H|$ and denote by P the Sylow p-subgroup of H. If G has a Sylow p-subgroup R such that $R > P$, then there exists $x \in R - H$ such that $P^x = P$. Then $N_G(P) \geq \langle H, x\rangle \geq F$, hence P is normal in F. As this cannot be true for every prime divisor of $|H|$ (and the respective Sylow subgroup of H), we conclude that H has a Sylow q-subgroup Q such that Q is also a Sylow q-subgroup of G. Furthermore, we may assume that $N_G(Q) = H$.

Now, if $H < T$ and T is a maximal subgroup of G, then T is subnormal and, in fact, normal in G. By Frattini-lemma (as Q is a Sylow q-subgroup of T), $G = TN_G(Q) = TH = T$, a contradiction. The proof is complete. $\square$

Lemma 3.3. *Let $H \cong D \times C$, where D is a dihedral 2-group and C is a nontrivial cyclic group of odd order. If $H_G = 1$, then $G' \leq N_G(H)$.*

PROOF: Let G be a minimal counterexample. As $H_G = 1$, it follows from Lemma 2.2 that $N_G(H) = H \times Z(G)$. By Lemma 3.2, $Z(G) > 1$. Let $z \in Z(G)$ and $|z| = r$, where r is a prime number. Then we consider the groups $\bar{G} = G/\langle z\rangle$ and $\bar{H} = H\langle z\rangle/\langle z\rangle$. If the core of $\bar{H}$ in $\bar{G}$ is trivial, then it follows that $G' \leq N_G(H\langle z\rangle)$. If the core is not trivial, then there exists a normal subgroup K of G such that $\langle z\rangle < K \leq H\langle z\rangle$. Now we look at the groups G/K and HK/K. By induction (or by using Lemma 2.4), we may conclude that $G' \leq N_G(HK) = N_G(H\langle z\rangle)$ with one exception: the case where $K = C \times \langle z\rangle$ has to be considered separately. As $H_G = 1$, we see that in this case $|C| = r$.

Then let Q be a Sylow 2-subgroup of G such that $D \leq Q$ and write $E = KN_G(Q) = (C \times \langle z\rangle)N_G(Q)$. Clearly, $H < E$. Let M be a maximal subgroup of G such that $E \leq M$. By Lemma 2.3, $M_G > 1$ and by using Lemma 2.6 or Lemma 2.7, $HM_G/M_G = M/M_G$ is subnormal in G/M_G. But then M is normal in G. By Frattini-lemma, $G = MN_G(Q) = M$, a contradiction. Thus we may assume that $E = G$, hence $G = KN_G(Q)$. If $g \in G$, then it is easy to see that $D^g \leq Q$. If we write $L = \langle D^g \mid g \in G\rangle$, then L is a normal subgroup of G contained in Q. Furthermore, HL/L is cyclic. Therefore, by Lemma 2.4, $G' \leq HL$. Now HL is normal in G and as K is normal in G, we conclude that $HL \cap K = C$ is normal in G, a contradiction.

Thus we may assume that in any case $G' \leq N_G(H\langle z\rangle) = S$. Clearly, $N_G(H) = H \times Z(G) \leq S$ and S is normal in G. If $r \neq 2$ and r does not divide $|C|$, then D and C are characteristic subgroups of $H\langle z\rangle$, whence $N_G(H) \geq S$. But then $G' \leq N_G(H)$.

Then assume that $r = 2$. Clearly, C is normal in S and as $H\langle z\rangle$ is normal in S, we have $S' \leq H\langle z\rangle$. Since $H_G = 1$, we conclude that S' is a 2-group. If P is a Sylow 2-subgroup of S, then $P \geq S'$ and P is normal in S. Thus P is normal in G and $P \geq D \times \langle z\rangle$. Consider the groups G/P and $HP/P \cong C$. By Lemma 2.4, $G' \leq HP = P \times C$. Now $P \times C$ is normal in G and we may conclude that C is normal in G, a contradiction.

Finally assume that $|z| = r$ divides $|C|$. Then D is normal in S and $S' \leq H\langle z\rangle$. In this case we also see that S' is an r-group. Denote by Π the set of those odd prime numbers that divide $|C|$. If R is a Hall Π-subgroup of S (the existence of such a Hall subgroup is guaranteed by the solvability of G and S), then $S' \leq R$, R is normal in S and $C \times \langle z\rangle \leq R$. Of course, R is normal in G. Let Q be a Sylow 2-subgroup of G such that $D \leq Q$. We write $E = RN_G(Q)$ and now we may proceed as in the second part of this proof. It follows that $G = E$ and if we write $L = \langle D^g \mid g \in G\rangle$, then $L \le Q$, HL is normal in G and $HL \cap R = C$ is normal in G, again a contradiction. This completes the proof. □

Theorem 3.4. *Let $H \cong D \times C$, where D and C are as in Lemma 3.3. Then $H_G > 1$.*

PROOF: If $|D| = 4$, then D is abelian and our claim follows from Lemma 2.5. Thus we may assume that D is a nonabelian dihedral 2-group (then $|D| = 2^n$, where $n \ge 3$).

Then assume that $H_G = 1$. By using Lemma 2.2 and the previous lemma, we have $G' \le N_G(H) = H \times Z(G)$. Then obviously $H \times Z(G)$ is normal in G. If $|Z(G)|$ is odd, then D is a normal subgroup of G contradicting $H_G = 1$. Thus we may assume that $Z(G)$ has a nontrivial Sylow 2-subgroup T. As $F = D \times T$ is a Sylow 2-subgroup of $H \times Z(G)$, it follows that F is normal in G. But then $F' = D' \times T' = D'$ is a nontrivial normal subgroup of G and F' is contained in H. This is a contradiction and therefore we may conclude that $H_G > 1$. □

After having proved the preceding results which are purely group theoretical, we can now consider the structure of finite loops and their inner mapping groups. By combining Theorems 2.1 and 3.4, we get

Theorem 3.5. *Let Q be a finite loop. Then $I(Q) \cong D \times C$, where D is a dihedral 2-group and C is a nontrivial cyclic group of odd order, is not possible.*

4. Final remarks and open problems

As loops are generalizations of groups, it is often very useful to see what loop theoretical results mean in group theory. If Q is a group, then $I(Q)$ is, in fact, the group containing all inner automorphisms of Q (denoted by $\mathrm{Inn}(Q)$ in group theory). The question which finite abelian groups are possible as inner automorphism groups of groups was completely solved by Baer [1]. The result is as follows:

Theorem 4.1. *Let G be a finite abelian group and let $G = C_1 \times \cdots \times C_n$ be the direct product of cyclic groups such that $|C_{i+1}|$ divides $|C_i|$ $(i = 1, \ldots, n-1)$. Then there exists a group H such that $\mathrm{Inn}(H) \cong G$ if and only if $n \ge 2$ and $|C_1| = |C_2|$.*

In our opinion the situation in loop theory, as concerns the structure of finite abelian inner mapping groups, is similar to the situation in group theory and therefore we introduce

Conjecture 1. *If Q is a loop and $I(Q) = C_1 \times \cdots \times C_n$ is a finite abelian group (written as in Theorem 4.1), then $n \ge 2$ and $|C_1| = |C_2|$.*

The reader should observe that the main result in [10] is a first step towards proving this conjecture in the case of abelian p-groups.

In the case of finite nonabelian groups we formulate a conjecture which reads as follows:

Conjecture 2. *Assume that H is a finite nonabelian group such that $H \cong I(Q)$ for some loop Q. Then $H \times C$, where C is a nontrivial finite cyclic group and* $\gcd.(|H|, |C|) = 1$*, is not isomorphic to the inner mapping group of a loop.*

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