The hyperbolic triangle centroid

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Abstract. Some gyrocommutative gyrogroups, also known as Bruck loops or K-loops, admit scalar multiplication, turning themselves into gyrovector spaces. The latter, in turn, form the setting for hyperbolic geometry just as vector spaces form the setting for Euclidean geometry. In classical mechanics the centroid of a triangle in velocity space is the velocity of the center of momentum of three massive objects with equal masses located at the triangle vertices. Employing gyrovector space techniques we find in this article that, in full analogy, the centroid of a hyperbolic triangle in relativity velocity space is the velocity of the center of momentum of three massive objects with equal rest masses located at the triangle vertices. Being guided by the relativistic mass correction of moving massive objects in special relativity theory, we express the hyperbolic triangle centroid in terms of the triangle vertices, resulting in a novel hyperbolic triangle centroid identity that captures remarkable analogies with its Euclidean counterpart.

Keywords: loops, gyrogroups, gyrovector spaces, hyperbolic geometry, Einstein addition, Möbius transformation

Classification: 20N05, 51P05, 83A05

1. Introduction

Einstein addition of relativistic velocities in special relativity theory and Möbius addition in the theory of complex functions are isomorphic binary operations in a grouplike structure that turns out to be a loop. Einstein addition is straightforwardly generalized in Section 5 into a binary operation $\oplus_{\mathrm{E}}$ in the open unit ball $\mathbb{B}$ of any real inner product space $\mathbb{V}$, giving rise to the Einstein groupoid $(\mathbb{B}, \oplus_{\mathrm{E}})$, presented in Section 5. We recall that a groupoid is a nonempty set with a binary operation. Möbius addition in the complex open unit disc is a Möbius transformation of the disc without rotation. It is generalized in [21] into a binary operation, $\oplus_{\mathrm{M}}$, in the open unit ball $\mathbb{B}$ of any real inner product space $\mathbb{V}$. It gives rise to the Möbius groupoid $(\mathbb{B}, \oplus_{\mathrm{M}})$, presented in Section 3.

Both Möbius addition and Einstein addition are neither commutative nor associative. Accordingly, the groupoids of Einstein and Möbius do not form groups. These two groupoids share a common grouplike structure called a *gyrocommutative gyrogroup*. The latter turns out to be equivalent to the Bruck loop. Unlike the Bruck loop definition [12], however, the gyrocommutative gyrogroup definition, Section 2, emphasizes analogies with groups.

Endowing a gyrocommutative gyrogroup with scalar multiplication and inner product, one obtains a gyrovector space. Remarkably, gyrovector spaces form the setting for hyperbolic geometry just as vector spaces form the setting for Euclidean geometry. In particular, the resulting Möbius and Einstein gyrovector spaces are studied in this article in order to set the stage for the study of hyperbolic geometry analytically, allowing the hyperbolic triangle centroid to be determined analytically in Sections 6 and 7. An earlier study of the hyperbolic triangle centroid is found in [2].

Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry, as shown in Sections 3 and 4. Similarly, Einstein gyrovector spaces form the setting for the Beltrami (also known as Klein) ball model of hyperbolic geometry, as shown in Sections 5 and 6.

The study of hyperbolic geometry in terms of its two isomorphic models of Poincaré and Beltrami is particularly useful. On the one hand, the Poincaré model is conformal, so that hyperbolic angles between intersecting geodesics have the same measure as Euclidean angles between corresponding intersecting tangent lines [23, Figure 6.14]. On the other hand, geodesics in the Beltrami model are Euclidean straight lines, allowing one to employ linear algebra in the determination of points of intersection of geodesics. Indeed, in this article we use linear algebra to obtain the triangle centroid as the point of intersection of the triangle medians in the Beltrami model, Figure 3, and translate the result into the Poincaré model, Figure 4.

2. Gyrogroups and gyrovector spaces

Definition 1 (Gyrogroups)**.** The groupoid $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

(G1) $\quad 0 \oplus a = a \qquad$ Left Identity

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each a in G there is an element $\ominus a$ in G, called a left inverse of a, with

(G2) $\quad \ominus a \oplus a = 0. \qquad$ Left Inverse

Moreover, for any $a, b, z \in G$ there exists a unique element $\mathrm{gyr}[a, b]z \in G$ such that

(G3) $\quad a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z. \qquad$ Left Gyroassociative Law

If $\mathrm{gyr}[a, b]$ denotes the map $gyr[a, b] : G \to G$ given by $z \mapsto \mathrm{gyr}[a, b]z$ then

(G4) $\quad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus) \qquad$ Gyroautomorphism

and $\mathrm{gyr}[a, b]$ is called the *Thomas gyration*, or the *gyroautomorphism* of G, *generated* by $a, b \in G$. The operation $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called the gyrooperation of G. Finally, the gyroautomorphism $\mathrm{gyr}[a,b]$ generated by any $a, b \in G$ satisfies

(G5) $\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]$. Left Loop Property

Various gyrogroup theorems are presented in [23]. Thus, for instance, any gyrogroup possesses a right identity and a right inverse as well, which are identical to their left counterparts. Furthermore, the resulting identity element and the inverse of any given element are unique. In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 2 (Gyrocommutative gyrogroups)**.** The gyrogroup $(G, \oplus)$ is *gyrocommutative* if for all $a, b \in G$

(G6) $a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$. Gyrocommutative Law

A gyrogroup is a loop [23], and the gyrocommutative gyrogroup is equivalent to the Bruck loop. Following Ungar [12], [20], the latter is also known as a K-loop. Furthermore, a gyrocommutative gyrogroup is also a special Bol loop that possesses the automorphic inverse property, $\ominus(a \oplus b) = \ominus a \ominus b$ [18], [10]. Gyrocommutative gyrogroups result from transversals to subgroups, as shown in [8], [9]. Indeed, transversals to subgroups [4] and transversals in loops [14] are important in loop theory.

Definition 3 (Inner product gyrovector spaces)**.** A(n inner product) gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that admits:

(1) Inner product , $\cdot$, (i) which gives rise to a positive definite norm $\|\mathbf{v}\|$, that is, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$, $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$, $|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$; and (ii) which is invariant under gyroautomorphisms, that is,

$$\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{u}\cdot \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$$

for all gyrovectors $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$;

(2) Scalar multiplication, $\otimes$, satisfying the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all gyrovectors $\mathbf{a}, \mathbf{b}, \mathbf{v} \in G$:

(V1) $1 \otimes \mathbf{v} = \mathbf{v}$,

(V2) $(r_1 + r_2) \otimes \mathbf{v} = r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}$, Scalar Distributive Law

(V3) $(r_1 r_2) \otimes \mathbf{v} = r_1 \otimes (r_2 \otimes \mathbf{v})$, Scalar Associative Law

(V4) $\dfrac{|r| \otimes \mathbf{v}}{\|r \otimes \mathbf{v}\|} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, Scaling Property

(V5) $\mathrm{gyr}[\mathbf{a},\mathbf{b}](r \otimes \mathbf{v}) = r \otimes \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}$, Gyroautomorphism Property

(V6) $\mathrm{gyr}[r_1 \otimes \mathbf{v}, r_2 \otimes \mathbf{v}] = I$; Identity Automorphism

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional 'vectors'

$$\|G\| = \{\pm\|\mathbf{v}\| : \mathbf{v} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in G$,

(V7) $\|r\otimes\mathbf{v}\| = |r|\otimes\|\mathbf{v}\|$, Homogeneity Property

(V8) $\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\|$. Gyrotriangle inequality

A gyrovector space $G = (G, \oplus, \otimes)$ is a gyrometric space, with gyrometric given by the distance function

$$d(\mathbf{u}, \mathbf{v}) = \|\ominus\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\ominus\mathbf{u}\| \tag{1}$$

satisfying the gyrotriangle inequality

$$\|\ominus\mathbf{u}\oplus\mathbf{w}\| \leq \|\ominus\mathbf{u}\oplus\mathbf{v}\|\oplus\|\ominus\mathbf{v}\oplus\mathbf{w}\| \tag{2}$$

verified below.

By a gyrogroup identity [23] we have

$$\ominus\mathbf{u}\oplus\mathbf{w} = (\ominus\mathbf{u}\oplus\mathbf{v})\oplus \operatorname{gyr}[\mathbf{u}, \ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{w}). \tag{3}$$

Hence, by the gyrotriangle inequality $(V8)$ we have

$$\begin{aligned} \|\ominus\mathbf{u}\oplus\mathbf{w}\| &= \|(\ominus\mathbf{u}\oplus\mathbf{v})\oplus \operatorname{gyr}[\mathbf{u}, \ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{w})\| \\ &\leq \|\ominus\mathbf{u}\oplus\mathbf{v}\|\oplus\| \operatorname{gyr}[\mathbf{u}, \ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{w})\| \\ &= \|\ominus\mathbf{u}\oplus\mathbf{v}\|\oplus\|\ominus\mathbf{v}\oplus\mathbf{w}\|. \end{aligned} \tag{4}$$

Our ambiguous use of $\oplus$ and $\otimes$, Definition 3, as operations in the gyrovector space $(G, \oplus, \otimes)$ and in the vector space $(\|G\|, \oplus, \otimes)$ should raise no confusion, since the sets in which these operations operate are always clear from the context. The operations in the former are nonassociative-nondistributive gyrovector space operations, and in the latter are associative-distributive vector space operations. Additionally, the gyro-addition $\oplus$ is gyrocommutative in the former and commutative in the latter.

An inner product gyrovector space possesses a weak form of a distributive law,

$$r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) = r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v}) \tag{5}$$

called the *monodistributive law*, which follows from $(V2)$ and $(V3)$,

$$\begin{aligned} r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) &= r\otimes\{(r_1 + r_2)\otimes\mathbf{v}\} \\ &= (r(r_1 + r_2))\otimes\mathbf{v} \\ &= (rr_1 + rr_2)\otimes\mathbf{v} \\ &= (rr_1)\otimes\mathbf{v}\oplus(rr_1)\otimes\mathbf{v} \\ &= r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_1\otimes\mathbf{v}). \end{aligned} \tag{6}$$

3. Möbius gyrovector spaces

Definition 4 (Möbius addition)**.** Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{B} = \{\mathbf{v}\in\mathbb{V} : \|\mathbf{v}\| < 1\}$ be the open unit ball of $\mathbb{V}$. Möbius addition $\oplus_{\mathrm{M}}$ in the ball $\mathbb{B}$ is a binary operation in $\mathbb{B}$ given by the equation

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \frac{(1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{7}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{B}$ inherits from its space $\mathbb{V}$.

To justify calling $\oplus_{\mathrm{M}}$ in Definition 4 a Möbius addition we note that it is a natural extension of a special Möbius transformation of the complex open unit disc, as explained in [21], [23]. In earlier studies by Ahlfors [1] and Ratcliffe [17], Möbius addition is treated as a hyperbolic translation. Möbius translation became Möbius addition in [21] following the discovery of the analogies it shares, as a gyrocommutative gyrogroup operation, with ordinary vector addition. Applications of Möbius addition and its hyperbolic geometry in quantum mechanics are found in [3], [15], [16], [24].

The groupoid $(\mathbb{B}, \oplus_{\mathrm{M}})$ is a gyrocommutative gyrogroup, as demonstrated in [23], called a Möbius gyrogroup. Furthermore, it admits scalar multiplication $\otimes$, turning itself into the Möbius gyrovector space $(\mathbb{B}, \oplus_{\mathrm{M}}, \otimes)$.

Definition 5 (Möbius scalar multiplication)**.** Let $(\mathbb{B}, \oplus_{\mathrm{M}})$ be a Möbius gyrogroup. The Möbius scalar multiplication $r\otimes\mathbf{v} = \mathbf{v}\otimes r$ in $\mathbb{B}$ is given by the equation

$$\begin{aligned} r\otimes\mathbf{v} &= \frac{(1 + \|\mathbf{v}\|)^r - (1 - \|\mathbf{v}\|)^r}{(1 + \|\mathbf{v}\|)^r + (1 - \|\mathbf{v}\|)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \tanh(r\ \tanh^{-1}\|\mathbf{v}\|)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{8}$$

where $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{B}$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes\mathbf{0} = \mathbf{0}$.

As an example we present the Möbius half,

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v} \tag{9}$$

where $\gamma_{\mathbf{v}} = (1 - \|\mathbf{v}\|^2)^{-1/2}$. In accordance with the scalar associative law of gyrovector spaces we have

$$\begin{aligned} 2\otimes(\tfrac{1}{2}\otimes\mathbf{v}) &= 2\otimes\frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v} \\ &= \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v}\oplus_{\mathrm{M}}\frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v} \\ &= \mathbf{v}. \end{aligned} \tag{10}$$

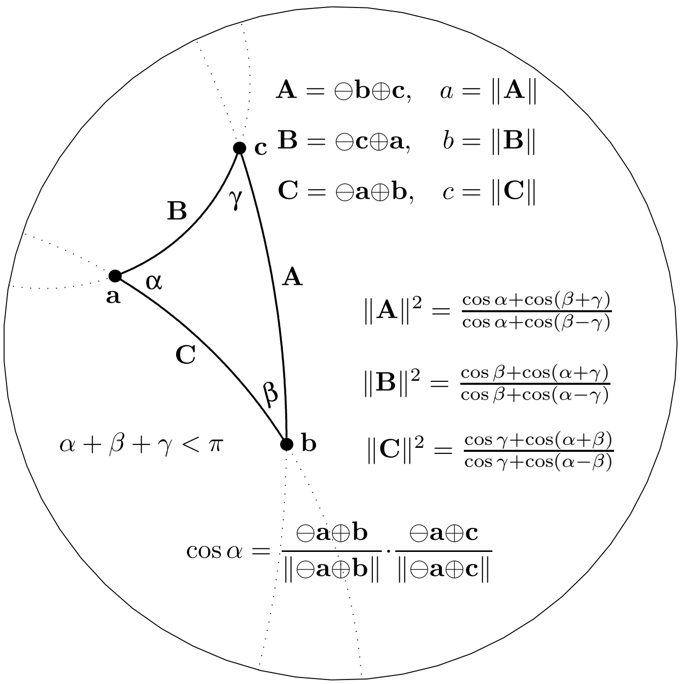

Figure 1: A Möbius triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector plane $(\mathbb{B}, \oplus, \otimes)$, $\oplus = \oplus_{\text{M}}$ is shown. Its sides are formed by geodesic segments that link its vertices $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, having the hyperbolic lengths a, b and c. The cosine of its angles are given by an identity that is fully analogous to its Euclidean counterpart. The Möbius gyrovector plane forms the setting for the Poincaré disc model of hyperbolic geometry just as the common vector plane forms the setting for the standard model of Euclidean plane geometry [13],[23]. Unlike the Euclidean triangle, the sides of the hyperbolic triangle are uniquely determined by its angles.

4. The Poincaré ball model of hyperbolic geometry

Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry, as demonstrated in [13], [23], just as vector spaces form the setting for the standard model of Euclidean geometry. Thus, the unique geodesic passing through the points $\mathbf{a}, \mathbf{b} \in \mathbb{B}$ in a Möbius gyrovector space $(\mathbb{B}, \oplus_{\text{M}}, \otimes)$ is given by the equation

$$\mathbf{a}\oplus_{\text{M}}(\ominus_{\text{M}}\mathbf{a}\oplus_{\text{M}}\mathbf{b})\otimes t \tag{11}$$

with the real parameter $t \in \mathbb{R}$. It passes through the point $\mathbf{a}$ at "time" $t = 0$ and, owing to the left cancellation law of gyrogroup theory, it passes through the point $\mathbf{b}$ at "time" $t = 1$. Several geodesics in the Poincaré disc model, generated by (11), are shown in Figures 1, 2 and 4.

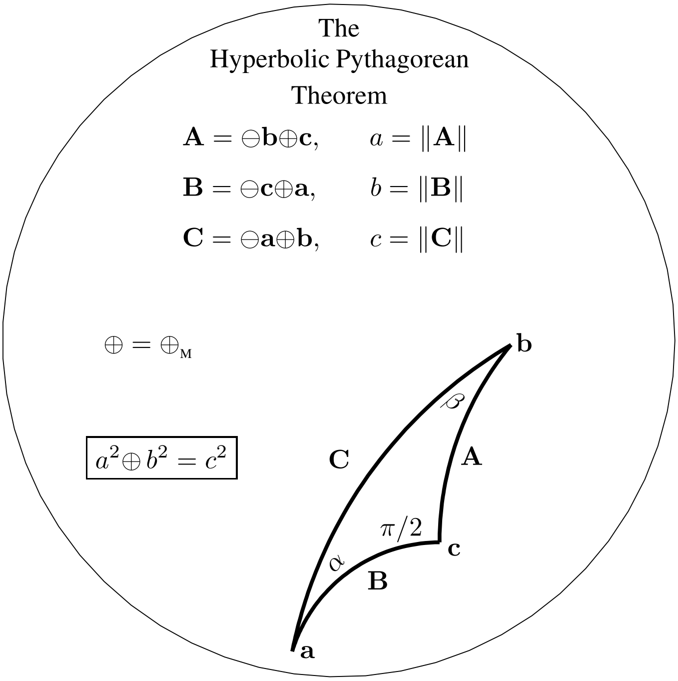

Figure 2: The Hyperbolic Pythagorean Theorem in the Poincaré disc model of hyperbolic geometry that is, equivalently, in the Möbius gyrovector plane $(\mathbb{B}, \oplus, \otimes)$, $\oplus = \oplus_{\mathrm{M}}$ [22],[23].

The hyperbolic midpoint $\mathbf{m}^{P}_{\mathbf{ab}}$ of $\mathbf{a}$ and $\mathbf{b}$ in the Poincaré model is obtained from (11) by selecting $t = 1/2$. It is the midpoint in the sense that $d(\mathbf{a}, \mathbf{m}^{P}_{\mathbf{ab}}) = d(\mathbf{b}, \mathbf{m}^{P}_{\mathbf{ab}})$ where $d(\mathbf{a}, \mathbf{b})$ is the hyperbolic distance function in the Poincaré model, given by the equation $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a}\ominus_{\mathrm{M}}\mathbf{b}\|$.

The cosine of the hyperbolic angle generated by two geodesics passing, respectively, through the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}, \mathbf{c}$ in the Möbius gyrovector space $(\mathbb{B}, \oplus_{\mathrm{M}}, \otimes)$, Figure 1, is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus_{\mathrm{M}}\mathbf{b}}{\|\ominus\mathbf{a}\oplus_{\mathrm{M}}\mathbf{b}\|} \cdot \frac{\ominus\mathbf{a}\oplus_{\mathrm{M}}\mathbf{c}}{\|\ominus\mathbf{a}\oplus_{\mathrm{M}}\mathbf{c}\|} \tag{12}$$

in full analogy with its Euclidean counterpart.

Three geodesic segments that form a triangle, and the triangle angles in the Möbius gyrovector plane are shown in Figure 1. The hyperbolic Pythagorean theorem in the Möbius gyrovector plane is shown in Figure 2. It shares visual analogies with its Euclidean counterpart, demonstrating the union of hyperbolic and Euclidean geometry [25]. The task we face in this article is to determine the centroid of the hyperbolic triangle in Figure 1.

5. Einstein gyrovector spaces

Definition 6 (Einstein addition)**.** Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{B} = \{\mathbf{v}\in\mathbb{V} : \|\mathbf{v}\| < 1\}$ be the open unit ball of $\mathbb{V}$. Einstein addition $\oplus_{\mathrm{E}}$ in the ball $\mathbb{B}$ is a binary operation in $\mathbb{B}$ given by the equation ([5], [6], [7], [19], [23], [3], [26])

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\mathbf{u}\cdot\mathbf{v}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{13}$$

where the vacuum speed of light is normalized to $c = 1$, where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{B}$ inherits from its space $\mathbb{V}$, and where $\gamma_{\mathbf{v}}$ is the Lorentz factor of $\mathbf{v}$,

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\|\mathbf{v}\|^2}}\,. \tag{14}$$

The groupoid $(\mathbb{B}, \oplus_{\mathrm{E}})$ is a gyrocommutative gyrogroup, as demonstrated in [23], called an Einstein gyrogroup. Furthermore, it admits scalar multiplication $\otimes$, turning itself into an Einstein gyrovector space $(\mathbb{B}, \oplus_{\mathrm{E}}, \otimes)$.

Definition 7 (Einstein scalar multiplication)**.** Let $(\mathbb{B}, \oplus_{\mathrm{E}})$ be an Einstein gyrogroup. The Einstein scalar multiplication $r\otimes\mathbf{v} = \mathbf{v}\otimes r$ in $\mathbb{B}$ is given by the equation

$$\begin{aligned} r\otimes\mathbf{v} &= \frac{(1+\|\mathbf{v}\|)^r - (1-\|\mathbf{v}\|)^r}{(1+\|\mathbf{v}\|)^r + (1-\|\mathbf{v}\|)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \tanh(r\ \tanh^{-1}\|\mathbf{v}\|)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{15}$$

where $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{B}$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes\mathbf{0} = \mathbf{0}$.

Useful identities that relate the Einstein scalar multiplication to the Lorentz factor (14) are [23, Chapter 3]

$$\gamma_{r\otimes\mathbf{v}} = \frac{1}{2}\gamma_{\mathbf{v}}^r\left\{(1+\|\mathbf{v}\|)^r + (1-\|\mathbf{v}\|)^r\right\} \tag{16}$$

and

$$\gamma_{r\otimes\mathbf{v}}r\otimes\mathbf{v} = \frac{1}{2}\gamma_{\mathbf{v}}^r\left\{(1+\|\mathbf{v}\|)^r - (1-\|\mathbf{v}\|)^r\right\}\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{17}$$

for $\mathbf{v} \neq \mathbf{0}$, of which the special case of $r = 2$ is of particular interest in this article,

(18) $$\gamma_{2\otimes\mathbf{v}} = 2\gamma_{\mathbf{v}}^2 - 1$$

and

(19) $$\gamma_{2\otimes\mathbf{v}} 2\otimes\mathbf{v} = 2\gamma_{\mathbf{v}}^2\mathbf{v}.$$

Interestingly, the scalar multiplication that Möbius and Einstein addition admit coincide. This is compatible with the fact that for parallel vectors in the ball, Möbius addition and Einstein addition coincide as well.

6. The Beltrami ball model of hyperbolic geometry

Einstein gyrovector spaces form the setting for the Beltrami ball model of hyperbolic geometry just as vector spaces form the setting for the standard model of Euclidean geometry [23]. Thus, the unique geodesic passing through the points $\mathbf{a}, \mathbf{b} \in \mathbb{B}$ in an Einstein gyrovector space $(\mathbb{B}, \oplus_{\mathrm{E}}, \otimes)$ is given by the equation

(20) $$\mathbf{a}\oplus_{\mathrm{E}}(\ominus_{\mathrm{E}}\mathbf{a}\oplus_{\mathrm{M}}\mathbf{b})\otimes t$$

with the real parameter $t \in \mathbb{R}$. It passes through the point $\mathbf{a}$ at "time" $t = 0$ and, owing to the left cancellation law of gyrogroup theory, it passes through the point $\mathbf{b}$ at "time" $t = 1$. Several geodesics in the Beltrami disc model, generated by (20), are shown in Figure 3.

The hyperbolic midpoint $\mathbf{m}_{\mathbf{ab}}^B$ of $\mathbf{a}$ and $\mathbf{b}$ in the Beltrami model is obtained from (20) by selecting $t = 1/2$. It is the midpoint in the sense that $d(\mathbf{a}, \mathbf{m}_{\mathbf{ab}}^B) = d(\mathbf{b}, \mathbf{m}_{\mathbf{ab}}^B)$ where $d(\mathbf{a}, \mathbf{b})$ is the hyperbolic distance function in the Beltrami model, given by the equation $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a}\ominus_{\mathrm{E}}\mathbf{b}\|$.

Interestingly, the hyperbolic midpoint $\mathbf{m}_{\mathbf{ab}}^B$ can be written in terms of ordinary, rather than Einstein, vector addition as

(21) $$\mathbf{m}_{\mathbf{uv}}^B = \mathbf{a}\oplus_{\mathrm{E}}(\ominus_{\mathrm{E}}\mathbf{a}\oplus_{\mathrm{M}}\mathbf{b})\otimes\tfrac{1}{2} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}\,.$$

The derivation of (21) follows from [23, Equation 3.41] and [23, Equation 1.40].

The hyperbolic midpoint $\mathbf{m}_{\mathbf{uv}}^B$, (21), in the Beltrami ball model of hyperbolic geometry is interesting. It can be interpreted, Figure 3, as the Newtonian velocity of the center of momentum of two objects with relativistically corrected masses $m\gamma_{\mathbf{u}}$ and $m\gamma_{\mathbf{v}}$, that move respectively with velocities $\mathbf{u}$ and $\mathbf{v}$ relative to some inertial frame,

(22) $$\mathbf{m}_{\mathbf{uv}}^B = \frac{m\gamma_{\mathbf{u}}\mathbf{u} + m\gamma_{\mathbf{v}}\mathbf{v}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}\,.$$

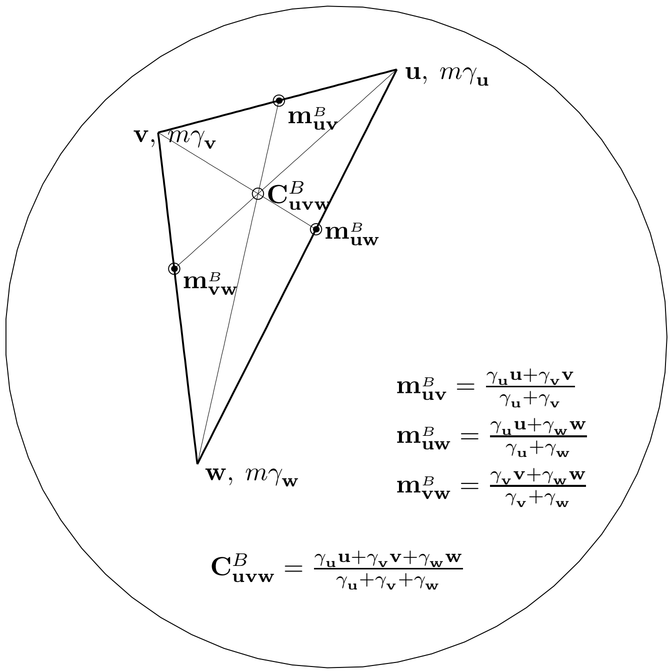

Figure 3: Hyperbolic midpoints, medians and a centroid of a triangle and its sides in the Beltrami model and its underlying Einstein gyrovector space. They possess a relativistic mechanical interpretation, analogous to the classical mechanical interpretation of their Euclidean counterparts in [11],[26].

The two masses $m\gamma_{\mathbf{u}}$ and $m\gamma_{\mathbf{v}}$ in (22), shown in Figure 3, are just the common relativistic masses of two objects with equal rest masses m and respective relative velocities $\mathbf{u}$ and $\mathbf{v}$.

Having identified the hyperbolic midpoint $\mathbf{m}^B_{\mathbf{uv}}$, (21), with the Newtonian velocity of the center of momentum of the two relativistically corrected masses, $m\gamma_{\mathbf{u}}$ and $m\gamma_{\mathbf{v}}$, (22), it is clear that the Newtonian velocity $\mathbf{m}^B_{\mathbf{uvw}}$ of the center of momentum of the three relativistically corrected masses $m\gamma_{\mathbf{u}}$, $m\gamma_{\mathbf{v}}$, and $m\gamma_{\mathbf{w}}$,

$$\mathbf{m}^B_{\mathbf{uvw}} = \frac{m\gamma_{\mathbf{u}}\mathbf{u} + m\gamma_{\mathbf{v}}\mathbf{v} + m\gamma_{\mathbf{w}}\mathbf{w}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}} + m\gamma_{\mathbf{w}}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}} \tag{23}$$

lies on the median connecting the point $\mathbf{m}^B_{\mathbf{uv}}$ to the point $\mathbf{w}$ of triangle $\Delta\mathbf{uvw}$ of Figure 3. By symmetry considerations, the Newtonian velocity $\mathbf{m}^B_{\mathbf{uvw}}$ of the center of momentum of the three relativistically corrected masses $m\gamma_{\mathbf{u}}$, $m\gamma_{\mathbf{v}}$, and $m\gamma_{\mathbf{w}}$ lies on the other two hyperbolic medians of triangle $\Delta\mathbf{uvw}$ as well. Hence, the point $\mathbf{m}^B_{\mathbf{uvw}}$ coincides with the centroid of the hyperbolic triangle $\Delta\mathbf{uvw}$, as shown in Figure 3. Hence, by elementary linear algebra, the hyperbolic centroid $\mathbf{C}^B_{\mathbf{uvw}}$ of the hyperbolic triangle $\Delta\mathbf{uvw}$ in the Beltrami ball model of hyperbolic geometry, Figure 3, is equal to the velocity $\mathbf{m}^B_{\mathbf{uvw}}$ in (23), that is,

$$\mathbf{C}^B_{\mathbf{uvw}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}}\,. \tag{24}$$

Formalizing our result in (24), and noting that an Einstein gyrovector space underlies the Beltrami ball model of hyperbolic geometry, we have the following

Theorem 8. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{B}$ be any three non-gyrocollinear points of a Beltrami ball model, $\mathbb{B}$, of hyperbolic geometry, where $\mathbb{B}$ is the ball of a real inner product space $\mathbb{V}$. The centroid $\mathbf{C}^B_{\mathbf{abc}}$ of the hyperbolic triangle $\Delta\mathbf{abc}$ in $\mathbb{B}$ is given by the equation*

$$\mathbf{C}^B_{\mathbf{avc}} = \frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{v} + \gamma_{\mathbf{c}}\mathbf{c}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}} + \gamma_{\mathbf{c}}}\,. \tag{25}$$

7. Triangle centroids in the Poincaré ball model of hyperbolic geometry

We wish, in this section, to translate the expression (25) of the centroid of a hyperbolic triangle in an Einstein gyrovector space and its associated Beltrami ball model of hyperbolic geometry into an expression describing the centroid of a hyperbolic triangle in a Möbius gyrovector space and its associated Poincaré ball model of hyperbolic geometry.

Let $G_e = (\mathbb{B}, \oplus_{\mathrm{E}}, \otimes)$ and $G_m = (\mathbb{B}, \oplus_{\mathrm{M}}, \otimes)$ be, respectively, the Einstein and the Möbius gyrovector spaces of the ball $\mathbb{B}$ of a real inner product space $\mathbb{V}$. They are gyrovector space isomorphic, with the isomorphism and its inverse isomorphism from G_m into G_e given by the equations [23]

$$\begin{aligned} \mathbf{v}_e &= 2\otimes\mathbf{v}_m, \\ \mathbf{v}_m &= \tfrac{1}{2}\otimes\mathbf{v}_e, \end{aligned} \tag{26}$$

$\mathbf{v}_e \in G_e$, $\mathbf{v}_m \in G_m$. Accordingly, the gyrogroup operations $\oplus_{\mathrm{E}}$ and $\oplus_{\mathrm{M}}$ in G_e and G_m are related to each other by the equation

$$\begin{aligned} \mathbf{u}_m\oplus_{\mathrm{M}}\mathbf{v}_m &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}_m\oplus_{\mathrm{E}}2\otimes\mathbf{v}_m), \\ \mathbf{u}_e\oplus_{\mathrm{E}}\mathbf{v}_e &= 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}_e\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes\mathbf{v}_e). \end{aligned} \tag{27}$$

Following (26) and (18) we have

$$\gamma_{\mathbf{v}_e} = \gamma_{2\otimes\mathbf{v}_m} = 2\gamma^2_{\mathbf{v}_m} - 1. \tag{28}$$

Similarly, following (26) and (19) we have

$$\gamma_{\mathbf{v}_e}\mathbf{v}_e = \gamma_{2\otimes\mathbf{v}_m}2\otimes\mathbf{v}_m = 2\gamma^2_{\mathbf{v}_m}\mathbf{v}_m\,. \tag{29}$$

Hence, by (21), (28) and (29) we have

$$\begin{aligned}\mathbf{m}^B_{\mathbf{u}_e\mathbf{v}_e} &= \frac{\gamma_{\mathbf{u}_e}\mathbf{u}_e + \gamma_{\mathbf{v}_e}\mathbf{v}_e}{\gamma_{\mathbf{u}_e} + \gamma_{\mathbf{v}_e}} \\ &= \frac{2\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + 2\gamma^2_{\mathbf{v}_m}\mathbf{v}_m}{(2\gamma^2_{\mathbf{u}_m} - 1) + (2\gamma^2_{\mathbf{v}_m} - 1)} \\ &= \frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} - 1}\end{aligned} \tag{30}$$

so that

$$\begin{aligned}\mathbf{m}^P_{\mathbf{u}_m\mathbf{v}_m} &= \tfrac{1}{2}\otimes\mathbf{m}^B_{\mathbf{u}_e\mathbf{v}_e} \\ &= \tfrac{1}{2}\otimes\frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} - 1}\,.\end{aligned} \tag{31}$$

We have thus obtained in (31) the following

Theorem 9. *Let* $\mathbf{a},\mathbf{b} \in \mathbb{B}$ *be any two points of a Poincaré ball model,* $\mathbb{B}$*, of hyperbolic geometry, where* $\mathbb{B}$ *is the ball of a real inner product space* $\mathbb{V}$*. The midpoint* $\mathbf{m}^P_{\mathbf{ab}}$ *of the hyperbolic segment* $\mathbf{ab}$ *joining the points* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbb{B}$ *is given by the equation*

$$\mathbf{m}^P_{\mathbf{ab}} = \tfrac{1}{2}\otimes\frac{\gamma^2_{\mathbf{a}}\mathbf{a} + \gamma^2_{\mathbf{b}}\mathbf{b}}{\gamma^2_{\mathbf{a}} + \gamma^2_{\mathbf{b}} - 1}\,. \tag{32}$$

In the same way we obtained (30) and (31) it follows, by (25), (28) and (29), that

$$\begin{aligned}C^B_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e} &= \frac{\gamma_{\mathbf{u}_e}\mathbf{u}_e + \gamma_{\mathbf{v}_e}\mathbf{v}_e + \gamma_{\mathbf{w}_e}\mathbf{w}_e}{\gamma_{\mathbf{u}_e} + \gamma_{\mathbf{v}_e} + \gamma_{\mathbf{w}_e}} \\ &= \frac{2\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + 2\gamma^2_{\mathbf{v}_m}\mathbf{v}_m + 2\gamma^2_{\mathbf{w}_m}\mathbf{w}_m}{(2\gamma^2_{\mathbf{u}_m} - 1) + (2\gamma^2_{\mathbf{v}_m} - 1) + (2\gamma^2_{\mathbf{w}_m} - 1)} \\ &= \frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m + \gamma^2_{\mathbf{w}_m}\mathbf{w}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} + \gamma^2_{\mathbf{w}_m} - \frac{3}{2}}\end{aligned} \tag{33}$$

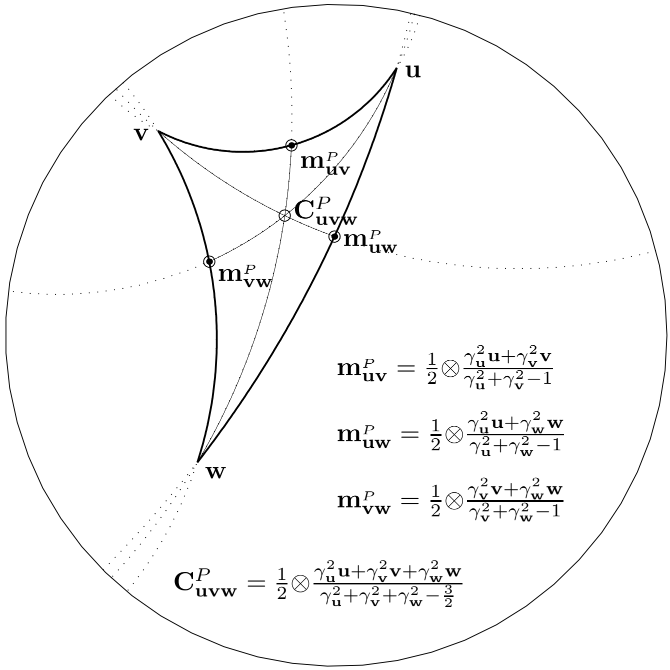

Figure 4: A triangle $\Delta\mathbf{uvw}$ in the Poincaré disc model of hyperbolic geometry is shown with the midpoints $\mathbf{m}^P_{\mathbf{uv}}$, $\mathbf{m}^P_{\mathbf{uw}}$ and $\mathbf{m}^P_{\mathbf{vw}}$ of its sides, and its medians, and centroid $\mathbf{C}^P_{\mathbf{uvw}}$.

so that

$$C^P_{\mathbf{u}_m\mathbf{v}_m\mathbf{w}_m} = \tfrac{1}{2}\otimes C^B_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e} \tag{34}$$

$$= \tfrac{1}{2}\otimes\frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m + \gamma^2_{\mathbf{w}_m}\mathbf{w}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} + \gamma^2_{\mathbf{w}_m} - \frac{3}{2}}$$

for any $\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m \in G_m$.

We have thus obtained in (34) the following

Theorem 10. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{B}$ be any three non-gyrocollinear points of a Poincaré ball model, $\mathbb{B}$, of hyperbolic geometry, where $\mathbb{B}$ is the ball of a real inner product space $\mathbb{V}$. The centroid $\mathbf{C}^P_{\mathbf{abc}}$ of the hyperbolic triangle $\Delta\mathbf{abc}$ in $\mathbb{B}$ is given by the equation*

$$C^P_{\mathbf{abc}} = \tfrac{1}{2}\otimes\frac{\gamma^2_{\mathbf{a}}\mathbf{a} + \gamma^2_{\mathbf{b}}\mathbf{b} + \gamma^2_{\mathbf{c}}\mathbf{c}}{\gamma^2_{\mathbf{a}} + \gamma^2_{\mathbf{b}} + \gamma^2_{\mathbf{c}} - \frac{3}{2}}\,. \tag{35}$$

The hyperbolic triangle centroid was also studied by O. Bottema [2]. The centroid of the hyperbolic triangle $\Delta\mathbf{abc}$ in the Poincaré disc model, as determined by (35), is shown in Figure 4.

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