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Essential P-spaces: a generalization of door spaces

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Abstract. An element f of a commutative ring A with identity element is called a *von Neumann regular element* if there is a g in A such that $f^2g = f$. A point p of a (Tychonoff) space X is called a *P-point* if each f in the ring $C(X)$ of continuous real-valued functions is constant on a neighborhood of p. It is well-known that the ring $C(X)$ is von Neumann regular ring iff each of its elements is a von Neumann regular element; in which case X is called a *P-space*. If all but at most one point of X is a P-point, then X is called an *essential P-space*. In earlier work it was shown that X is an essential P-space iff for each f in $C(X)$, either f or $1 - f$ is von Neumann regular element. Properties of essential P-spaces (which are generalizations of J.L. Kelley's door spaces) are derived with the help of the algebraic properties of $C(X)$. Despite its simple sounding description, an essential P-space is not simple to describe definitively unless its non P-point η is a G_δ, and not even then if there are infinitely many pairwise disjoint cozerosets with η in their closure. The general case is considered and open problems are posed.

Keywords: P-point, P-space, essential P-space, door space, F-space, basically disconnected space, space of minimal prime ideals, SV-ring, SV-space, rank, von Neumann regular ring, von Neumann local ring, Lindelöf space

Classification: 54D, 54G, 13F30, 16A30

1. Introduction

All spaces X considered in this note are assumed to be Tychonoff, $C(X)$ will denote the ring of continuous real-valued functions with domain X, and all rings considered are assumed to be commutative and have an identity element.

In problem B of Chapter II of his book *General Topology* [K55], J.L. Kelley calls a topological space in which every subset is open or closed a *door space*, and notes that every (Hausdorff) door space has at most one nonisolated point. In [HWi92], such spaces are called *almost discrete*. For each $f \in C(X)$, let the zero set of f, $Z(f) = \{x \in X : f(x) = 0\}$, and let its cozeroset $\mathrm{coz}(f) = X - Z(f)$. The family of zerosets (resp. cozerosets) of X is denoted by $\mathcal{Z}(X)$ (resp. $\mathrm{Coz}(X)$). A point $p \in X$ such that for every $f \in C(X)$, $f(p) = 0$ implies $p \in \mathrm{int}\, Z(f)$ is called a *P-point*, and X is called a *P-space* if each of its points is a P-point. It is known that a space X is a P-space if and only if any G_δ-set in X is open, see [GJ76]. Every isolated point is a P-point but not conversely as is noted in [GJ76]. In this paper, we study spaces in which all but one point is a P-point because for such spaces X, the ring $C(X)$ has an easily described algebraic characterization. For most notation and undefined terms the reader is referred to [GJ76].

A commutative ring R is called a *von Neumann regular ring* (*VNR-ring*) if for each $a \in R$ there exists $b \in R$ such that $a = a^2b$. It is shown in [GJ76] that $C(X)$ is a VNR-ring if and only if X is a P-space if and only if for each $f \in C(X)$, $Z(f)$ is open. In [AAH04], a commutative ring R with identity element 1 is called a *von Neumann Local ring* (*VNL-ring*) if for each $a \in R$ there exists $b \in R$ such that $a = a^2b$ or $1 - a = (1 - a)^2b$. There, it is shown that:

1.1 Proposition ([AAH04]). *The following statements are equivalent:*

(a) *the ring $C(X)$ is a VNL-ring;*
(b) *if $f \in C(X)$, then $Z(f)$ or $Z(1 - f)$ is open;*
(c) *at most one point of X fails to be a P-point.*

Spaces with these equivalent properties are called *essential P-spaces*; a class that includes (Tychonoff) door spaces properly. This note is devoted to exploring further the properties of this class of spaces. One question we address is when an essential P-space X is basically disconnected, especially because such a space cannot be a dense subspace of an extremally disconnected space of nonmeasurable cardinality unless X is a door space.

We begin by reviewing some basic definitions and terminology. Our basic sources will be [GJ76], [E89], and [HJ65].

βX denotes the Stone-Čech compactification of X, υX the real compactification of X, and $C^*(X)$ the subring of all bounded functions in $C(X)$. Note that $C(X)$ and $C(\upsilon X)$ are isomorphic as are $C^*(X)$ and $C(\beta X)$. A subspace X is said to be *z-embedded in a space Y* if the map $Z \to Z \cap X$ is a surjection of $\mathcal{Z}(Y)$ onto $\mathcal{Z}(X)$, and if every $f \in C(X)$ (resp. $C^*(X)$) has a continuous extension to $C(Y)$ (resp. $C^*(Y)$), then X is said to be C- (resp. C^*-) *embedded in Y*. It is clear that C-embedded subspaces are C^*-embedded, and C^*-embedded subspaces are z-embedded. In [We75] and [B76], it is shown that if X is either a cozeroset or a Lindelöf subspace of Y, then X is z-embedded in Y.

For any space X, let $P(X)$ denote its subspace of P-points, and if X is an essential P-space that is not a P-space, it will be called a *proper essential P-space* and its non P-point will be denoted by η.

For each $p \in \beta X$, let $M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ and let $O^p = \{f \in C(X) : p \in \mathrm{Int}_{\beta X} \mathrm{cl}_{\beta X} Z(f)\}$. If $p \in X$, one often writes M_p for M^p and O_p for O^p. As is shown in Chapter 7 of [GJ76], every (proper) prime ideal of $C(X)$ lies between O^p and M^p for precisely one point $p \in \beta X$, and it follows from Theorem 5.6(e) of [AAH04] that:

1.2 Proposition. *If X is a proper essential P-space, and P is a nonmaximal prime ideal of $C(X)$, then $P \subset M_\eta$.*

1.3 Definition. The set of minimal prime ideals of $C(X)$ that are not maximal ideals is called the *minimal core of* X and is denoted by $\mathrm{mc}(X)$.

By 1.2, if X is a proper essential P-space, then $\mathrm{mc}(X)$ is a nonempty anti-chain of ideals contained in M_η. Below, we will use it to describe the structure of X.

We are indebted to S. Larson for some valuable comments.

2. Essential P-spaces X such that $\{\eta\}$ is a G_δ

Recall from [HVW87] that a space in which every cozeroset is C^*-embedded is said to be an *F-space*, and if this must hold only for dense cozerosets, it is called a *quasi F-space*. In Chapter 14 of [GJ76], it is shown that X is an F-space if and only if O^p is prime for every $p \in \beta X$. A space X is called *basically disconnected* if $\mathrm{cl}(\mathrm{coz}(f))$ is open for each $f \in C(X)$. It is shown in [GJ76] that X is basically disconnected if and only if βX is basically disconnected. (Thus, if X is a P-space, then both X and βX are basically disconnected.)

See [HVW87] for a discussion of the algebraic properties of quasi F-spaces. It is shown in Proposition 3.2 of [HVW87] that a dense subspace of a quasi F-space is z-embedded in it if and only if it is C^*-embedded.

For each commutative ring R, let $\mathrm{Max}(R)$ denote the set of all maximal ideals and $\mathrm{Min}(R)$ the set of all minimal prime ideals both equipped with the hull kernel topology. As is noted in [HJ65]; $\mathrm{Min}(R)$ is always a zero-dimensional Hausdorff space, and is countably compact. If $R = C(X)$ for some space X, then $\mathrm{Min}\, C(X)$ need not be compact, but it is known that if $\mathrm{Min}\, C(X)$ is compact, then it is basically disconnected. If a point p of X is such that $f \in C(X)$ and $p \in Z(f)$ imply $\mathrm{int}(Z(f)) \neq \emptyset$, then p is called an *almost P-point*. It is shown in 5.5 of [HMW03] that if $\mathrm{Min}(C(X))$ is compact, then any almost P-point of X is a P-point, see also [HM93]. So, the one-point compactification of an uncountable discrete space is an essential P-space with a space of minimal prime ideals that fails to be compact.

Some structure theorems follow.

2.1 Proposition. *If $P(X)$ is dense and z-embedded in X (in particular if $P(X)$ is either a cozeroset of X or a Lindelöf subspace), then $\mathrm{Min}(C(X))$ is compact.*

Proof: As noted above, in either case $P(X)$ is z-embedded in X. In 7.6 of [M70], it is shown that the minimal prime ideal space of $C(X)$ is homeomorphic to the minimal prime ideal space of $C(T)$ for any dense z-embedded subspace T. Because, as is noted in [HJ65] and [HM93], the space of minimal prime ideals of a von Neumann regular ring is compact, the conclusion follows. □

2.2 Theorem. *Suppose X is a proper essential P-space. Then:*

(a) $\mathrm{Min}\, C(X)$ *is compact if and only if its unique non P-point is a G_δ;*
(b) *if X is a quasi F-space and $\{\eta\}$ is a G_δ, then X is basically disconnected.*

PROOF: (a) If $\{\eta\}$ is a G_δ, then $P(X) = X \backslash \{\eta\}$ is a cozeroset, so $\operatorname{Min} C(X)$ is compact by 2.1. Otherwise, η is an almost P-point that is not a P-point, so $\operatorname{Min} C(X)$ is not compact by the remarks preceding 2.1.

(b) As noted in the proof of (a), $P(X)$ is a cozeroset of the quasi F-space X and hence is z-embedded in it. So, by the remarks preceding 2.1, $P(X)$ is C^*-embedded in X which implies that βX and hence X is basically disconnected. $\square$

Hence we have:

2.3 Corollary. *If X is a proper essential P-space and $\{\eta\}$ is a G_δ, then the following are equivalent:*

(a) *X is a quasi F-space;*
(b) *X is basically disconnected;*
(c) *X is an F-space.*
(d) *$|\operatorname{mc}(X)| = 1$.*

Any of the above imply that $\operatorname{Min} C(X)$ is compact and basically disconnected.

PROOF: Clearly (a) implies (b) by 2.2(b). That (b) implies (c) and (c) implies (a) is obvious. If (c) holds, then O_η is prime and hence is the only prime ideal which is not maximal in $C(X)$, so (d) holds. That (d) implies (c) is obvious. $\square$

Remark. The validity of the equivalences above depend essentially on the assumption that $\{\eta\}$ is a G_δ; see Example 3.2 below.

We are also able to say something about essential P-spaces such that $1 < |\operatorname{mc}(X)| < \infty$. To do so, we need to review some definitions and recall some background from other papers. An integral domain D such that whenever $0 \neq a, b$ are in D, then $a|b$ or $b|a$ is called a *valuation domain*. Any field is a valuation domain, as is the subring of the field of rational numbers consisting of all (reduced) fractions with odd denominator.

2.4 Definitions. (i) Let A be a commutative ring with identity. If A/P is a valuation domain for every (proper) prime ideal P, then A is called an *SV-ring*.

(ii) If M is a maximal ideal of A, then the *rank of M* is the number k of minimal prime ideals of A that are contained in M, where $1 \leq k \leq \infty$, and the *rank of A* is the supremum of the ranks of M for every maximal ideal M of A.

(iii) If $A = C(X)$ for a space X and $x \in X$, then by the *rank of x in X*, we mean the rank of M_x.

It is shown in 3.1 of [L97] or [HLMW94] that the rank of x in X is $k < \infty$ if there is a family of k pairwise disjoint cozerosets with x in their closure, and no such family of $k+1$ cozerosets.

To show that a commutative semiprime ring A with identity is an SV-ring, it suffices to show that A/P is a valuation domain for every minimal prime ideal P.

SV-rings that are subdirect products of totally ordered rings are studied extensively in [HLMW94], [L97], and in special cases in [HWi(a)92] and [HWi(b)92]. If $C(X)$ is an SV-ring, then X is called an *SV-space*.

From the above, and Theorem 5.6 of [AHH03], we have:

2.5 Theorem. *If X is a proper essential P-space with non P-point η, then the rank of $C(X)$ is the rank of M_η. Moreover, X is an SV-space if and only if $C(X)/P$ is a valuation domain for every minimal prime ideal contained in M_η.*

Our next result describes precisely when an essential P-space X is an SV-space.

2.6 Theorem. *An essential P-space X is an SV-space if and only if it has finite rank, that is if and only if it is a P-space or X is not a P-space and there is a positive integer k such that there are no more than k pairwise disjoint cozerosets each of which contains η in its closure.*

Proof: It is shown in 4.1 and 4.2 of [HLMW94] that if X is an SV-space, then X has finite rank. By 2.5, the rank of a proper essential P-space is its rank at η, and by the remarks following 2.4, this latter is the maximal number of pairwise disjoint cozerosets each of which contains η in its closure. By 1.3 of [L03], the rank of X is the same as that of βX. In 3.11 of [L97], it is shown that a normal space of finite rank whose set of points of rank greater than 1 is countable and discrete must be an SV-space. It follows then because βX is normal, that if X is an essential P-space of finite rank, then βX, and hence X is an SV-space. □

We may summarize many of the results given above to obtain:

2.7 Corollary. *If X is a Tychonoff space, then the following are equivalent:*

(a) *$C(X)$ is a von Neumann local ring such that $C(X)/P$ is a valuation domain whenever P is a prime ideal of $C(X)$;*
(b) *at most one point of X fails to be a P-point and no maximal ideal of $C(X)$ contains infinitely many minimal prime ideals.*

2.8 Nonnormal essential P-spaces.

It is well known that every door space is normal; see [HWi(a)92]. As is noted in 7.7 in [GH54] and in Sections 4 and 6 of [LR81], there are nonnormal P-spaces. The topological sum of a nonnormal P-space and any proper essential P-space is an essential P-space.

Another way to construct a nonnormal essential P-space follows: Given an infinite P-space T and an infinite index set Γ, let T_α denote a copy of T for each $\alpha \in \Gamma$. Let $S = \bigoplus_{\alpha\in\Gamma} T_\alpha$ denote the topological sum of all of the T_α, and let $S(w) = S \cup \{w\}$, where $w \notin S$. If $s \in S$, basic neighborhoods of s in $S(w)$ will be those of the topological sum S, while neighborhoods of w will contain w together with all but finitely many of the T_αs. It is easy to verify that with this topology, $S(w)$ is an essential P-space with non P-point w. Moreover, a finite union of the sets T_α is closed, so if T is chosen to be nonnormal, then so is $S(w)$. (Note also that any set that meets infinitely many of the T_αs in one point is neither closed nor open.) Finally, it is not difficult to see that $S(w)$ has infinite rank.

2.9 Definition. A space X such that βX is a finite union of closed F-spaces is said to be *finitely an F-space*.

It follows easily from 2.8 of [HWi(a)92] that:

2.10 Proposition. *If X is finitely an F-space, then it is an SV-space.*

Recently a rather complicated example of an SV-space that is not finitely an F-space was constructed by S. Larson in [L03], but we do not know if there is such an example that is an essential P-space.

Note that by 2.3 and the remarks following 2.4, an essential P-space in which $\{\eta\}$ is a G_δ is a finite union of basically disconnected spaces as is observed in 3.1 of [HWi(b)92]. If, however,the converse of 2.10 holds for SV-spaces that are essential P-spaces of finite rank, we would have a characterization of such spaces.

2.11 Example: Spaces that are finitely an F-space.

Suppose k is a positive integer and $\{X_i : 1 \leq i \leq k\}$ are basically disconnected spaces such that $(X_i \backslash P(X_i)) = \eta_i$ has exactly one point. The space X obtained from the topological sum of the X_i by collapsing all of the η_i to one point η, and imposing the quotient topology is finitely an F-space, as is its topological sum with any P-space.

We conjecture that all essential P-spaces that are finitely an F-space can be obtained in this way.

3. Miscellaneous results when $\{\eta\}$ need not be a G_δ

What little we know about (proper) essential P-spaces when $\{\eta\}$ fails to be a G_δ is recorded below.

3.1 Remark. If X is an essential P-space (more generally, if $X \backslash P(X)$ is finite), then every prime z-ideal of $C(X)$ is minimal or maximal, see 2.5 of [HMW03].

Next, we give the example promised after Corollary 2.3.

3.2 Example: An essential P-space that is an F-space without being basically disconnected.

Let $X = D \cup \{p_1\}$ denote the one-point Lindelöfication of a discrete space of cardinality $\aleph_1$. (Neighborhoods of p_1 intersect D in a co-countable set.) Let $Y = N \cup \{p_2\}$, where N is a countable discrete space and $p_2 \in \beta N \backslash N$ considered as a subspace of βN, and let W denote the space obtained by pasting X and Y together at the common point p obtained by collapsing p_1 and p_2 together. Because p is the only nonisolated point of W, W is an essential P-space. It is an exercise to verify that W is an F-space. $N \subset W$ is a cozeroset of W whose closure $N \cup \{p\}$ is not open in W, so W is not basically disconnected. □

It can be shown that $\operatorname{Min} C(W)$ is locally compact by modifying slightly the argument given in Example 5.7 in [HJ65].

As is well known, a locally compact P-space is discrete. For essential P-spaces we have:

3.3 Theorem. *X is a locally compact proper essential P-space if and only if it is the topological sum of a one-point compactification of an infinite discrete space and a discrete space.*

Proof: The sufficiency is clear from the definition of an essential P-space. For the converse, we may assume X is proper, in which case $X \backslash \{\eta\}$ is a locally compact P-space and so is discrete, since it is an open subspace of X. Let U be a compact set containing η, then U is a one point compactification of the infinite discrete set $U - \{\eta\}$, see 5.6 in [AAH04]. Hence $X = U \oplus (X - U)$. □

In [HP86] *L-closed* spaces are defined to be spaces in which Lindelöf subspaces are closed. The authors showed that every P-space is L-closed.

The analogous result for essential P-spaces is:

3.4 Theorem. *Every Lindelöf subspace of a proper essential P-space X that contains η is closed.*

Proof: Suppose A is a Lindelöf subspace of X containing η and $x \in X - A$. For each $a \in A$ there exist two open disjoint sets U_a, V_a such that $x \in U_a$ and $a \in V_a$. Then the open cover $\{V_a\}_{a \in A}$ of the Lindelöf subspace A has a countable subcover $\{V_{a_i}\}$ whose union is disjoint from $\bigcap_{i=1}^{\infty} U_{a_i}$. Because x is a P-point, this latter set is an open neighborhood of x, so A is closed. □

Much of the rest of this section is concerned with determining when projection maps in product spaces are closed. In addition to the explicit references made below, the reader should examine [N69].

3.5 Remarks. (a) The complement of $\{\eta\}$ in the one-point compactification of a countably infinite discrete space shows that not every Lindelöf subspace of an essential P-space need be closed.

(b) In [HWo88], an example is given of an L-closed space with no P-points. So no sort of converse of 3.4 can hold.

(c) It is an exercise to verify that any quotient space of an essential P-space is an essential P-space.

(d) Because an essential P-space has at most one non P-point, a topological sum of essential P-spaces is an essential P-space if and only if at most one of them is proper.

(e) It is well known that if X is Lindelöf, Y is a P-space, and $p : X \times Y \to Y$ is the projection map, then p is closed. (See, for example, 8B of [Wa74].) The analog for essential P-spaces is:

3.6 Theorem. *If X is Lindelöf, Y is a proper essential P-space, $p : X \times Y \to Y$ is the projection map, $A \subset X$ is closed, and $\eta \in p(A)$, then $p(A)$ is closed.*

Proof: Let $y \in Y \backslash p(A)$. Then $p^{-1}(y) = X \times \{y\} \subseteq X \times Y \backslash A$. For each $x \in X$, there exist two open sets $O_x \subseteq X$ and $V_x \subseteq Y$ such that $(x, y) \in O_x \times V_x \subseteq X \times Y \backslash A$ and $\eta \notin Cl_Y V_x$. Then, since X is Lindelöf, there is a countable subfamily $\{O_{x_i}\}$ of $\{O_x : x \in X\}$ such that $X = \bigcup_{i=1}^{\infty} O_{x_i}$. If $V = \bigcap_{i=1}^{\infty} V_{x_i}$, then V is a G_δ-set in Y not containing η and so is open. Now, $p^{-1}(V) = X \times V \subseteq X \times Y \backslash A$ which implies that $V \cap p(A) = \emptyset$. Hence $p(A)$ is closed in Y. □

The next example shows that the projection map in the previous theorem needs not be closed.

3.7 Example. Let $X = [0, \infty)$ with its usual topology and let $Y = [0, \infty)$ with the topology obtained from X by making all points in $(0, \infty)$ isolated. Then both X and Y are Lindelöf, and Y is a proper essential P-space. Then $A = \{(x, y) \in X \times Y : xy = 1\}$ is closed in $X \times Y$, but $p(A) = (0, \infty)$ is not closed in Y. □

It is well known that while the product of two Lindelöf spaces need not be Lindelöf, the product of two Lindelöf spaces one of which is a P-space is Lindelöf. Our final results strengthen this.

3.8 Theorem. *If X, Y are Lindelöf spaces and the set Y_0 of points of Y that fail to be P-points is a countable closed set, then $X \times Y$ is a Lindelöf space.*

Proof: Let $\tilde{O}$ denote an open cover of $X \times Y$. If $z \in Y_0$ and $x \in X$, there exist open sets $U_x^z \subset X$, $V_x^z \subset Y$, and $O_x^z \in \tilde{O}$ such that $(x, z) \in U_x^z \times V_x^z \subset O_x^z$. Since X is a Lindelöf space, it follows that $X = \bigcup_{x \in X} U_x^z = \bigcup_{i=1}^{\infty} U_{x_i}^z$ for some countable subfamily $\{U_{x_i}^z\}_{i=1}^{\infty}$ of $\{U_x^z\}_{x \in X}$. If $V_z = \bigcap_{i=1}^{\infty} V_{x_i}^z$, then $X \times V_z \subseteq \bigcup_{i=1}^{\infty} O_{x_i}^z$. Let $Y_1 = Y - \bigcup_{z \in Y_0} V_z$ and note that $Y_1 \cap Y_0 = \emptyset$ since $Y_0 \subseteq \bigcup_{z \in Y_0} V_z$. If $y \in Y_1$, then for each $x \in X$, there exist open sets $U_x^y \subset X, V_x^y \subset Y - Y_0$, and $O_x^y \in \tilde{O}$ such that $(x, y) \in U_x^y \times V_x^y \subset O_x^y$. Since $X = \bigcup_{x \in X} U_x^y = \bigcup_{i=1}^{\infty} U_{x_i}^y$ it follows that $X \times \{y\} \subseteq \bigcup_{i=1}^{\infty} (U_{x_i}^y \times V_{x_i}^y) \subseteq \bigcup_{i=1}^{\infty} O_{x_i}^y$. If $V_y = \bigcap_{i=1}^{\infty} V_{x_i}^y$, then we have $X \times V_y \subseteq \bigcup_{i=1}^{\infty} (U_{x_i}^y \times V_{x_i}^y) \subseteq \bigcup_{i=1}^{\infty} O_{x_i}^y$. But V_y is an open subspace of $Y - Y_0$ since it is a G_δ-set contained in the open P-space $Y - Y_0$. Because $Y_1 = Y - \bigcup_{z \in Y_0} V_z = Y - \bigcup_{z \in Y_0} \left(\bigcap_{i=1}^{\infty} V_{x_i}^z\right) = Y - \bigcap_{i=1}^{\infty} \left(\bigcup_{z \in Y_0} V_{x_i}^z\right) = \bigcup_{i=1}^{\infty} \left(Y - \left(\bigcup_{z \in Y_0} V_{x_i}^z\right)\right) = \bigcup\limits_{i=1}^{\infty} \left(\bigcap_{z \in Y_0} (Y - V_{x_i}^z)\right)$, it is an F_σ-sets in the Lindelöf space Y and so is a Lindelöf space. But $Y_1 \subseteq \bigcup_{y \in Y_1} V_y$ and therefore $Y_1 \subset \bigcup_{j=1}^{\infty} V_{y_j}$.

If $p : X \times Y \to Y$ is the projection map, then we have since $Y_1 \subseteq \bigcup_{j=1}^{\infty} V_{y_j}$,

$$\begin{aligned}
X \times Y = p^{-1}(Y) &= p^{-1}\Big(\Big(Y - \bigcup_{z \in Y_0} V_z\Big) \cup \Big(\bigcup_{z \in Y_0} V_z\Big)\Big) \\
&= p^{-1}(Y_1) \cup p^{-1}\Big(\bigcup_{z \in Y_0} V_z\Big) \\
&\subset p^{-1}\Big(\bigcup_{j=1}^{\infty} V_{y_j}\Big) \cup p^{-1}\Big(\bigcup_{z \in Y_0} V_z\Big) \\
&= \Big(\bigcup_{j=1}^{\infty} p^{-1}(V_{y_j})\Big) \cup \Big(\bigcup_{z \in Y_0} p^{-1}(V_z)\Big) \\
&= \Big(\bigcup_{j=1}^{\infty} (X \times V_{y_j})\Big) \cup \Big(\bigcup_{z \in Y_0} (X \times V_z)\Big) \\
&\subset \Big(\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} O_{x_i}^{y_j}\Big) \cup \Big(\bigcup_{z \in Y_0} \bigcup_{i=1}^{\infty} O_{x_i}^{z}\Big) \\
&\subset X \times Y.
\end{aligned}$$

Hence $X \times Y$ is a Lindelöf space. □

3.9 Corollary. *If X, Y are Lindelöf spaces and Y is an essential P-space, then $X \times Y$ is a Lindelöf space.*

4. What remains to be done

Reading the above reveals that despite their simple description, essential P-spaces can be complex in character. A lot is known about them when their non P-point η is a G_δ and any family of pairwise disjoint cozerosets with η in their closure is finite, less if this latter family is allowed to be infinite, and even less if $\{\eta\}$ fails to be a G_δ. Among the questions that arise are:

What can be said about $\operatorname{Min} C(X)$ if X is an essential P-space? Must it always be locally compact? Must it always be basically disconnected?

Is there a characterization of essential P-spaces such that $P(X)$ is a Lindelöf space?

Which of the results presented above extend to (Tychonoff) spaces such that $X \backslash P(X)$ is finite?

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