Orientations and 3-colourings of graphs

Vincent Chouinard-Prévost, Alexandre Côté, Claude Tardif

Abstract. We provide the list of all paths with at most 16 arcs with the property that if a graph G admits an orientation $\vec{G}$ such that one of the paths in our list admits no homomorphism to $\vec{G}$, then G is 3-colourable.

Keywords: graph colourings, paths, Hasse-Gallai-Roy Theorem

Classification: 05C15

1. Introduction

Let $\vec{P}_n$ denote the directed path with vertices $0, 1, \dots, n$ and arcs $(0, 1)$, $(1, 2)$, $\dots$, $(n-1, n)$. One well-known result on graph colourings states that if a graph G admits an acyclic orientation $\vec{G}$ containing no copy of $\vec{P}_n$, then G is n-colourable. It seems to have been discovered independently by Gallai [1], Hasse [2], Roy [5] and possibly others. In terms of homomorphisms, the result can be stated as follows: If there is no homomorphism from $\vec{P}_n$ to $\vec{G}$, then G is n-colourable. In fact, a little more can be said: The absence of homomorphism from $\vec{P}_n$ to $\vec{G}$ implies that there exists a homomorphism from $\vec{G}$ to the transitive tournament $\vec{T}_n$ on n vertices; the n-colouring of G can then be recovered from this homomorphism. This is an example of a "homomorphism duality" between $\vec{P}_n$ and $\vec{T}_n$. More generally, the following holds:

Theorem 1 ([3]). *For every directed tree $\vec{T}$, there exists a directed graph $\vec{D}_{\vec{T}}$ such that for every directed graph $\vec{G}$, there exists a homomorphism from $\vec{G}$ to $\vec{D}_{\vec{T}}$ if and only if there is no homomorphism from $\vec{T}$ to $\vec{G}$.*

Thus, if for some directed tree $\vec{T}$ we know that $\chi(\vec{D}_{\vec{T}}) \leq n$, we get the following variation of the Gallai-Hasse-Roy theorem:

> Let G be a graph which admits an orientation $\vec{G}$ such that there is no homomorphism from $\vec{T}$ to $\vec{G}$. Then $\chi(G) \leq n$.

We will call a path $\vec{T}$ an *n-colourability certificate* if $\chi(\vec{D}_{\vec{T}}) \leq n$. In particular, the following is known:

Theorem 2 ([4]). *For integers k, n such that $2 \leq k \leq n$, let $\vec{P}_{k,n}$ denote the directed path with k forward edges followed by one backward edge and $n+2-k$ forward edges. Then $\vec{P}_{k,n}$ is an n-colourability certificate.*

For $k = 2$ or n, this gives a stronger test than the Hasse-Gallai-Roy theorem; the other tests are independent. It would be interesting to obtain a general characterisation of the n-colourability certificates. So far, the results of Theorem 2 are the only ones available in this direction. In this note, we present the results of a computer search for the 3-colourability certificates with at most 16 arcs, in the hope that it will lead to further theoretical development in the field.

2. The results

We will use binary notation to represent paths, a 0 denoting a backward arc, and a 1 a forward arc. Thus $\vec{P}_{3,3}$ and $\vec{P}_{2,3}$ are represented respectively by 111011 and 110111. If there exists a homomorphism from the path $\vec{T}$ to the path $\vec{T}'$, then there also exist a homomorphism from $\vec{D}_{\vec{T}}$ to $\vec{D}_{\vec{T}'}$. Thus if $\vec{T}'$ is an n-colourability certificate, then so is $\vec{T}$. In listing the 3-colourability certificates with at most 16 arcs, we need only to list those which are maximal with respect to the homomorphism order, that is, those who do not admit homomorphisms to other paths in the list.

In [4], the dual $\vec{D}_{\vec{T}}$ of a tree $\vec{T}$ is characterized as follows: The vertices of $\vec{D}_{\vec{T}}$ are the functions from the vertex set of $\vec{T}$ to the arc set of $\vec{T}$ such that $f(u)$ is incident to u for all u. There is an arc from f to g in $\vec{D}_{\vec{T}}$ if for every arc (u, v) of $\vec{T}$, we have $f(u) \neq (u, v)$ or $g(v) \neq (u, v)$ (or both). Thus for paths $\vec{T}$ with up to 16 arcs, $\vec{D}_{\vec{T}}$ can have as many as 2^{15} vertices. Švejdarová [6] proved that apart from the isolated vertices, $\vec{D}_{\vec{T}}$ has only one connected component. We can of course remove all the isolated vertices, along with the vertices which are "dominated" in the sense that their in-neighbourhood and out-neighbourhood are contained it those of other vertices. In this way it is generally possible to dismantle $\vec{D}_{\vec{T}}$ down to a size which can be managed by a graph colouring program.

There are 2^{16} strings of zeroes and ones which correspond to paths with 16 arcs. However most paths admit two encodings, obtained respectively by reading it from left to right and from right to left. Moreover every path $\vec{T}$ admits an "inverse" $\vec{T}^{-1}$ obtained by reversing the orientation of every arc; $\vec{D}_{\vec{T}^{-1}}$ is then obtained by reversing the arcs of $\vec{D}_{\vec{T}}$. Therefore $\chi(\vec{D}_{\vec{T}^{-1}}) = \chi(\vec{D}_{\vec{T}})$, hence $\vec{T}^{-1}$ belongs to our list if and only if $\vec{T}$ does. The task of filtering through the remaining candidates is still considerable, but it is possible to reduce it repeatedly by removing strings containing substrings such as 1111, 1110111 and others, which correspond to paths whose dual is known not to be 3-colourable. We eventually obtained the list presented in the following table.

Paths	Number of vertices in the dual
00010100100, 11101011011	20
00010010100, 11101101011	13
00100010100, 11011101011	15

The maximal 3-colouring certificates with at most 16 arcs

These paths each consist of a copy of $\vec{P}_3$ and two copies of $\vec{P}_2$, joined together by a fence of length 1 and a fence of length 3 in all possible ways. Thus our family has a nice structure, and perhaps the most surprising fact is that all the maximal 3-colouring certificates with at most 16 arcs have exactly 11 arcs. Thus there is some hope that a nice characterisation of the n-colouring certificates may be found.

For the moment, it is not known whether a 3-colouring certificate can contain more than one copy of $\vec{P}_3$. In this direction we are able to prove negative results concerning two families of paths. Let $\vec{F}_{1,n}$ denote the path 11101010...010111 (with n zeroes) consisting of two copies of $\vec{P}_n$ joined by a "fence". Similarly, let $\vec{F}_{2,n}$ denote the path 1110011001100...001100111 (with $2n$ zeroes).

Proposition 3. *None of the paths $\vec{F}_{1,n}$, $\vec{F}_{2,n}$, $n \geq 1$ are 3-colouring certificates.*

PROOF: We will show that for every path $\vec{F}_{1,n}$, $\vec{F}_{2,n}$, there exists a 4-chromatic oriented graph $\vec{G}$ such that there is no homomorphism from $\vec{F}_{1,n}$ or $\vec{F}_{2,n}$ to $\vec{G}$. For $m \geq 1$, let A_m be the graph with vertex set $\{0, 1, \ldots, 6m+2\}$ and edges $[i-1, i]$, $i = 1, \ldots, 6m+2$ and $[i-1, i+1]$, $i = 1, \ldots, 6m+1$. Then A_m is uniquely 3-colourable; we fix the 3-colouring $c : A_m \mapsto \{0, 1, 2\}$ defined by $c(i) = i \bmod 3$, and we give an orientation $\vec{A}_m$ to A_m by pointing each edge towards its endpoint with the largest colour. We then add a vertex t to $\vec{A}_m$, along with the arcs $(0, t), (t, 6m+2)$, to get a new oriented graph $\vec{B}_m$. Note that $\vec{B}_m$ is also uniquely 3-colourable, with $c(t) = 1$; in particular, $c(t) = c(3m+1)$; independently of the labels of the colours.

The oriented graph $\vec{G}_m$ is obtained from two copies $\vec{B'}_m$, $\vec{B''}_m$ of $\vec{B}_m$, by identifying the vertices $3m+1'$ and $3m+1''$ and adding the arc (t', t''). We then have $\chi(\vec{G}_m) = 4$; and the only copy of $\vec{P}_3$ contained in $\vec{G}_m$ is the one spanned by the vertices $0', t', t'', 6m+2''$. However any fence of type 01010...010 or 001100...001100 joining $6m+2''$ to $0'$ in $\vec{G}_m$ must pass through the identified vertex $6m+1' = 6m+1''$, hence have at least as many vertices as the diameter of $\vec{A}_m$, which is roughly $3m+1$. Thus for any fixed n, there exists an integer m such that there is no homomorphism from $\vec{F}_{1,n}$ or $\vec{F}_{2,n}$ to $\vec{G}_m$. □

3. Algorithmic considerations

The verification that there is no homomorphism from a path $\vec{T} = x_1 x_2 \dots x_n$ to an oriented graph $\vec{G}$ amounts to (boolean) matrix multiplication: Let A be the adjacency matrix of $\vec{G}$. Consider the product $M = \prod_{i=1}^{n} M_i$, where M_i is A if $x_i = 1$ and A^T if $x_i = 0$. Then $M = 0$ if and only if there is no homomorphism from $\vec{T}$ to $\vec{G}$. Of course, with large paths or families of paths, a substantial amount of time may be saved by storing subproducts that will be used repeatedly. Thus, it is possible to implement quick n-colourability tests using n-colouring certificates.

Of course, the question as to whether such an implementation would be worth the while depends on the strength of the n-colourability certificates that can be characterized. It is unlikely that a perfectly reliable family of tests can be characterized, since 3-colouring is an NP-complete problem. However, even a lower reliability could be useful for practical purposes, and intriguing from a theoretical point of view.

We found it interesting to submit our small family of paths to some tests to see how reliable it was. The input graphs were subgraphs of the cartesian product $G \times H$, where G is the Petersen graph and H is the graph obtained from the 9-cycle by adding edges between vertices with a common neighbour. That is, the vertices of $G \times H$ are the couples (u, v) with $u \in V(G)$ and $v \in V(H)$, and the edges of $G \times H$ are the pairs $[(u, v), (u', v')]$ such that $u = u'$ and $[v, v'] \in E(H)$ or $[u, u'] \in E(G)$ and $v = v'$. Then $G \times H$ is 3-colourable, but its 3-colourings are highly constrained, since H is uniquely 3-colourable and G has few 3-colourings. Thus the greedy algorithm seldom succeeds in 3-colouring it.

For our first test, we selected subgraphs of $G \times H$ and oriented them as follows: starting from a random ordering $x_1, x_2, \dots x_{90}$ of the vertices of $G \times H$, we coloured them using the greedy algorithm until we reached a first vertex x_k which needed a fourth colour. We then took the subgraph X of $G \times H$ induced by $\{x_1, \dots, x_k\}$, and oriented it by pointing each edge towards its endpoint with the largest colour. Because of the vertex x_k, $\vec{X}$ necessarily contains a copy of $\vec{P}_3$ hence the original Gallai-Hasse-Roy fails to prove that X is 3-colourable. In 100000 instances, at least one of our 3-colouring certificates proved that X was 3-colourable 30% of the time, and 1.5% of the time one of the 3-colouring certificates with eleven arcs detected 3-colourability while none of the 3-colouring certificates with six arcs did. Individually, the 3-colouring certificates with eleven arcs each detected 3-colourability in 10% to 25% of the cases. The two paths 00100010100 and 11011101011 were the most efficient, each detecting 3-colourability in 25% of the cases. However, every time one of these two paths detected 3-colourability, at least one other path also detected 3-colourability. Every other path with 11 arcs was sometimes alone in detecting 3-colourability.

This behaviour of the paths 00100010100 and 11011101011 prompted us to do a second test. It was very much like the first, except that the greedy colouring proceeded until we reached a second vertex that could not be coloured with the first three colours. This time, in 100000 instances, the performance of our 3-colouring certificates dropped to 4.8%, and 1.2% of the time one of the 3-colouring certificates with eleven arcs detected 3-colourability while none of the 3-colouring certificates with six arcs did. Each path detected 3-colourability in 0.5% to 3.5% of the cases, and this time each path with 11 arcs was sometimes alone in detecting 3-colourability.

It is interesting to note that while the reliability of our family of 3-colouring certificates decreased considerably in the second test, the relative efficiency of the paths with 11 arcs increased notably when compared to that of the paths with 6 arcs. In the case of general graphs, the reliability of our family of certificates is probably much lower still, but the relative increase indicates that it is probably worth the while to seek larger families of n-colourability certificates.

Now, any colouring algorithm derived from a finite family of n-colourability certificates can be made constructive: For any tree $\vec{T}$, there is a polynomial algorithm which finds a homomorphism from $\vec{G}$ to $\vec{D}_{\vec{T}}$ whenever there is no homomorphism from $\vec{T}$ to $\vec{G}$. Therefore it suffices to store an n-colouring of $\vec{D}_{\vec{T}}$ to derive an n-colouring of $\vec{G}$. This situation would change if one could find an infinite family of n-colouring certificates. On one hand the n-colouring tests derived from this family could be much more reliable, but the n-colouring tests derived from this family would no longer be necessarily constructive.

References

[1] Gallai T., *On directed paths and circuits*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 115–118.

[2] Hasse M., *Zur algebraischen Begründung der Graphentheorie. I.*, Math. Nachr. **28** (1964/1965), 275–290.

[3] Nešetřil J., C. Tardif C., *Duality Theorems for Finite Structures* (*Characterizing Gaps and Good Characterizations*), J. Combin. Theory Ser B **80** (2000), 80–97.

[4] Nešetřil J., C. Tardif C., *A dualistic approach to bounding the chromatic number of a graph*, preprint, 2001, 9 pages ms.

[5] Roy B., *Nombre chromatique et plus longs chemins d'un graphe*, Rev. Francaise Informat. Recherche Opérationelle **1** (1967), 129–132.

[6] Švejdarová I., *Colouring of graphs and dual objects* (in Czech), Thesis, Charles University, 2003.

Royal Military College of Canada, PO Box 17000 Stn Forces, Kingston,
ON K7K 7B4, Canada

(*Received* August 29, 2003, *revised* February 2, 2004)